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Interest Rate Dynamics, Variable-Rate Loans, and the Business Cycle*

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Abstract

The interest rate at which US firms borrow funds has two features: (*i*) it moves in a countercyclical fashion and (*ii*) it is an inverted leading indicator of real economic activity: low interest rates forecast booms in GDP, consumption, investment, and employment. We show that a Kiyotaki-Moore model accounts for both properties when business-cycle movements are driven, in a significant way, by animal spirit shocks to credit-financed investment demand. The credit-based nature of such self-fulfilling equilibria is shown to be essential: the dynamic correlation between current loanable funds rate and future aggregate economic activity depends critically on the property that the loan has a variable-rate component. In addition, Bayesian estimation of our benchmark DSGE model on US data 1975-2010 shows that movements in investment driven by animal spirits are quantitatively important and result in a better fit to the data than both standard RBC models and Kiyotaki-Moore type models with unique equilibrium.

Keywords: Endogenous Borrowing Constraints, Collateral, Variable-Rate Loans, Multiple Equilibria, Sunspot Shocks.

JEL codes: E21, E22, E32, E44, E63.

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1 Introduction

The inverted leading indicator property of the borrowing cost is a long-standing puzzle. In US data, low real interest rates are negatively correlated with both current and future investment and output booms. However, standard real business-cycle (RBC thereafter) models deliver the opposite relationship: high investment and output are associated with high interest rate (see, e.g., King and Watson, 1996). The reason behind such counterfactual predictions of one-sector RBC settings is simple. In such models the real interest rate is dictated by the marginal product of capital, which is proportional to the output-to-capital ratio. Given that output is more volatile than the capital stock, high output thus always implies high interest rate regardless of the source of shocks. Solutions to such a puzzle are so scarce that, in fact, we know of only one:[1] the two-sector, representative-agent RBC model of Boldrin, Christiano and Fisher (2001).

In this paper, we take a different route to solve this long-standing puzzle, as we introduce a credit market that channels funds from lenders to borrowers. Due to borrowing constraints *à la* Kiyotaki and Moore (1997) - KM thereafter - the credit market friction creates a wedge between credit supply and credit demand. However, this wedge by itself is not sufficient for the loanable funds rate to be countercyclical because in equilibrium credit demand still depends on the rate of return to capital: the cost of borrowing is still dictated by the benefit of borrowing and investing, that is, by the marginal product of capital, so that high credit demand (associated with high capital returns) results in high interest rates. Our main theoretical finding is that if the loan is such that the loanable funds rate is not pre-determined, or set when the loan is negotiated, but instead responds to changes in credit market conditions when the loan payment is due (as in the case of variable-rate loans), then this market structure featuring adjustable loanable funds rate leads to an interesting property of the credit market: when the demand for loans increases, the supply of loans increases by more in response to the higher credit demand, so that the equilibrium interest rate falls instead of rising, leading to countercyclical real interest rates. This also suggests that the low-rate-based economic boom can be self-fulfilling: in the absence of any fundamental shocks, the very anticipation by borrowers of a lower debt payment due to a lower expected interest rate can stimulate credit demand and aggregate investment, resulting in an economic boom and fulfilling the initial optimistic expectations. Conversely, expectations of high debt interest payments can trigger a recession and an interest rate hike in the credit market, as if a higher credit risk had materialized and had reduced loanable funds even though it is in fact not the case.

The fact that the borrowing cost faced by US firms is countercyclical has far-reaching macroeconomic consequences. When the borrowing cost is low, financing investment is easier and the economy booms. Figures 1 and 2 report the impulse response functions (IRFs thereafter), at quarterly frequency, of real land price, the inverse relative price of capital, real consumption, real investment, real business debt, hours worked, real GDP, and real borrowing interest rate faced by corporate and noncorporate firms. Those IRFs are obtained from two vector autoregressive (VAR) models, using Cholesky decomposition and ordering first either land price (figures

[1] Backus, Kehoe and Kydland (1994) addressed a similar puzzle arising from international trade data, using a two-country RBC model.

1) or investment (figure 2).[2] Both figures make clear that all variables are procyclical, except the borrowing interest rate. When there is a positive shock to either land price or investment, the interest rate stays below trend for several quarters while all variables boom. To the extent that both credit demand (by firms) and credit supply (by investors and financial intermediaries) are procyclical, this evidence suggests that changes regarding the supply of loanable funds dominate those of demand for loans. While data clearly shows that the borrowing cost is countercyclical, standard RBC models counterfactually predict that the interest rate is procyclical, as noticed above.[3] Since there is no credit market in the standard one-sector RBC model, one might wonder whether or not theoretical predictions agree with empirical evidence in meaningful extensions of the textbook model.

In this paper, we consider various versions of dynamic models that incorporate a credit market and endogenous collateral constraints following the seminal contribution of KM, whose setting has become a building block of current DSGE models with financial frictions. Our main contribution is to show that the loanable funds rate is countercyclical only in versions of the model such that the unique steady state is indeterminate, which in turn happens if loans have a variable-rate component. In other words, collateralized lending with predetermined borrowing cost delivers a procyclical interest rate that is at odds with data while, in sharp contrast, collateralized loans with variable rates accord with empirical evidence. A striking implication of our results is therefore that self-fulfilling swings, and in particular investment fluctuations caused by "animal spirits" and supported by collateralized borrowing, are an important driver behind actual business cycles both in theory and in the data.

Our focus on collateral requirements is dictated by the fact that they are a prominent feature of loans in many economies around the world, both in developed and in developing countries. It is well understood both in practice and in theory that contractual agreements involving some form of collateral brought by borrowers mitigate the consequences of asymmetric information in debtor-creditor relationships (see for example the textbook by Tirole, 2006, chapter 4). In particular, because collateralized borrowing reduces default risk, conventional wisdom holds that financial institutions that rely more on secured debt - and less on unsecured debt - should be less prone to financial crisis.[4] This paper shows, however, that such conventional wisdom is not necessarily correct: even collateralized lending can itself be a source of self-fulfilling credit cycles and financial instability. This finding is thus surprising for two reasons: (*i*) it is against the common view that secured borrowing is safer and thus promotes macroeconomic stability; (ii) it is a salient feature of KM-type models.

Collateralized borrowing hinges on market values, yet such market values are endogenous to the economy and out of control by creditors and debtors. Thus, intuition tells us that endogenous collateral constraints may subject the economy to speculation and self-fulfilling financial crisis. When the market value of collateral is above trend, for example, the practice of collateralized borrowing stimulates, instead of curtailing, credit lending, fueling the asset boom. Conversely, when the market value of collateral is below trend, collateralized borrowing restricts credit lending instead of relaxing it, exacerbating the crisis in a downturn. Hence, the market value of

[2] The first source of shocks is consistent with the collateral channel documented by Chaney, Sraer and Thesmar (2012) while the second embodies the keynesian notion of investment booms and busts driven by animal spirits, which turns out to be empirically relevant in our setting as we will show.

[3] Of course, such a negative correlation between the market cost of borrowing and aggregate variables is at the heart of countercyclical policies, which aim at lowering interest rates in recessions so as to boost investment. Our results suggest such monetary policies - that set the nominal short term rate - may not be the full story behind countercyclical real interest rate movements.

[4] For recent theoretical models that shows the inherent instability of financial institutions under uncollateralized lending practices, see Gu, Mattesini, Monnet, and Wright (2013), Azariadis, Kaas, and Wen (2015).

Figure 1: IRFs from VAR model with land price ordered first - one standard deviation shock

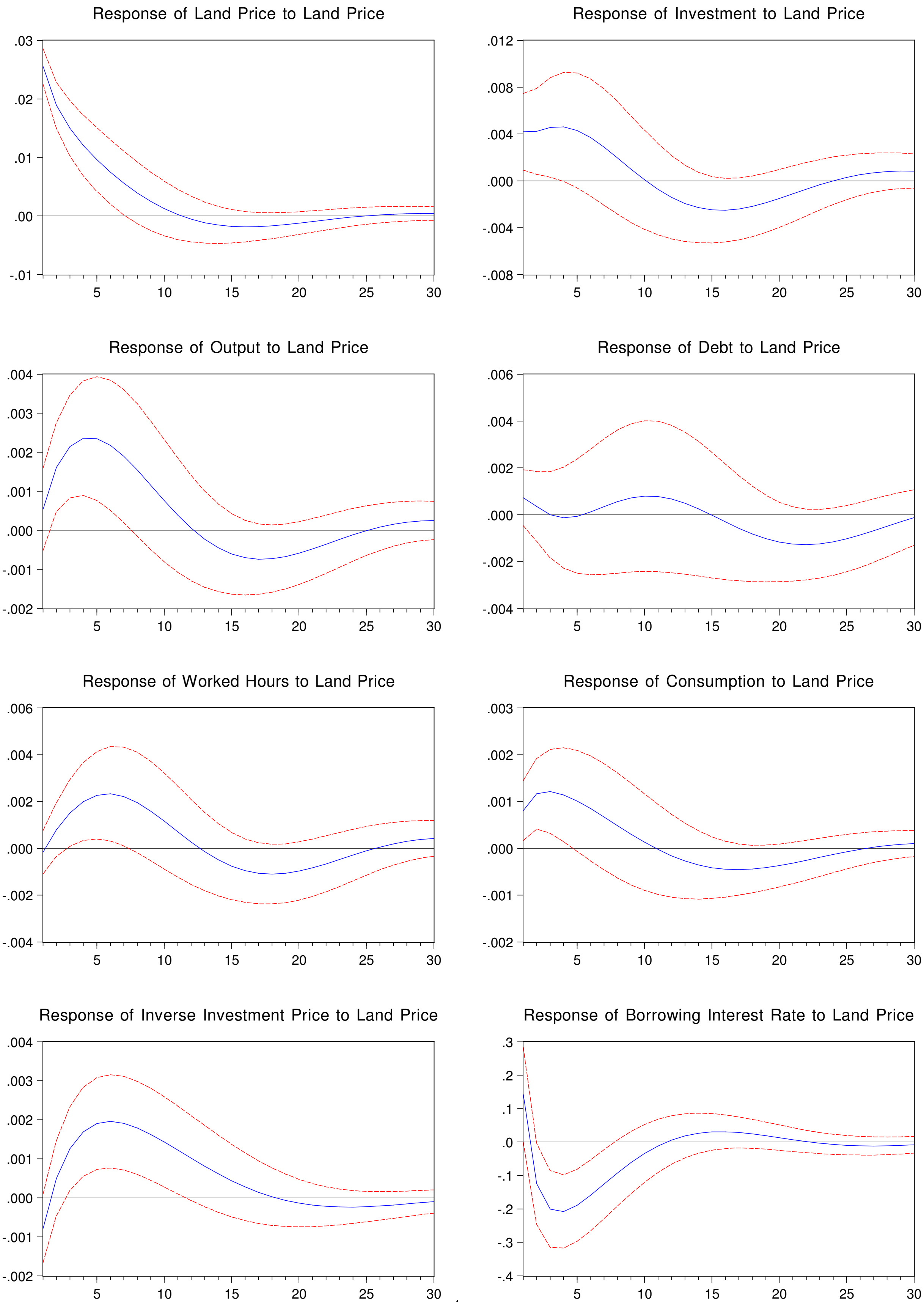
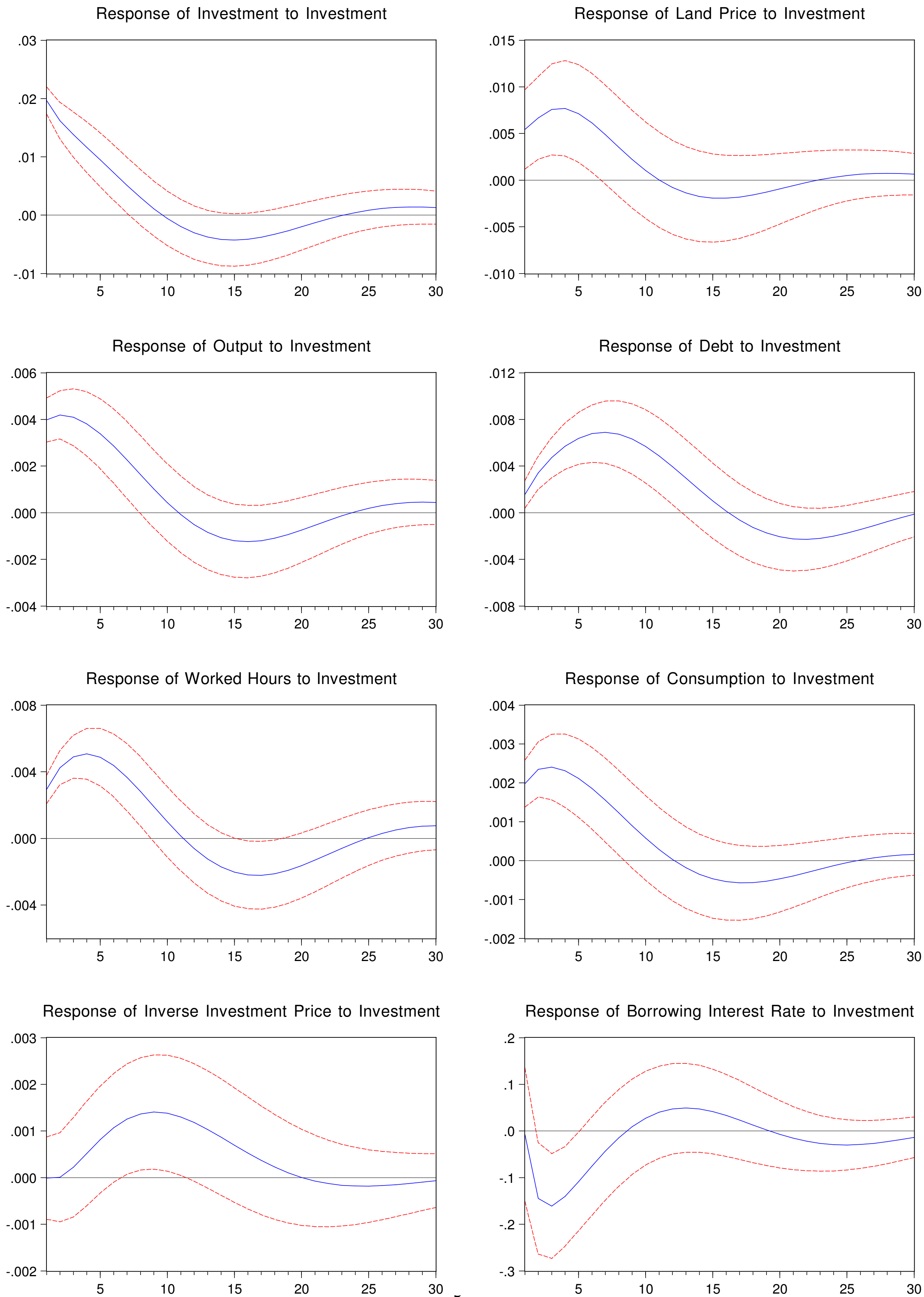


Figure 2: IRFs from VAR model with investment ordered first - one standard deviation shock



collateral generates an externality that serves not only to amplify and propagate business cycle shocks, but may also make expected changes in asset prices self-fulfilling, creating business-cycle movements even without any fundamental shocks to the economy. Of course, the amplification and propagation mechanism of collateralized borrowing through such an externality has long been noticed in the literature, and the seminal contribution by KM precisely emphasized such a mechanism. However, this literature shows that the KM constraint alone is not sufficient for generating self-fulfilling business cycles, unless additional features or frictions such as fixed cost of production or transaction are added in conjunction with collateralized borrowing to generate self-fulfilling business cycles (see e.g. Benhabib and Wang, 2013, Liu and Wang, 2014).

The contribution of this paper is twofold. On the theory side, we show that borrowing constraints of the KM type are sufficient to generate self-fulfilling business cycles in asset prices and aggregate output, even in simple versions of the original model with realistic parameter values. The intuition is straightforward: simply relaxing the borrowing constraint via a higher value of the collateral does not by itself generate a higher demand for loans if the interest repayment on loans is expected to rise. Hence, once the credit market is in an equilibrium, an expectation of a higher asset value cannot be self-fulfilling unless the loanable funds rate is countercyclical. Therefore, key to our results is the assumption that loans have a variable-rate (also labeled adjustable-rate) component, which typically offers to borrowers the possibility to benefit from market interest rates that are falling over time. Vickery (2008) documents that US firms have been relying to a large extent on variable-rate borrowing over the last four decades. Although less important since the 2007-08 financial crisis, adjustable-rate mortgages have been a major source of financing for US households over the same time period (see Moench, Vickery, and Aragon, 2010). We show in this paper that collateralized loans with adjustable interest rates produce financial instability, as they generate sunspot equilibria for virtually all parameter values.

On the empirical side, we perform a Bayesian estimation of the extended model on US data 1975-2010 and we show that sunspot shocks to investment are quantitatively important and add up to the contributions of fundamental financial shocks that have been stressed by previous quantitative studies. In addition, our estimation results establish that data overwhelmingly favor the variable-rate (indeterminate) model over the traditional predetermined-rate (determinate) model à la KM, and that the former produces the S-shaped pattern of dynamic correlations between the borrowing cost and aggregate variables found in the data while the latter does not.

Regarding our theoretical contribution, we show that while adjustable-rate loans lead to self-fulfilling, multiple equilibria near the steady state, loans with predetermined (or constant) interest rates do not. Multiplicity arises in our model because of an *aggregate credit-demand externality*: equilibria with lower interest rate implies lower debt repayment, making larger loan amounts affordable, which in turn imply larger investment demand and higher asset prices, benefit the lenders and encourage them to issue more loans to push down the interest rate. Intuitively, everything else equal, the expectation of a higher price of collateral is unable to induce a higher demand for loans unless the interest rate on loan payment is simultaneously lowered, which nonetheless cannot happen in a fixed-rate environment, thus preventing the original optimistic expectation of an asset boom to be self-fulfilling. Such multiplicity is shown to be very pervasive both in the basic model and in the extended quantitative model that we consider next, as it happens for virtually all parameter values. The technical reason why indeterminacy is so pervasive is easy to grasp, if not trivial. Essentially, moving from the fixed-rate economy to the variable-rate economy involves moving the time index of the interest one period ahead. Therefore, when the interest rate is predetermined, shocks that occur in period t do not affect the interest payment due in the same period, in contrast with what happens in the variable-rate economy. More formally, this means that both economies share identical steady states and identical eigenvalues at their linearizations, but the variable-rate economy has one more jump variable - since the interest rate is no longer predetermined - compared to the fixed-rate economy, which obviously leads to one-dimensional indeterminacy.

Not surprisingly, multiplicity generates endogenous persistence of iid shocks and it is associated with different impulse responses to fundamental shocks as well as with a new role for sunspot shocks in triggering volatility of the asset price and other aggregates.

This stark distinction between fixed-rate economies that are immune from sunspot shocks and variable-rate economies that are highly prone to sunspot disturbances has eluded the literature, largely because most contributions assume that the interest rate is either exogenous (as in KM and more recently Mendoza, 2010, among others) or predetermined (as in Iacoviello, 2005, Iacoviello and Neri, 2010, Liu, Wang and Zha, 2013, Guerrieri and Iacoviello, 2013, Justiniano, Primiceri and Tambalotti, 2015a,b, among others). We argue that our results point at expectation-driven movements as a potential empirically relevant force behind credit booms and busts, since variable-rate loans are a widespread form of borrowing in the US economy. This mechanism is tightly related to the recent work by Benhabib, Wang and Wen (2015), who show that self-fulfilling equilibria arise naturally when producers make production decisions based on expected demand and consumers make consumption decision based on expected labor income, yet production takes place before goods markets clear and before real wages are realized, in otherwise standard RBC models. We add to their contribution by showing in a dynamic model that a similar insight applies to credit markets where lenders make loans based on expected collateral value of the borrowers and the borrowers make borrowing decisions based on expected interested payment, yet the volume of loans are negotiated in advance based on adjustable interest rates, that is, when the interest rate on loan payments is allowed to fluctuate according to changes in credit market conditions. In such a natural environment with rational expectations, credit-led boom-bust cycles can become self-fulfilling: suppose the lender anticipates an investment boom with higher collateral value and thus unleashes more loans into the credit market, then a lowered interest rate would induce more demand for loans, which enables the borrowers to finance more investment and, consequently, increases their collateral value, thus fulfilling the lender's original optimistic expectations.

As a first step towards addressing the question of whether or not indeterminacy matters in quantitative terms, we use as a benchmark setup the more elaborated model of Liu, Wang and Zha (2013) in which there is a unique steady state that is determinate. We show that, just as in our basic model, determinacy is due to the assumption that the interest rate is predetermined in the bond market formulation used by those authors. When the interest rate is assumed to be fixed or predetermined, a pecuniary externality (of the sort analyzed in Bianchi, 2011, and the references therein) is not sufficient for generating self-fulfilling asset price and investment fluctuations because the demand for credit depends not only on borrower's collateral value but also on the anticipated interest rate because of debt repayments. However, allowing variable-rate loans leads to indeterminacy for virtually all parameter values also in Liu, Wang and Zha (2013) since the borrowing cost falls in booms, which enables borrowers to borrow and invest more even though the collateralizable asset may be fixed. We perform a Bayesian estimation of the extended quantitative model, using the same US data 1975-2010 as in Liu, Wang and Zha (2013). We estimate both the determinate model that obtains when the fraction of fixed-rate loans in the economy is large enough, and the indeterminate model (using the technique proposed in Farmer, Khramov, and Nicoló, 2015) when the fraction of variable-rate loans in the economy is not too small. Our main findings are as follows. First, we show that sunspot shocks to investment are quantitatively important to explain output, investment, credit, and labor hours along US business and credit cycles. Second, we show that the sunspot/indeterminate model has a much better fit than the determinate model: the latter is overwhelmingly rejected against the former. This is, to our knowledge, the first set of evidence showing why sunspot shocks matter quantitatively in a DSGE model with financial frictions.

In policy terms, the main implication of our results is that asset-backed credit markets are likely to experience boom-bust patterns driven by expectations when loans have a large variable-rate component, as in the US or the UK. Conversely, fixed-interest loans that are common practice in many continental Europe countries are an efficient tool to rule out self-fulfilling equilibria. Therefore, how the fraction of variable-rate loans evolves over time should be a key indicator for monetary/prudential authorities.

Related Literature: Our results are arguably reminiscent of earlier and famous views about how capitalist economies work. In particular, the main mechanism that is formalized in this paper can be viewed as the outcome of combining Keynes' idea of "animal spirits" as important drivers of investment decisions, on the one hand, and Minsky's views on financial instability driven by debt accumulation, on the other. This paper connects, of course, to more recent strands of research. We very much follow Backus, Kehoe and Kydland (1994) (see also, more recently, Gomme, Kydland and Rupert, 2001, and Kydland, Rupert and Sustek, 2015) by considering how the model matches not only contemporaneous correlations in the data but also dynamic lead-lag relationships, in our case between the borrowing cost and aggregate variables. In so doing, we provide a theoretical interpretation of the leading indicator property of interest rates pointed out by King and Watson (1996), that we also document for US firms borrowing cost. There is by now a large literature, to which this paper also belongs, about whether credit cycles are mostly explained by fundamental shocks, expectation - sunspot - shocks or a combination of the two, which remains an unsettled issue and calls for further evidence both to understand the mechanisms at work and to guide sound policy. As part of the ongoing research agenda that tries to address this issue, a large literature has developed, building upon the seminal contributions of Bernanke and Gertler (1989) and KM.[5] On the one hand, a robust result that several attempts to fit DSGE models with fundamental disturbances to data share is that financial shocks are important (Kiyotaki, Michaelides and Nikolov, 2011, Liu, Wang and Zha, 2013, Justiniano, Primiceri and Tombalotti, 2015a, among others). More precisely, land demand shocks, and to a lesser extent leverage shocks, are key drivers that help account for business-cycle data. In line with such an approach, Pintus and Wen (2013) have provided quantitative results showing how simple variants

[5] This strand of literature has shown how endogenous borrowing constraints amplify shocks and generate excess-volatility that would not materialize absent credit markets. Early papers include Carlstrom and Fuerst (1997), Krishnamurthy (2003), Cooley, Marimon, and Quadrini (2004), Iacoviello (2005), Campbell and Hercowitz (2006), Boháček and Rodríguez Mendizábal (2007), Christiano, Motto, and Rostagno (2010) among many others.

of KM's setting indeed produce significant and robust amplification of productivity and financial shocks that is line with evidence on credit booms, thus addressing early criticism about the plausibility of the collateral channel (e.g. Kocherlakota, 2000, Cordoba and Ripoll, 2004). On the other hand, in addition to amplifying fundamental shocks, endogenous borrowing constraints have been shown to originate multiple equilibria, as the early numerical examples in Cordoba and Ripoll (2004) have revealed in a simple RBC setup. In this approach, the emphasis is on sunspot shocks as a possible driver of credit cycles. Building on these early examples, Benhabib and Wang (2013) and Liu and Wang (2014) have further examined how various forms of fixed costs - and the associated increasing returns - make indeterminacy and self-fulfilling business cycles more likely than the model without fixed cost analyzed by Cordoba and Ripoll (2004).[6] In contrast with Benhabib and Wang (2013) and Liu and Wang (2014), we do not introduce fixed costs. Multiplicity is shown to be very pervasive both in our basic model and in the extended quantitative model that we consider next, as it happens for virtually all parameter values. This is in sharp contrast with Benhabib and Wang (2013) and Liu and Wang (2014), who show that the indeterminacy parameter region such is rather small. In addition, the novelty of our paper, compared to earlier studies, is to provide estimation results about the quantitative importance of sunspot shocks in US data. Last but no least, our main results also connect to the growing literature about debt deflation and redistribution (e.g. Calza, Monacelli and Stracca, 2013, Gomes, Jermann and Schmid, 2014, Auclert, 2016, Kaplan, Moll and Violante, 2016). Our results show that even if monetary policy is able to perfectly anchor inflation, redistribution between lenders and borrowers may still occur as long as credit instruments allow for floating debt repayments. Financial innovation is an obvious force behind the development of such instruments and our results show that the associated redistributive effects are quite important for the business cycle.

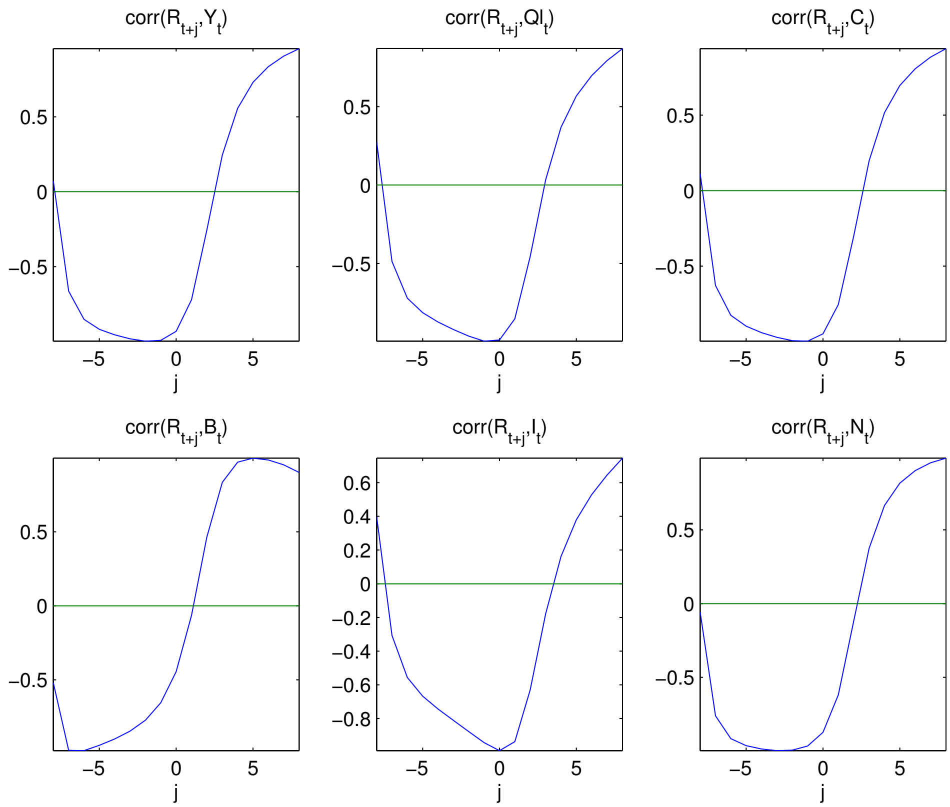
In what follows, Section 2 reports some empirical motivation of the paper. Section 3 presents a basic setup with collateralized variable-rate loans and it shows that such model generates local indeterminacy and self-fulfilling collateral cycles for virtually all parameter values. Section 4 shows that local indeterminacy is robustly pervasive by considering extensions of the basic model that we use to conduct our quantitative analysis and to show that sunspot shocks matter. Section 5 concludes the paper with remarks for future research, and an Appendix gathers proofs.

2 Empirical Motivation: Lead-Lag Correlations from Aggregate Data

We first present some stylized facts about the dynamic relationships between macroeconomic variables at quarterly frequency. More precisely, we report the lead-lag correlations of all variables with the interest rate, which we construct from the time series generated by the impulse responses in Figures 1 and 2. In all figures of this section, all variables are real, with R denoting the interest rate, Ql land price, C consumption, B corporate and noncorporate nonfinancial firms' debt, I capital investment, N working hours. The dynamic correlations that we obtain are therefore conditional on either a land price shock (Figure 3) or an investment shock (Figure 4). The most striking feature in both Figure 3 and Figure 4 is that the empirical dynamic correlations of the

[6] More recently, He, Wright and Zhu (2014) have shown that bubbly and cyclical patterns driven by expectations arise in search environments subject to KM constraints. In addition, labor and credit market frictions interact to create indeterminacy in the model of Kaas, Pintus, and Ray (2014).

Figure 3: Empirical dynamic correlations from VAR with land price ordered first



interest rate with all other variables have an S-shaped pattern. While King and Watson (1996) reported a similar pattern for the rate on three month Treasury bills, which is a policy instrument, our VAR results extend their findings to a measure of market borrowing cost faced by US firms. Consistent with the IRFs reported above, the contemporaneous correlations of the interest rate with virtually all variables are negative. So as to get a first sense of how empirically relevant the settings developed and estimated in the next sections are, in the next two figures we report the dynamic correlations that are predicted by our two competing models.

Figure 4: Empirical dynamic correlations from VAR with investment ordered first

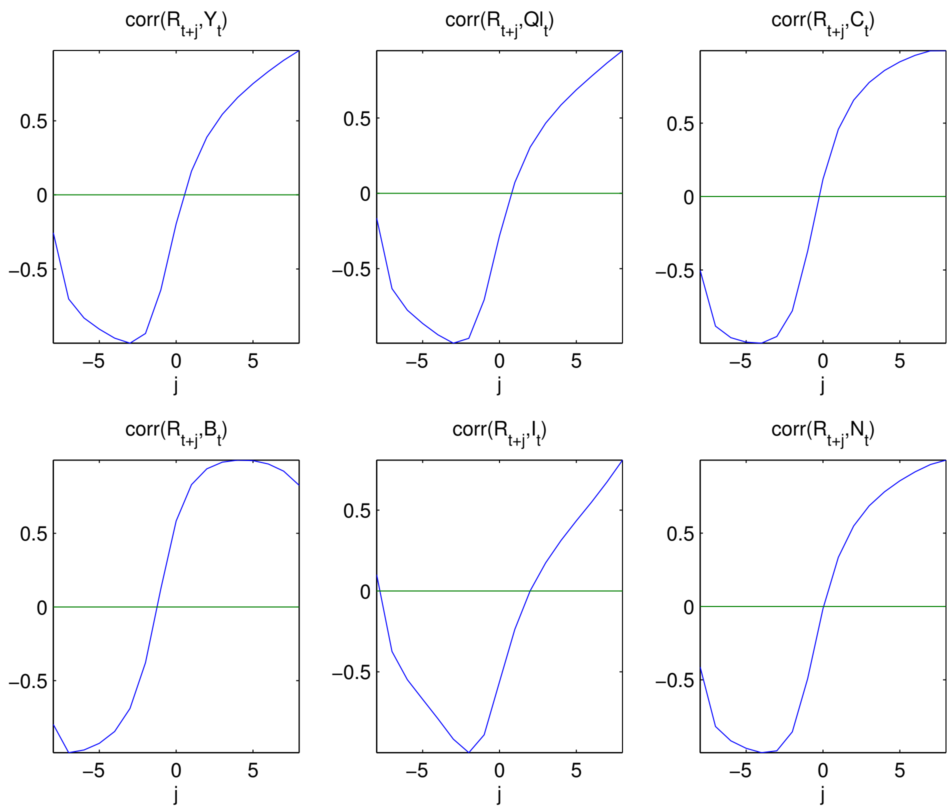


Figure 5 reports the theoretical lead-lag correlations that are produced by the determinate model with predetermined interest rate, when a positive shock to household's land demand hits and triggers a boom. Dynamic correlations in Figure 6 arise in the indeterminate model with variable-rate loans, when a positive sunspot shock increases investment.

Inspection of Figures 5 and 6 clearly shows that while the determinate model does not produce the S-shape pattern that is a feature of the data in view of Figures 3 and 4, the sunspot model is more successful in that respect.[7] This is because while both models predict that credit demand and credit supply go up in booms, they reach opposite conclusions regarding the net effect of those changes. The determinate model predicts that the interest rate is procyclical, which suggests that changes in the rate that is charged in the credit market are mainly determined by a rise of credit demand during good times. In contrast, the interest rate is countercyclical in the sunspot model, which means that supply changes dominate demand changes so that the interest rate falls during booms. The evidence from both VAR models and dynamic correlations reported in this section suggests that the sunspot model with variable-rate loans is more in line with the data than the determinate model with predetermined interest rate. In particular, the sunspot model not only correctly predicts that contemporaneous correlations between the interest and macroeconomic variables are negative but also that low levels of borrowing cost predicts future booms. We examine more formally those aspects in the following sections, which develop and estimate both models, where we show that the sunspot model does a good job along other dimensions as

[7] We have checked that similar conclusions are reached under other sources of shocks.

Figure 5: Theoretical dynamic correlations from determinate model with land price shock

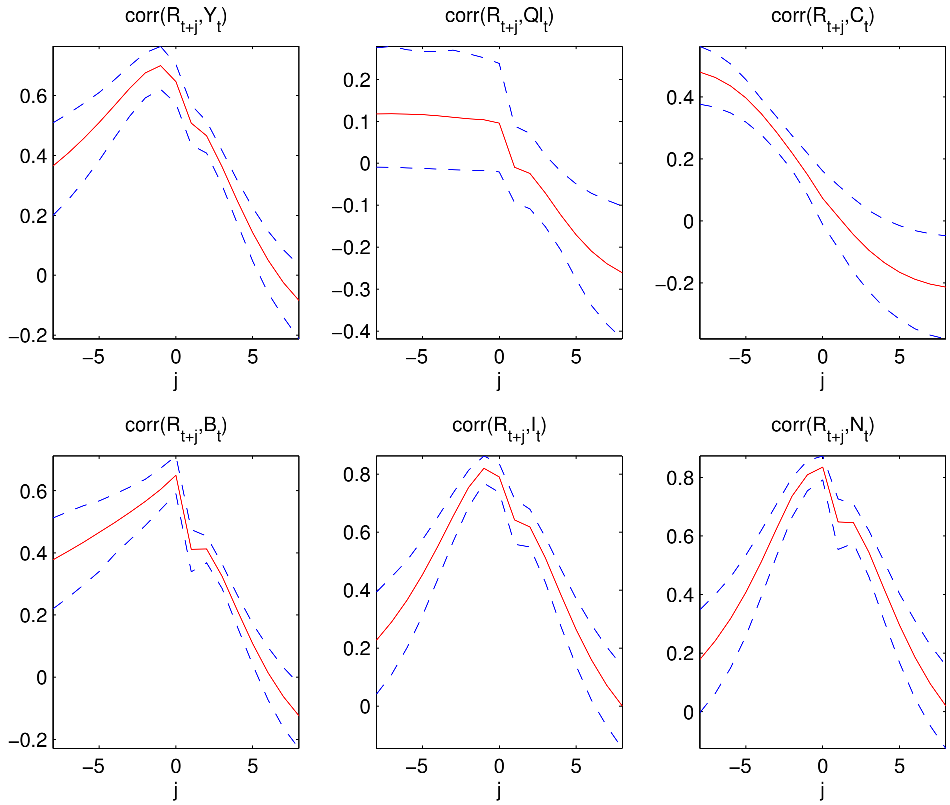
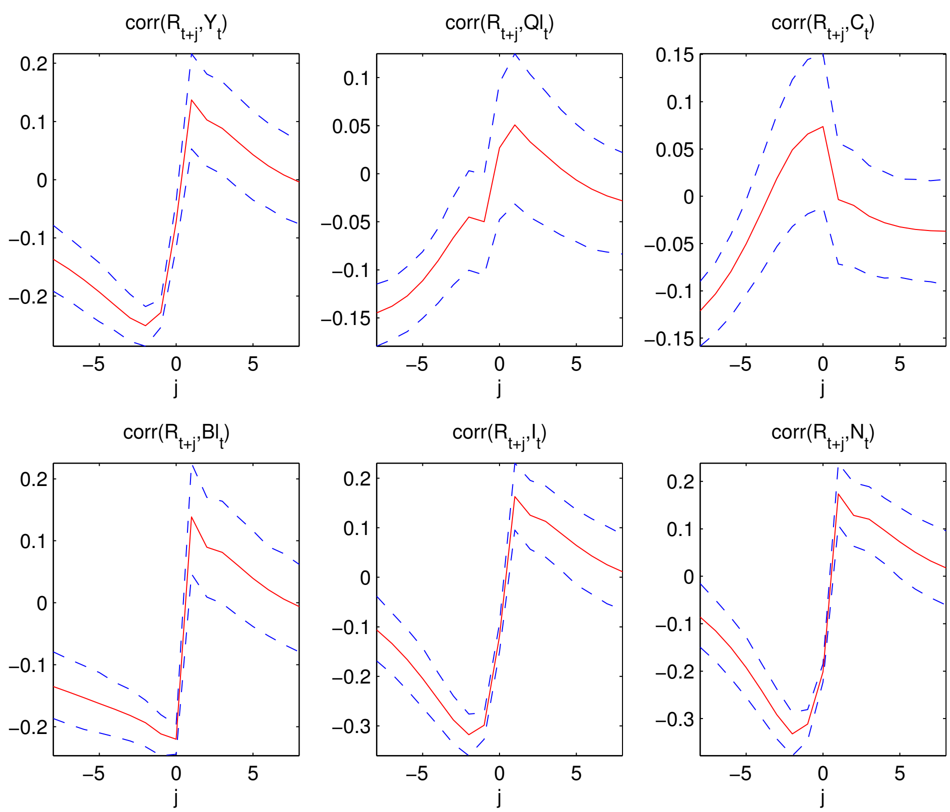


Figure 6: Theoretical dynamic correlations from indeterminate model with investment sunspot shock



well.

3 Sunspot Equilibria in a Basic Model with Variable-Rate Loans

In this section we show that incorporating variable-rate loans into the small-scale setup developed in Pintus and Wen (2013) triggers an aggregate credit-demand externality that leads to steady-state indeterminacy for virtually all parameter values. We provide a simple example in which the existence of global sunspot equilibria is derived analytically and then show how local sunspot equilibria occur in the linearized version of the more general version of the model.

3.1 Assumptions

There are two types of infinitely-long lived agents in the economy, lenders and borrowers. Lenders do not produce, but provide loans (credit) to borrowers. In this sense, lenders serve the role of banks or financial intermediaries in the economy. The type of credit provided by lenders are one-period loans that can be used to finance consumption and land investment. Lenders derive utilities from consumption and land,[8] do not accumulate fixed capital, and use interest income from payment on previous loans to finance current consumption and land investment. The budget constraint of a representative lender is given by

$$\tilde{C}_t + Q_t(\tilde{L}_{t+1} - \tilde{L}_t) + B^l_{t+1} \leq R_t B^l_t \tag{1}$$

where $\tilde{C}_t$ denotes consumption, $\tilde{L}_t$ the amount of land owned by the lender in the beginning of period t, Q_t the relative price of land, B^l_{t+1} the amount of new loans (credit lending) generated in period t, and R_t the gross real interest rate. The instantaneous utility function of the lender is given by

$$U_L = \frac{\tilde{C}_t^{1-\sigma_L}}{1-\sigma_L} + \psi_t \frac{\tilde{L}_t^{1-\sigma_W}}{1-\sigma_W}, \qquad \{\sigma_L, \sigma_W\} \geq 0 \tag{2}$$

and the time discounting factor is $\tilde{\beta} \in (0,1)$. In addition, ψ_t is a scale parameter affecting lender's preference for land, that we will use as a source of fundamental shocks to the land price.

We later use both TFP and financial shocks in our quantitative analysis and we use changes in the loan-to-value ratio and in the scale factor of land utility as proxies for financial shocks that affect the land price.

Borrowers can produce goods using land and capital.[9] The production technology is given by

$$Y_t = A_t K_t^{\alpha} L_t^{\gamma}, \quad \alpha, \gamma \in (0,1), \alpha + \gamma < 1; \tag{3}$$

where A_t is TFP, L_t denotes the amount of land owned by the borrower, and K_t denotes capital stock. Capital is reproducible but the total amount of land is in fixed supply,

$$L_t + \tilde{L}_t = \bar{L}. \tag{4}$$

[8] As in Iacoviello (2005), the lender's asset demand comes from utility attached to land.

[9] Labor supply is assumed to be fixed in the basic model and elastic labor will be introduced in Section 4.

We allow land in the model so as to study asset price and collateral movements and also to keep the model comparable to KM and the related literature.

A representative borrower in each period needs to finance consumption C_t, land investment $L_{t+1} - L_t$, capital investment $K_{t+1} - (1-\delta)K_t$, and loan repayment $R_t B_t^l$, where $\delta \in (0,1)$ is the depreciation rate of capital. The budget constraint of the borrower is given by

$$C_t + K_{t+1} - (1-\delta)K_t + Q_t(L_{t+1} - L_t) + R_t B_t^l \leq B_{t+1}^l + A_t K_t^\alpha L_t^\gamma \tag{5}$$

An important feature of the budget constraint is that the debt repayment is not predetermined in period t, as the endogenous interest rate adjusts to fundamental and possibly sunspot shocks. The per-period utility function of the representative borrower is given by

$$U_B = \frac{C_t^{1-\sigma_B}}{1-\sigma_B}, \qquad \sigma_B \geq 0 \tag{6}$$

and her discount factor is $\beta \in (0,1)$. Borrowers are assumed to be less patient than lenders, that is, their time discounting factor satisfies $\beta < \tilde{\beta}$.

The *ex-ante* borrowing constraint faced by the borrower is

$$\mathbb{E}_t R_{t+1} B_{t+1}^l \leq \theta_t \mathbb{E}_t Q_{t+1} L_{t+1} \tag{7}$$

where θ_t is the loan-to-value ratio and reflects shocks to terms of loans or current financial conditions. For example, a positive shock to θ_t implies that creditors are willing to lend more with the same collateral value of land. Following KM, reproducible capital does not have collateral value in our model but relaxing this assumption does not affect our results.[10] The borrowing constraint imposes that the amount of debt in the beginning of the next period cannot exceed a fraction θ_t (≤ 1) of the collateral value of assets owned by the borrower next period. The rationale for this constraint is that, due to lack of contractual enforceability, the lender has incentives to lend today only if the loan is secured by the value of the collateral that will be realized tomorrow. Therefore, the lender has to forecast in period t both the debt obligations that will be redeemed and the market value of collateral that will prevail in $t+1$. In contrast with KM, who assume a fixed interest rate, the fact that the interest rate is variable is a key feature for our results.

Before we analyze competitive equilibria with debt constraint, it is perhaps useful to recall that in the first-best allocation, absent such constraint, the dynamics of the model is very similar to that of a standard RBC model. Equilibrium near steady-state is unique and there is no hump-shaped cyclical propagation mechanism in such a model for realistic parameter values on the lender side (see Pintus and Wen, 2013).

3.2 Competitive Equilibrium with Borrowing Constraints

Denoting $\tilde{\Lambda}_t$ the Lagrangian multiplier of the constraint (1), the first-order conditions of the lender with respect to consumption, land investment, and lending are given, respectively, by

$$\tilde{C}_t^{-\sigma_L} = \tilde{\Lambda}_t \tag{8}$$

[10] Section 4 develops a model with collateralizable capital.

$$Q_t\tilde{\Lambda}_t = \tilde{\beta}\mathbb{E}_t Q_{t+1}\tilde{\Lambda}_{t+1} + \tilde{\beta}\psi_t \tilde{L}_{t+1}^{-\sigma_W} \tag{9}$$

$$\tilde{\Lambda}_t = \tilde{\beta}\mathbb{E}_t R_{t+1}\tilde{\Lambda}_{t+1}. \tag{10}$$

Denoting $\{\Lambda_t, \Phi_t\}$ the Lagrangian multipliers of constraints (5) and (7), respectively, the first-order conditions of the borrower with respect to consumption, land investment, capital investment, and borrowing are given, respectively, by

$$C_t^{-\sigma_B} = \Lambda_t \tag{11}$$

$$Q_t\Lambda_t = \beta\mathbb{E}_t Q_{t+1}\Lambda_{t+1} + \beta\gamma\mathbb{E}_t \frac{Y_{t+1}}{L_{t+1}}\Lambda_{t+1} + \theta_t\Phi_t\mathbb{E}_t Q_{t+1} \tag{12}$$

$$\Lambda_t = \beta\mathbb{E}_t\Lambda_{t+1}\left[\alpha\frac{Y_{t+1}}{K_{t+1}} + 1 - \delta\right] \tag{13}$$

$$\Lambda_t = \mathbb{E}_t R_{t+1}(\beta\Lambda_{t+1} + \Phi_t). \tag{14}$$

A rational expectations competitive equilibrium is a sequence of allocations $\left\{C_t, \tilde{C}_t, B_{t+1}^l, K_{t+1}, L_{t+1}, \tilde{L}_{t+1}\right\}_{t=0}^{\infty}$ and prices $\{Q_t, R_t\}_{t=0}^{\infty}$ such that, given exogenous processes for θ_t, ψ_t, A_t:

(i) $\left\{C_t, \tilde{C}_t, B_{t+1}^l, K_{t+1}, L_{t+1}, \tilde{L}_{t+1}\right\}_{t=0}^{\infty}$ satisfies the first-order conditions (8)-(14), the transversality conditions, $\lim_{t\to\infty}\beta^t\Lambda_t L_{t+1} = \lim_{t\to\infty}\beta^t\Lambda_t K_{t+1} = \lim_{t\to\infty}\tilde{\beta}^t\tilde{\Lambda}_t\tilde{L}_{t+1} = 0$, and the complementarity condition, $\Phi_t\left[\theta_t\mathbb{E}_t Q_{t+1}L_{t+1} - \mathbb{E}_t(1+R_{t+1})B_{t+1}^l\right] = 0$ for all $t \geq 0$, given $\{Q_t, R_t\}_{t=0}^{\infty}$ and the initial endowments $L_0 \geq 0, \tilde{L}_0 \geq 0, B_0^l \geq 0, K_0 \geq 0$

(ii) The good and asset markets clear for all t, $C_t + \tilde{C}_t + K_{t+1} - (1-\delta)K_t = A_t K_t^{\alpha} L_t^{\gamma}$ and $L_t + \tilde{L}_t = \bar{L}$, respectively.

The model has a determinate steady-state equilibrium and a unique saddle-path that converges to it, for which the borrower is credit-constrained, i.e., equation (7) binds for all t, provided that the number of stable eigenvalues is equal to the number of predetermined variables. In contrast, the steady-state is an indeterminate saddle with a binding credit-constraint if there are more stable eigenvalues than predetermined variables. It turns out that, as shown in Section 3.4, the indeterminacy is one-dimensional, which means that there is only one more stable eigenvalue.

We abstract from any corner solutions with zero credit and, for simplicity, we assume that the steady state value of $\theta_t = 1$ in the benchmark model.[11] In the steady state, equation (10) indicates that the interest rate is determined by the lender's time discounting factor, $R = \tilde{\beta}^{-1}$. This interest rate of loanable funds is lower than the return determined by the firm's marginal product of capital. Equation (14) then implies $\Phi = (\tilde{\beta} - \beta)\Lambda > 0$, suggesting that the borrowing constraint binds around the steady state. Equation (13) implies that the capital-to-output ratio is given by $\frac{K}{Y} = \frac{\beta\alpha}{1-\beta(1-\delta)}$. The capital-to-output ratio determines the return from capital, which is equal to the loanable funds rate if $\beta = \tilde{\beta}$; or, as in the first-best economy, if there exists perfect risk sharing without borrowing constraints. Since $\theta = 1$, equation (12) implies $Q = (1-\tilde{\beta})^{-1}\beta\gamma\frac{Y}{L} = \sum_{j=0}^{\infty}\tilde{\beta}^j\beta\gamma\frac{Y}{L}$,

[11] Our results remain qualitatively the same if $\theta < 1$ as shown below.

suggesting that the price of land is determined by the present value of its marginal products. If $\theta < 1$, the price of land, $Q = (1 - \beta - \theta(\tilde{\beta} - \beta))^{-1}\beta\gamma\frac{Y}{L} < \left(1 - \tilde{\beta}\right)^{-1}\beta\gamma\frac{Y}{L}$, is adjusted downward by the loan-to-collateral ratio because, other things equal, a tighter credit constraint decreases incentives for accumulating land. If $\theta = 1$, the lender's budget constraint implies $\tilde{C} = \left(1 - \tilde{\beta}\right)QL = \beta\gamma Y$, suggesting that the lender's consumption level is just the interest income, which is proportional to aggregate output. The borrower's budget constraint implies $C + [\delta K + \beta\gamma Y] = Y$, where the bracketed term denotes savings, and part of the savings, $\beta\gamma Y = \left(1 - \tilde{\beta}\right)QL$, is used to finance the loan and equals the lender's interest income. This indicates that the lender serves essentially as a bank and the borrower's total business investment can deviate from own savings because of bank's credit lending. In addition, since the value of γ is small, lender's consumption share ($\beta\gamma$) will be small, so lender does not play a direct role in aggregate consumption and this is what we have in mind for the financial sector. All of the great ratios (e.g., capital-to-output ratio, land-to-output ratio, consumption-to-output ratio) are determined as functions of the model's structural parameters only. Once the steady-state distribution of land is determined, the steady-state values of all other variables are determined through the great ratios. Because equation (12) is the demand curve of land and equation (9) gives the supply curve of land, the steady-state distribution of land across agents is determined uniquely by the implicit equation,

$$\beta\gamma\frac{Y(L)}{L} = \tilde{\beta}\psi\left(\bar{L} - L\right)^{-\sigma_W}\tilde{C}(L)^{\sigma_L}, \tag{15}$$

where the left-hand side decreases in L and the right-hand side increases in L, resulting in a unique solution for land allocation in steady state.

3.3 Global Sunspot Equilibria: an Analytical Example

The purpose of this section is to provide a constructive proof that global sunspot equilibria exist in the variable-rate economy and that they do not in the corresponding fixed-rate economy. All fundamental shocks are shut down in this section. So as to get closed-form decision rules and analytical results, we assume away capital and we simplify preferences and technology. More precisely, technology is linear (that is, $\gamma = 1$), the lender has linear utility (that is, $\sigma_L = \sigma_W = 0$) while the borrower has log utility (that is, $\sigma_B = 1$). In addition, the loan-to-collateral ratio, TFP and the scale parameter for land utility are constant over time, with $A = 1$ so that $Y_t = L_t$, $\theta = 1$, whereas ψ is fixed so as to ensure that a steady-state exists.

Under $\sigma_L = \sigma_W = 0$, the first-order conditions (8)-(10) imply that both the expected interest rate and the land price are constant over time, that is $\mathbb{E}_t R_{t+1} = \tilde{\beta}^{-1}$ and, using the steady-state expressions for (12) and (14), $Q = \beta/(1-\tilde{\beta})$ is the constant land price.[12] In addition, (14) can be solved for Φ_t, the expression of which can then be plugged into (12) to get:

$$\Lambda_t = \mathbb{E}_t X_{t+1}\Lambda_{t+1} \tag{16}$$

where $X_t \equiv 1 + Q(1 - \tilde{\beta}R_t)$ represents borrower's income net of interest payment, that is therefore available for consumption and down-payment for land investment. In addition, the binding credit constraint gives $B^l_{t+1} =$

[12]In addition, (9) imposes that $\psi = \beta/\tilde{\beta} < 1$.

$\tilde{\beta}QL_{t+1}$, which we plug into the borrower's budget constraint to get:

$$C_t + Q(1-\tilde{\beta})L_{t+1} = X_t L_t \tag{17}$$

It is then easy to show that the lender's consumption and land demand have closed-form solutions that are given by $C_t = (1-\beta)X_t L_t$ and $Q(1-\tilde{\beta})L_{t+1} = \beta X_t L_t$. It follows that equilibria are in this simplified economy given by $L_{t+1} = X_t L_t$ with $X_t \equiv 1 + Q(1-\tilde{\beta}R_t)$ and $\mathbb{E}_t R_{t+1} = \tilde{\beta}^{-1}$. Therefore, sunspot equilibria are simply constructed as solutions to $L_{t+1} = [1 + Q(1-\tilde{\beta}R_t)]L_t$ and $\tilde{\beta}R_t = 1 + \varepsilon_t$, where the sunspot innovation ε_t is any i.i.d. random variable with zero mean, given initial value $L_0 > 0$. The dynamics of consumption, output and borrowing all follow from that of the land stock allocated to the borrower.[13] Therefore, we can state the following result.

Proposition 1 (An Analytical Example of Global Sunspot Equilibria)

Suppose there is no capital, technology is linear ($\gamma = 1$), lender has linear utility ($\sigma_L = \sigma_W = 0$), borrower has log utility ($\sigma_B = 1$) and $A = 1$, $\theta = 1$, $\psi = \beta/\tilde{\beta}$. In addition, all fundamental shocks are shut down.
Then there exist global sunspot equilibria such that the dynamics of the land stock allocated to the borrower follows the stochastic difference equation $L_{t+1} = [1 + Q(1-\tilde{\beta}R_t)]L_t$ for all $t \geq 0$, given initial value $L_0 > 0$, where the gross interest rate is given by $R_t = \tilde{\beta}^{-1}(1+\varepsilon_t)$ and the sunspot innovation ε_t is an i.i.d. random variable with zero mean.

Before moving on to the analysis of the basic model in its general form that is developed in the next section, it is interesting to contrast the above results with what happens in the economy with fixed-rate loans. By this we mean that the borrower's budget and credit constraints are now:

$$C_t + Q(L_{t+1} - L_t) + R_{t-1}B_t^l \leq B_{t+1}^l + L_t \tag{18}$$

$$R_t B_{t+1}^l \leq QL_{t+1} \tag{19}$$

while the lender's budget constraint is:

$$\tilde{C}_t + Q(\tilde{L}_{t+1} - \tilde{L}_t) + B_{t+1}^l \leq R_{t-1}B_t^l \tag{20}$$

so that the interest payment due in period t is now predetermined while the interest rate that enters the credit constraint is variable but now known in period t. It is then easy, using again the lender's first-order condition (10), to show that the interest rate is constant over time, that is, $R_t = \tilde{\beta}^{-1}$, so that $X_t = 1$ at all dates and the economy is forever in steady state absent fundamental shocks.
In contrast Proposition 1 shows that the variable-rate economy is subject to global sunspot equilibria, with both the growth rate X and output $Y = L$ driven by sunspot innovations. As an illustrative example, Figure 7

[13] Obviously, the logarithm of land stock follows a random walk in the analytical example of this section, with absorbing barriers at the values 0 and $\bar{L}$ for L. Standard theory of random processes shows that the probability of non-absorbing states tends to zero geometrically fast as time goes to infinity. Alternatively, reflecting (or more generally elastic) barriers at the end points of the feasible interval can be introduced so as to ensure permanent fluctuations along a sunspot equilibrium.

reports a sample path of such an economy when $\beta = 0.95$ and $\tilde{\beta} = 0.99$, while the sunspot innovations has zero mean and standard deviation close to zero. Figure 7 makes clear that the variable-rate economy is subject to sunspot innovations, whereas the fixed-rate economy is not and stays at steady state. The main mechanism is that a falling interest rate triggers a boom, as we now explain in a more detailed way.

Dissecting the Mechanism: Self-fulfilling Countercyclical Changes in the Interest Rate The small-scale model of this section is useful to develop some intuition about the mechanisms at work and why those differ in the fixed-rate and variable-rate economies. It is perhaps enlightening to proceed in two steps, by considering first the simplified version of this section, so as to extract the essence of the main channels, and to turn next to the general version discussed in Section 3.4.

In the model without capital and risk-neutral lender of this section, let us notice first that land price is fixed, as it equals $Q = \beta/(1-\tilde{\beta})$, which means that the existence of sunspot equilibria is not due to the pecuniary externality (through asset price) that has been stressed by the existing literature. In addition, in this simplified model output is split between borrower and lender since there is no aggregate investment in capital, so that any change in borrower's consumption crowds out lender's, that is, $\tilde{C}_t = L_t - C_t$.

Now suppose that the borrowing cost that governs interest payments in period t, that is, R_t, is expected to go down because of a sunspot shock. Because the stock of outstanding debt B_t^l is predetermined, the borrower benefits from some additional income, hence she raises her consumption C_t and land holdings L_{t+1} so that output goes up. Larger land investment is partly financed by borrowing. This can be seen by defining credit demand as:

$$B_{t+1}^d = \tilde{\beta} Q L_{t+1} \tag{21}$$

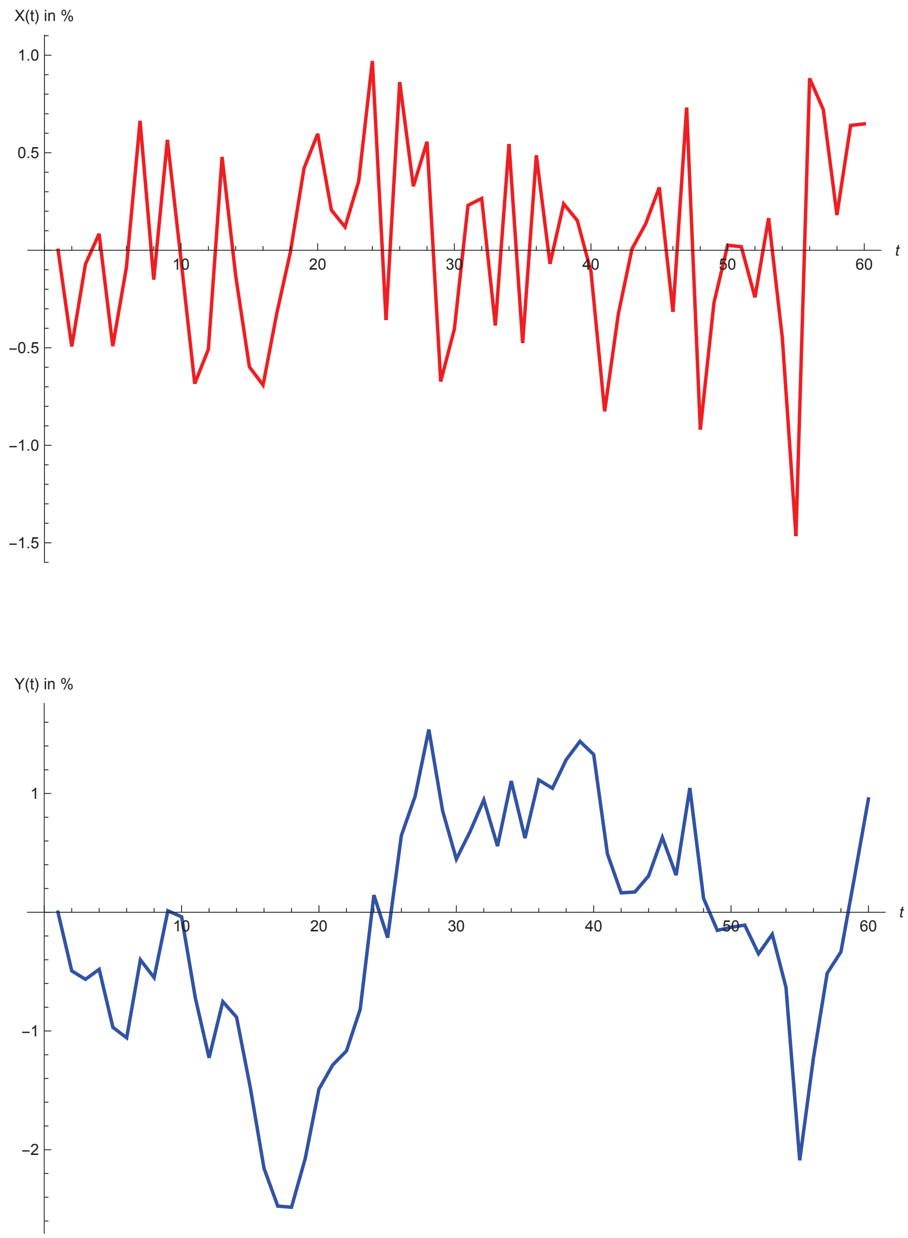
which is simply (7) rewritten as an equality using that $\mathbb{E}_t R_{t+1} = \tilde{\beta}^{-1}$ when the lender is risk neutral, in view of (10), and $\theta = 1$. Of course, that the borrower is able to get more land is consistent with equilibrium reallocation from lender to borrower in the land market and the associated rise in borrower's collateral. However, if the rise in credit demand would stand alone, the market interest rate that clears the market would go up, thus invalidating the initial expectation of a falling borrowing cost. We now show that a self-fulfilling, countercyclical interest rate arises in a boom only if the supply of credit goes up as well. To see this, define credit supply from (1), using that $\tilde{C}_t = L_t - C_t$ and replacing borrower's consumption $C_t = (1-\beta)X_t L_t$, as:

$$B_{t+1}^s = Q L_{t+1} - \beta X_t L_t \tag{22}$$

where $X_t \equiv 1 + Q(1 - \tilde{\beta} R_t)$.

Now suppose again that the borrower expects the interest rate to go down. Then the borrower increases consumption and land investment L_{t+1}. In addition to being a shifter of credit demand through the collateral channel - see (21) - L_{t+1} is also a shifter of credit supply through land reallocation to the borrower - see (22). As can be seen from Figure 8, the net effect is a fall of the interest rate. This is because in view of equations (21) and (22), the credit supply curve shifts to the right by more than the credit demand curve when L_{t+1} goes up: when the borrower's land demand goes up by ΔL_{t+1}, the lender's land holdings go down by the

Figure 7: Global sunspot equilibria in the simplified model with risk-neutral lender and linear technology. The top panel reports the growth rate of output while the bottom panel reports the level of output (both in percentage deviations from their steady-state values).

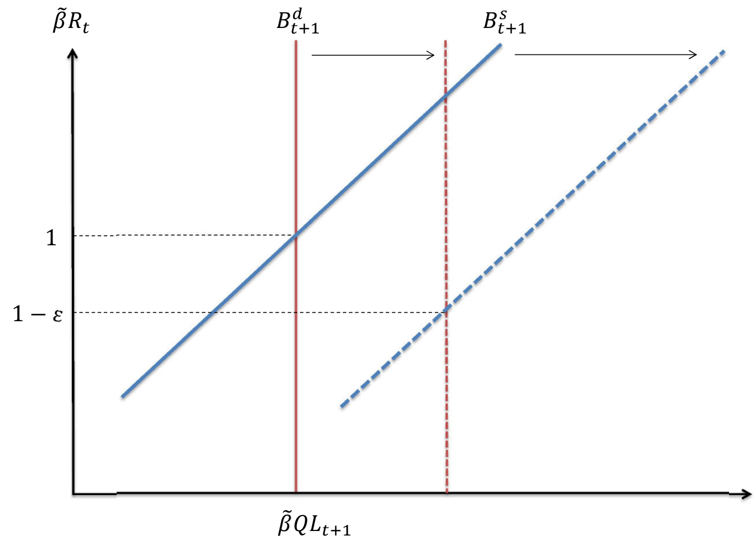


same amount since land is in fixed supply, which means that the lender's savings in the form of lending goes up by $Q\Delta L_{t+1}$. On the other hand, borrower's credit demand goes up by $\tilde{\beta}Q\Delta L_{t+1}$, that is, by a little less since the loan-to-value ratio is smaller than one. The bottom line is that the interest rate goes down and the initial expectation is fulfilled. In other words, the interest rate is countercyclical in the indeterminate model.[14] Appendix 6.1 shows that global sunspot equilibria survive when, more realistically, both fixed and variable-rate loans are used, provided that the constant share of variable-rate loans is larger than 0.5. In contrast, the economy with predetermined interest rate stays in steady state forever, absent fundamental shocks, because the interest rate is constant through time and there is no reallocation of land that can trigger shifts in credit supply or demand. It turns out that sunspot equilibria are also ruled out in the simple economy with predetermined interest rate even if we allow the land price to move over time, typically in a procyclical fashion, and despite the associated pecuniary externality. An easy way to see this analytically is to suppose that while the lender has linear utility in consumption ($\sigma_L = 0$) so that the interest rate is constant, she now has logarithmic utility for land ($\sigma_L = 1$). Then the land price is no longer constant but moves over time because land reallocation between lender and borrower changes lender's marginal utility from land and this reflects into asset price. It is then not difficult to show that in the deterministic version of such an economy, the land price dynamic equation can be solved forward and delivers a unique Q_t so that sunspot equilibria are ruled out. This is in line with the intuition stated earlier. With a constant interest rate, the pecuniary externality is not strong enough to generate self-fulfilling equilibria because a high land price in a boom relaxes the borrowing constraint but at the same time makes collateral more expensive. In contrast, with a constant land price but variable interest rate, self-fulfilling equilibria arise because the borrowing cost is countercyclical so that the borrower can afford more credit and invest more in a boom. Of course, the simplified model of this section obtains under extreme assumptions but it turns out that the intuition developed in this context carries through in the more general model with capital and risk-averse lender to which we come back next, in Section 3.4. Before doing so, a few remarks about the welfare consequences of sunspot equilibria are in order. In the context of Proposition 1, it may seem that variable-rate contracts should not be proposed by competitive lenders as long as they make borrowers worse-off compared to fixed-rate contracts. And indeed in this simple example, it is possible that the risk-averse borrower prefers the fixed-rate loan, that is immune from sunspot innovations, to the variable-rate loan that leads to volatile outcomes, while the risk-neutral lender is indifferent.[15] Deriving the optimal contract is beyond the scope of this paper so we think of the credit market as a centralized market allocating loans among atomistic participants. However, even if one think in terms of loan contract, it is not necessarily the case in our simple model that variable-rate contracts make agents worse-off. For instance, results not reported here show that occasionally binding credit constraints generate asymmetric sunspot-driven business cycles, as in Guerrieri and Iacoviello (2013), that make the lender better-off because booms are less pronounced than busts so that on average lenders enjoy higher consumption and more land compared to the steady state. In that case, the lender would have incentives to propose variable-rate contracts as a take-or-leave-it offer. In addition, it is also easy to show that in an enriched version of the example in which there are fundamental

[14] Note that if one plugs the expression of optimal land holdings by the borrower, that is, $L_{t+1} = X_t L_t$, into equation (22) and equates credit demand to credit supply, then the current interest rate R_t disappears. This is why one needs the additional equation $\mathbb{E}_t R_{t+1} = \tilde{\beta}^{-1}$ to pin down the interest rate.

[15] We thank Fernando Broner for drawing our attention to this point.

Figure 8: Both credit demand B^d and credit supply B^s shift rightward when the borrower expects a fall in interest rate and invests more in land so that L_{t+1} goes up, resulting in a self-fulfilling fall in R_t.



shocks that correlate with sunspot innovations, the borrower may be better-off under variable-rate loans even in the linearized model. In general, therefore, there is no reason why variable-rate contracts would not materialize in equilibrium. Elaborating further on this issue seems promising but beyond the scope of our analysis, which focuses on the macroeconomic consequences of variable-rate collateralized loans, especially in view of Carlstrom, Fuerst and Paustian (2016) who show in an different environment under which conditions the optimal lending contract has indeed variable loan repayment.

3.4 Steady-State Indeterminacy and Local Sunspot Equilibria

Going back to the full model described in Section 3.1, the model's stationary equilibrium path is solved by log-linearizing the model around the interior steady state (see the equations in Appendix 6.2). Under the assumption of rational expectations, we check the uniqueness of a determinate equilibrium near-steady state or the multiplicity of a continuum of indeterminate equilibria by the eigenvalue method. Because the dynamical system is high-dimensional, we resort to numerical methods to compute eigenvalues for the determinate model.

For all parameter values, the unique steady state turns out to be locally indeterminate. More precisely, the difference between the number of stable eigenvalues and the number of predetermined variables equals one so that local indeterminacy is one-dimensional. This leaves room for one jump variable to be affected by non-fundamental or sunspot shocks and in the remainder of the paper we will assume that this variable is investment.[16] To show that indeterminacy arises, it is useful to go back to Pintus and Wen (2013), who develop

[16]Which variable is subject to sunspot shocks is irrelevant for the model impulse responses, though not for the estimation results

a version of the very same model in which the interest rate is predetermined and the steady state is determinate. More precisely, in Pintus and Wen (2013) the lender's and borrower's budget constraints and the borrowing constraint are respectively:

$$\tilde{C}_t + Q_t(\tilde{L}_{t+1} - \tilde{L}_t) + B^l_{t+1} \leq R_{t-1}B^l_t \tag{23}$$

$$C_t + K_{t+1} - (1-\delta)K_t + Q_t(L_{t+1} - L_t) + R_{t-1}B^l_t \leq B^l_{t+1} + A_t K^\alpha_t L^\gamma_t \tag{24}$$

$$R_t B^l_{t+1} \leq \theta_t \mathbb{E}_t Q_{t+1} L_{t+1} \tag{25}$$

Given the above budget and borrowing constraints, equations (10) and (14) are replaced by, respectively:

$$\tilde{\Lambda}_t = \tilde{\beta} R_t \mathbb{E}_t \tilde{\Lambda}_{t+1}. \tag{26}$$

$$\Lambda_t = R_t \mathbb{E}_t(\beta \Lambda_{t+1} + \Phi_t). \tag{27}$$

It is straightforward to show that this formulation is in fact equivalent to assuming that, in period t, the borrower issues one-period bonds in quantity $B_{t+1} \equiv R_t B^l_{t+1}$, at given unit price $1/R_t$, and has to repay $B_t \equiv R_{t-1}B^l_t$. In both interpretations, the key feature is that what the borrower has to repay in period t is predetermined, hence does not adjust to shocks that occur at t. Therefore, the market interest rate is fixed when the lender and borrower agree on the level of credit and one can think of such an arrangement as a *fixed-rate* loan.

In contrast, the agreement that is described in Sections 3.1-3.3 is a *variable-rate* loan, whereby the borrower's repayment in t adjusts to any shock, fundamental or sunspot, that occurs within the period and the lender has to forecast *both* the market value of collateral and the debt obligations that will prevail in $t+1$. In other words, one goes from the fixed-rate economy, described in the last paragraph, to the variable-rate economy simply by moving the time index of the interest rate one period ahead. More precisely, on the one hand the fixed interest rate R_{t-1} that has to be paid in period t becomes variable R_t, while on the other hand what is R_t in the fixed-rate economy becomes $\mathbb{E}_t R_{t+1}$, which is now the relevant expression to determine how much to lend today in the variable-rate economy.

Of course, the fixed-rate economy and the variable-rate economy have in common equations (8), (9), (11), (12) and (13), since these do not have the interest rate in them. Not surprisingly, therefore, comparing the constraints and first-order conditions (23)-(27) and their analogs in Sections 3.1 and 3.2 reveals that the variable-rate economy has an additional *jump* variable, compared to he fixed-rate economy, because the interest rate is now nonpredetermined. Since it is easily established that both economies have identical steady-states and identical eigenvalues at their linearizations, the following proposition follows.

Proposition 2 (Local Indeterminacy in the Variable-Rate Economy)

Suppose that first-order conditions (8), (9), (11), (12) and (13) hold. Moreover, define the fixed-rate *economy by the additional conditions (23) to (27) while the* variable-rate *economy is defined by the alternative set of additional conditions (1), (5) (7), (10) and (14).*

Then both economies have identical steady states and their linearizations at that unique steady state have identical

that we report in the next section.

eigenvalues. It follows that whenever the unique steady state is locally determinate in the fixed-rate economy, it is locally indeterminate in the variable-rate economy and local indeterminacy is one-dimensional. Therefore, local sunspot equilibria can be constructed near the steady state in the latter case.

Pintus and Wen (2013) have shown that the unique steady state of the fixed-rate economy is locally determinate so it follows that the steady state is locally indeterminate in the variable-rate economy.[17] In Section 4 we show that Proposition 2 can be adapted to a larger model with elastic labor and many additional features that are required to take the model to the data, with identical conclusions. Before doing so, we now come back to the intuition developed in Section 3.3 and show that it carries through in the linearized model of this section.

Although relaxing the assumptions of Proposition 1 rules out closed-from solutions, it is still possible to derive analytical expressions for credit demand and credit supply arising in the linearized version of this section's model. In particular, suppose we slightly modify the model without capital of Section 3.3 by allowing now the lender's utility for consumption to be logarithmic (that is, $\sigma_L = 1$), so that the endogenous land price is no longer constant. Manipulating the linearized equations in Appendix 6.2, we get among other conditions that $(1-\beta)c_t + \beta\tilde{c}_t = l_t$, $q_t = \tilde{c}_t$, $\mathbb{E}_t r_{t+1} = \mathbb{E}_t[q_{t+1} - q_t]$ and $b_{t+1} = l_{t+1} + q_t$ (where lowercase letters denote deviations from steady state), which together with the lender's budget constraint, allow to derive credit demand and supply as respectively:

$$b^d_{t+1} = l_{t+1} + q_t \tag{28}$$

$$b^s_{t+1} = l_{t+1}[Q - \beta(1-\tilde{\beta})]/(\tilde{\beta}Q) + (r_t + q_{t-1})(Q-\beta)/(\tilde{\beta}Q) \tag{29}$$

where steady-state land price is $Q = \beta/(1-\tilde{\beta})$. The borrower's budget constraint simplifies to:

$$(1-\beta)c_t + \beta(1-\tilde{\beta})l_{t+1} = -\beta r_t \tag{30}$$

if the economy was in steady state prior to period t, that is, $l_t = q_{t-1} = 0$. Suppose that the borrower expects at date t that the interest rate goes below its steady-state value, that is, she expects $r_t < 0$. Then she benefits here again from a positive income shock and it is straightforward to show that she decides to consume more and invest more in land, in view of equation (30). But then from equations (28) and (29), we see that expectations of lower interest rate again shift rightward both credit demand and credit supply, and it is not difficult to show that the latter shifts by more than the former. Therefore, in the linearized version with risk-averse lender, the intuition for self-fulfilling interest rate expectation that is pictured in Figure 8 still holds. Of course, a major difference is that land price now reacts to sunspot shocks to the interest rate. In view of the linearized equations reported above, one sees that $q_t = \tilde{c}_t$ so that a falling interest rate leads to declines of both land price and lender's consumption at impact of equal magnitudes (although both go above steady state in the following periods so that land price is overall procyclical). This means that credit demand shifts rightward but by less than the full amount of land reallocation.[18] This accords with the above intuition that credit demand shifts rightward by

[17] To be more precise, there is a typo in the published paper that sets the time index for the interest rate incorrectly. All figures reported in Pintus and Wen (2013) are produced by a code that assumes a predetermined interest rate. The correction appears as an online appendix available at the Review of Economic Dynamics.

[18] It is not difficult to show, by manipulating the linearized equations, that the fall in land price is lower than the lender's land holdings increase so that debt goes up.

less than credit supply, which then extends to the case with risk-averse lender. In addition, a similar intuition holds if capital is added. In summary, while countercyclical interest rate changes can be self-fulfilling in the indeterminate model with variable-rate loans, interest rate changes are procyclical in the determinate model with fixed-rate loans and fundamental shocks only. In view if the evidence reported in Section 2, the former turns out to be more in line with data than the latter. Before we move on to a quantitative assessment of such a claim, it is perhaps instructive to inquire into the robustness of, and also the very reason behind indeterminacy. This is what we do next. In particular, next section takes a broader perspective and asks why indeterminacy is so pervasive in variable-rate economies.

3.5 Aggregate Credit-Demand Externality and Local Indeterminacy

The purpose of this section is to explain why interest rate variability is a key feature that is required for indeterminacy to occur for all parameter values. It is perhaps useful to start with the early numerical results reported in Cordoba and Ripoll (2004). Their model is a simpler variant of our basic economy without capital accumulation and, most importantly, with a different formulation of the credit constraint that assumes a predetermined interest rate. That is, Cordoba and Ripoll (2004) model the credit market as a bond market and, using their notation, p_t is the unit price of a bond issued in period t and that promises to repay one unit of the consumption good in period $t+1$. In each period t borrowers have to redeem, using their notation, a_t units of bonds and issue new bonds with value $p_t a_{t+1}$ subject to the credit constraint $a_{t+1} \leq q_{t+1} k_{t+1}$, where k is land in their notation. In other words, they assume a fixed-rate economy since the bond market formulation is equivalent to a loan market with predetermined interest rate. Results reported in Cordoba and Ripoll (2004) suggest that indeterminacy occurs in their setting for a small set of parameter values that turn out to be unrealistic. For example, their figure 4 shows that local indeterminacy arises when the land share is larger than 0.8 and provided relative risk aversion is larger than 25 (or equivalently if the consumption elasticity of intertemporal substitution is close to zero). This also explains why indeterminacy does not show up under the calibration used in Pintus and Wen (2013), which excludes those extreme values.

As stated in the previous section, the result that the steady state is unlikely to be indeterminate in the settings of Cordoba and Ripoll (2004) and Pintus and Wen (2013) hinges upon the fact that the interest rate is assumed to be predetermined. This is clearly the fact in the bond formulation used by Cordoba and Ripoll (2004), in which a_t is a predetermined variable that can be thought of as debt repayments including interest. Similarly, the code used in Pintus and Wen (2013) implicitly assumes that the interest rate is predetermined. These observations, taken together, suggest that when one changes in our basic model the credit constraint (7) to:

$$R_t B^l_{t+1} \leq \theta_t \mathbb{E}_t Q_{t+1} L_{t+1} \tag{31}$$

and both budget constraints (1) and (5) accordingly as in Section 3.4 (see equations (23) and (24)), then determinacy prevails in the fixed-rate economy.[19] Comparing (7) and (31) then indicates that two types of externalities are at play, with different outcomes regarding the determinacy properties of the unique steady

[19] In other words, our basic model is under (31) an extension of that in Cordoba and Ripoll (2004) with capital accumulation, in which the price of bonds and the interest rate relate through the equality $p_t = 1/R_t$.

state. Under credit constraint (31), a pecuniary externality is at work, such that borrowers do not take into account that whenever they decide to accumulate more land today so as to borrow more tomorrow, their action triggers an upward pressure on the land price that relaxes the borrowing constraint for everybody else, including those inactive agents that chose no to invest more in the first place. When the pecuniary externality alone is at work, indeterminacy either does not occur[20] or arises for a small set of unrealistic parameter values.

To further check that the intuition reported above is correct, we have experimented with alternative formulations of the credit constraints, such as the following:

$$B^l_{t+1} \leq \theta_t \mathbb{E}_t Q_{t+1} L_{t+1} \qquad \text{or} \qquad \frac{B^l_{t+1}}{R_t} \leq \theta_t \mathbb{E}_t Q_{t+1} L_{t+1} \tag{32}$$

while the lender's and borrower's budget constraint are modified in a consistent way. In both formulations in (32) there is no role for expectations about the interest rate in the borrowing constraint but indeterminacy still arises for some parameter values. We have also checked that the intuition developed above still holds under (32) provided that $\theta < 1$. As under credit constraint (7), the credit market equilibrium is susceptible to self-fulfilling swings. Here again, if one expects the interest rate to be low, then there is ample credit available so that high levels of investment, economic activity, land price and collateral materialize, which is consistent with one's expectations. However, such an expectation-driven boom coexists with another equilibrium where the interest rate is high, which depresses investment and puts downward pressure on land price so that collateral and credit are scarce resources. In other words, an aggregate credit-demand externality comes to life. Of course, in this case i.i.d. sunspot shocks can drive land price, credit and economic activity. In addition, responses to fundamental shocks are possibly altered by the very property that the steady state is indeterminate. More precisely, local indeterminacy typically originates endogenous persistence so that fundamental shocks with zero autocorrelation can nevertheless have persistent effects. These are the issues we address in the next section within a more elaborated model.

4 A Quantitative Sunspot Model with Variable-Rate Loans

This section shows that indeterminacy is pervasive also in extensions of the basic model. To do so we introduce variable-rate loans in the medium-scale model of Liu et al. (2013), that originally deals with fixed-rate loans. Such an extended setup is useful in quantitatively assessing whether or not indeterminacy and sunspot shocks are relevant and we show that they are. More precisely, we perform a Bayesian estimation of both the determinate and the indeterminate models (following Farmer et al., 2015, for the latter). Investment sunspot shocks are shown to be quantitatively more important than land demand shocks to explain US business and credit cycles. In addition the determinate model is rejected against the sunspot model according to the Bayes factor criterion.

4.1 Determinate Economy with Fixed-Rate Loans

So as to make clear how and why variable-rate loans modify the analysis, we first expose briefly the original model of Liu et al. (2013) in which the interest rate is predetermined and the steady state is determinate, using

[20] Cordoba and Ripoll (2004) also report that indeterminacy is not robust in their setting, in the sense that it does not survive the introduction of heterogeneity in risk aversion between lender and borrower. This is not the case with our indeterminacy results.

the same notation as in their paper, including the end-of-period convention for stock variables.

Household: The infinitely-long lived representative household consume and supply both labor and credit in each period. They take decisions that maximize lifetime utility, defined as:

$$\max \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t A_t (\ln(C_{ht} - \gamma_h C_{ht-1}) + \varphi_t \ln L_{ht} - \psi_t N_{ht}) \right] \tag{33}$$

where C_{ht} is consumption, L_{ht} is the land stock, and N_{ht} represents labor hours. Parameter $\beta \in (0,1)$ denotes the discount factor and consumption habits are measured by parameter $\gamma_h \in (0,1)$. Preferences are subject to three shocks, as follows. An intertemporal preference shock is denoted by $A_t = A_{t-1}(1 + \lambda_{at})$, with $\ln \lambda_{at} = \rho_a \ln \lambda_{at-1} + (1 - \rho_a) \ln \bar{\lambda}_a + \sigma_a \varepsilon_{a,t}$, $\bar{\lambda}_a > 0$, $\rho_a \in (-1,1)$, and ε_{at} is i.i.d. and normally distributed with mean zero and unit variance so that $\sigma_a > 0$ is the standard deviation of the innovation. In addition, a shock to land utility is denoted by ϕ_t such that $\ln \varphi_t = \rho_\varphi \ln \varphi_{t-1} + (1 - \rho_\varphi) \ln \bar{\varphi} + \sigma_\varphi \varepsilon_{\varphi t}$, $\bar{\varphi} > 0$, $\rho_\varphi \in (-1,1)$, and $\varepsilon_{\varphi t}$ is i.i.d. and normally distributed with mean zero and unit variance so that $\sigma_\varphi > 0$ denotes the innovation's standard deviation. Finally, a labor supply shock is denoted by ψ_t such that $\ln \psi_t = \rho_\psi \ln \psi_{t-1} + (1 - \rho_\psi) \ln \bar{\psi} + \sigma_\psi \varepsilon_{\psi_t}$, $\bar{\psi} > 0$, $\rho_\psi \in (-1,1)$, and ε_{ψ_t} is i.i.d. and normally distributed with mean zero and unit variance while $\sigma_\psi > 0$ is the innovation's standard deviation.

Households are subject to their budget constraint:

$$C_{ht} + q_{lt}(L_{ht} - L_{ht-1}) + \frac{S_t}{R_t} \leq w_t N_{ht} + S_{t-1} \tag{34}$$

where q_{lt} is the relative land price in terms of the produced good, R_t is the debtor gross interest rate, w_t is the real wage, and S_t denotes the quantity of uncontigent bonds that each pays one consumption unit in period $t+1$.

Defining μ_{ht} as the Lagrange multiplier attached to (34), it is straightforward to derive the following first-order conditions with respect to consumption demand, labor demand, land demand and credit supply:

$$\mu_{ht} = A_t \left(\frac{1}{C_{ht} - \gamma_h C_{ht-1}} - \mathbb{E}_t \left[\frac{\beta \gamma_h}{C_{ht+1} - \gamma_h C_{ht}} (1 + \lambda_{at+1}) \right] \right) \tag{35}$$

$$w_t = \frac{A_t \psi_t}{\mu_{ht}} \tag{36}$$

$$q_{lt} = \beta \mathbb{E}_t \left[\frac{\mu_{ht+1}}{\mu_{ht}} q_{lt+1} \right] + \frac{A_t \varphi_t}{\mu_{ht} L_{ht}} \tag{37}$$

$$1 = \beta \mathbb{E}_t \left[\frac{\mu_{ht+1}}{\mu_{ht}} \right] R_t \tag{38}$$

Entrepreneur: The representative entrepreneur is also infinite-long lived and runs the productive technology that uses capital, labor and land and delivers a good that can be either consumed or used for investment. Her consumption, investment and borrowing decisions maximize lifetime utility, as defined by:

$$\max \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \ln(C_{et} - \gamma_e C_{et-1}) \right] \tag{39}$$

where C_{et} is consumption and the habit parameter $\gamma_e \in (0,1)$. Entrepreneur operate under four types of constraints.

(i) a technological constraint:

$$Y_t = Z_t(L_{et-1}^{\phi} K_{t-1}^{1-\phi})^{\alpha} N_{et}^{1-\alpha} \tag{40}$$

where Y_t is output produced out of capital K_{t-1}, labor N_{et} and land L_{et-1}, with $\alpha \in (0,1)$ and $\phi \in (0,1)$. Total factor productivity Z_t is stochastic and subject to a temporary component ν_{zt} and a permanent component Z_t^p, with $Z_t = \nu_{zt} Z_t^p$, $Z^p = Z_{t-1}^p \lambda_{zt}$, $\ln \lambda_{zt} = \rho_z \ln \lambda_{zt-1} + (1-\rho_z)\bar{\lambda}_z + \sigma_z \varepsilon_{zt}$, $\ln \nu_{zt} = \rho_{\nu_z} \ln \nu_{zt-1} + \sigma_{\nu_z} \varepsilon_{\nu_z t}$. It follows that $\bar{\lambda}_z$ denotes the growth rate of productivity, parameters ρ_z and ρ_{ν_z} belong to $(0,1)$, parameters $\sigma_z > 0$ and $\sigma_{\nu_z} > 0$ denote standard deviations, while ε_{zt} and $\varepsilon_{\nu_z t}$ are i.i.d. and normally distributed with zero mean and unit variance.

(ii) a capital accumulation constraint:

$$K_t = (1-\delta)K_{t-1} + \left(1 - \frac{\Omega}{2}\left(\frac{I_t}{I_{t-1}} - \bar{\lambda}_I\right)^2\right) I_t \tag{41}$$

where I_t denotes investment, $\bar{\lambda}_I$ is the steady-state growth rate of investment, and $\Omega > 0$ measures the cost of adjusting the investment flow.

(iii) a budget constraint:

$$C_{et} + q_{lt}(L_{et} - L_{et-1}) + B_{t-1} = Y_t - \frac{I_t}{Q_t} - w_t N_{et} + \frac{B_t}{R_t} \tag{42}$$

where B_t denotes uncontingent debt that matures in period t, Q_t denotes stochastic investment-specific technological change, with $Q_t = Q_t^p \nu_{qt}$. The permanent component Q_t^p follows an autoregressive process, that is, $Q^p = Q_{t-1}^p \lambda_{qt}$, $\ln \lambda_{qt} = \rho_q \ln \lambda_{qt-1} + (1-\rho_q)\bar{\lambda}_q + \sigma_q \varepsilon_{qt}$, $\ln \nu_{qt} = \rho_{\nu_q} \ln \nu_{qt-1} + \sigma_{\nu_q} \varepsilon_{\nu_q t}$. Parameter $\bar{\lambda}_q$ denotes the growth rate of Q_t^p, parameters ρ_q and ρ_{ν_q} belong to $(0,1)$, parameters $\sigma_q > 0$ and $\sigma_{\nu_q} > 0$ denote standard deviations, while ε_{qt} and $\varepsilon_{\nu_q t}$ are i.i.d. and normally distributed with zero mean and unit variance.

(iv) an endogenous collateral requirement:

$$B_t \leq \theta_t \mathbb{E}_t[q_{lt+1} L_{et} + q_{kt+1} K_t] \tag{43}$$

where q_{kt+1} is tomorrow's shadow price of capital expressed in units of the produced good, and θ_t denotes stochastic loan-to-value ratio, with $\ln \theta_t = \rho_\theta \ln \theta_{t-1} + (1-\rho_\theta)\ln \bar{\theta} + \sigma_\theta \varepsilon_{\theta t}$, $\bar{\theta} > 0$ is the steady-state value of the loan-to-value ratio, $\rho_\theta \in (-1,1)$, and $\varepsilon_{\theta t}$ is i.i.d. and normally distributed with mean zero and unit variance while $\sigma_\theta > 0$ is the innovation's standard deviation.

Defining μ_{et}, μ_{kt}, μ_{bt} as the respective Lagrange multipliers of (41), (42), and (43), it follows that relative price of capital in terms of the consumption good satisfies $q_{kt} = \frac{\mu_{kt}}{\mu_{et}}$ and the first-order conditions with respect to demands for consumption, labor, investment, capital, land and credit are:

$$\mu_{et} = \frac{1}{C_{et} - \gamma_e C_{et-1}} - \mathbb{E}_t\left[\frac{\beta\gamma_e}{C_{et+1} - \gamma_e C_{et}}\right] \tag{44}$$

$$w_t = (1-\alpha)Y_t/N_{et} \tag{45}$$

$$q_{kt} = \frac{\mu_{kt}}{\mu_{et}} \tag{46}$$

$$\frac{1}{Q_t} = q_{kt}\left(1 - \frac{\Omega}{2}\left(\frac{I_t}{I_{t-1}} - \bar{\lambda}_I\right)^2 - \Omega\left(\frac{I_t}{I_{t-1}} - \bar{\lambda}_I\right)\frac{I_t}{I_{t-1}}\right) + \beta\Omega\mathbb{E}_t\left[\frac{\mu_{et+1}}{\mu_{et}}q_{kt+1}\left(\frac{I_{t+1}}{I_t} - \bar{\lambda}_I\right)\left(\frac{I_{t+1}}{I_t}\right)^2\right] \tag{47}$$

$$q_{kt} = \mathbb{E}_t\left[\beta\frac{\mu_{et+1}}{\mu_{et}}\left((1-\phi)\alpha\frac{Y_{t+1}}{K_t} + q_{kt+1}(1-\delta)\right) + \frac{\mu_{bt}}{\mu_{et}}\theta_t q_{kt+1}\right] \tag{48}$$

$$q_{lt} = \beta\frac{\mu_{et+1}}{\mu_{et}}\left(\phi\alpha\frac{Y_{t+1}}{L_{et}} + q_{lt+1}\right) + \frac{\mu_{bt}}{\mu_{et}}\theta_t q_{lt+1} \tag{49}$$

$$1 = \mathbb{E}_t\left[\beta\frac{\mu_{et+1}}{\mu_{et}} + \frac{\mu_{bt}}{\mu_{et}}\right]R_t \tag{50}$$

Finally, in all period t the market clearing conditions are $Y_t = C_t + \frac{I_t}{Q_t}$ for the goods market, where $C_t = C_{ht} + C_{et}$ denotes aggregate consumption, $N_{et} = N_{ht}$ for the labor market, $L_{ht} + L_{et} = \bar{L}$ for the land market, where land is in fixed supply given by parameter $\bar{L} > 0$, and finally $B_t = S_t$ for the credit market.

The stationary version of the model, its linearization at the unique steady state and the calibration strategy are given in Appendix 6.3.1.

4.2 Sunspot Economy with Variable-Rate Loans

As explained in Section 3, the bond formulation used by Liu et al. (2013) is equivalent to a loan. We switch to the latter and then move the time index of the interest rate one period ahead, in order to introduce variable-rate loans. Consider the situation where the borrower repays $R_t B_{t-1}^l \equiv B_{t-1}$ and gets loanable fund $B_t^l \equiv B_t/R_{t+1}$ in period t, which means he will have to repay $R_{t+1}B_t^l \equiv B_t$ in period $t+1$. All conditions can now be expressed in terms of the amount borrowed B^l and moving from the fixed-rate economy in Section 4.1 to the variable-rate economy implies the following changes in equations:

(34)→

$$C_{ht} + q_{lt}(L_{ht} - L_{ht-1}) + S_t^l \leq w_t N_{ht} + R_t S_{t-1}^l \tag{51}$$

(38)→

$$1 = \beta\mathbb{E}_t\left[\frac{\mu_{ht+1}}{\mu_{ht}}R_{t+1}\right] \tag{52}$$

(42)→

$$C_{et} + q_{lt}(L_{et} - L_{et-1}) + R_t B_{t-1}^l = Y_t - \frac{I_t}{Q_t} - w_t N_{et} + B_t^l \tag{53}$$

(43)→

$$\mathbb{E}_t[R_{t+1}]B_t^l \leq \theta_t\mathbb{E}_t[q_{lt+1}L_{et} + q_{kt+1}K_t] \tag{54}$$

Table 1: Parameter values

Structural parameters	g_γ	λ_q	γ_h	γ_e	Ω	$\bar{L}$	α		
	1.004221	1.012126	0.4976	0.6584	0.1753	1	0.3		
Targeted steady state values	$\frac{I}{K}$	$\frac{K}{Y}$	$\frac{q_l L_e}{Y}$	$\frac{q_l L_h}{Y}$	N	θ			
	0.052325	4.6194	2.6	5.8011	0.25	0.75			
Shock parameters	ρ_a	ρ_z	ρ_{ν_z}	ρ_q	ρ_{ν_q}	ρ_ψ	ρ_θ	ρ_φ	
	0.9055	0.4263	0.0095	0.5620	0.2949	0.9829	0.9804	0.9997	
	σ_a	σ_z	σ_{ν_z}	σ_q	σ_{ν_q}	σ_ψ	σ_θ	σ_φ	σ_{sun}
	0.1013	0.0042	0.0037	0.0042	0.0029	0.0073	0.0112	0.0462	0.0462

(50)→

$$1 = \mathbb{E}_t \left[\left(\beta \frac{\mu_{et+1}}{\mu_{et}} + \frac{\mu_{bt}}{\mu_{et}} \right) R_{t+1} \right] \tag{55}$$

It is straightforward to show that the fixed-rate and variable-rate economies have the same steady state. In addition, the linearized system for the variable-rate economy obtains from that of the fixed-rate economy by replacing $\hat{R}_t$ with $\hat{R}_{t+1}$, $\hat{B}_{t-1}$ with $\hat{R}_t + \hat{B}^l_{t-1}$ and $\hat{B}_t$ with $\hat{R}_{t+1} + \hat{B}^l_t$. The resulting linearized system appears in Appendix 6.3.2.

4.3 Comparing Determinate and Sunspot Economies: Impulse Responses

We calibrate the model following Liu et al. (2013) to match some key ratios and use their posterior means for the other deep parameters. More details about the calibration strategy are given in Appendix 6.3.1. Parameter values are set according to Table 1.

Determinate Economy: Since it is the most important shock in Liu et al. (2013), we activate only the (fundamental) land demand shock while all other shocks are shut down in the determinate economy with fixed-rate loans. Figure 9 reports the corresponding IRFs while Table 2 reports moment statistics. A noticeable and surprising feature of Figure 9 is that although the shock to the lender's utility for land is positive, which implies that households are willing to hold more land, it turns out that land is reallocated to the entrepreneur at impact, as explained in Liu et al. (2013). This happens because land price goes up, which relaxes the entrepreneur's credit constraint and generates a boom that initially reallocates land to the borrower. Because the shock to land utility is very persistent (see Table 1), the response of land price and other aggregates are also very persistent. In addition, the interest rate looks procyclical in Figure 9, which is confirmed by Table 2 and is at odds with evidence reported in Section 2.

Sunspot Economy: There is one-dimensional indeterminacy in the variable-rate economy and we assume, following earlier literature, that sunspot innovations affect investment (though similar results obtain if, for instance, either the land price or the interest rate reacts to extrinsic uncertainty instead). We first investigate what happens in the indeterminate economy when sunspots are inactive while a land demand shock hits. Figure 10 reports the IRFs to a land demand shock and Table 3 reports moment statistics. In Figure 10, a positive land demand shock generates a recession, opposite to what happens in fixed-rate economy (see Figure 9). This is because land now goes to the household when she experiences a positive land demand shock. This means that

Figure 9: IRFs to a positive land demand shock in the fixed-rate - determinate - economy ($\rho_\varphi = 0.9997$, $\sigma_\varphi = 0.0462$)

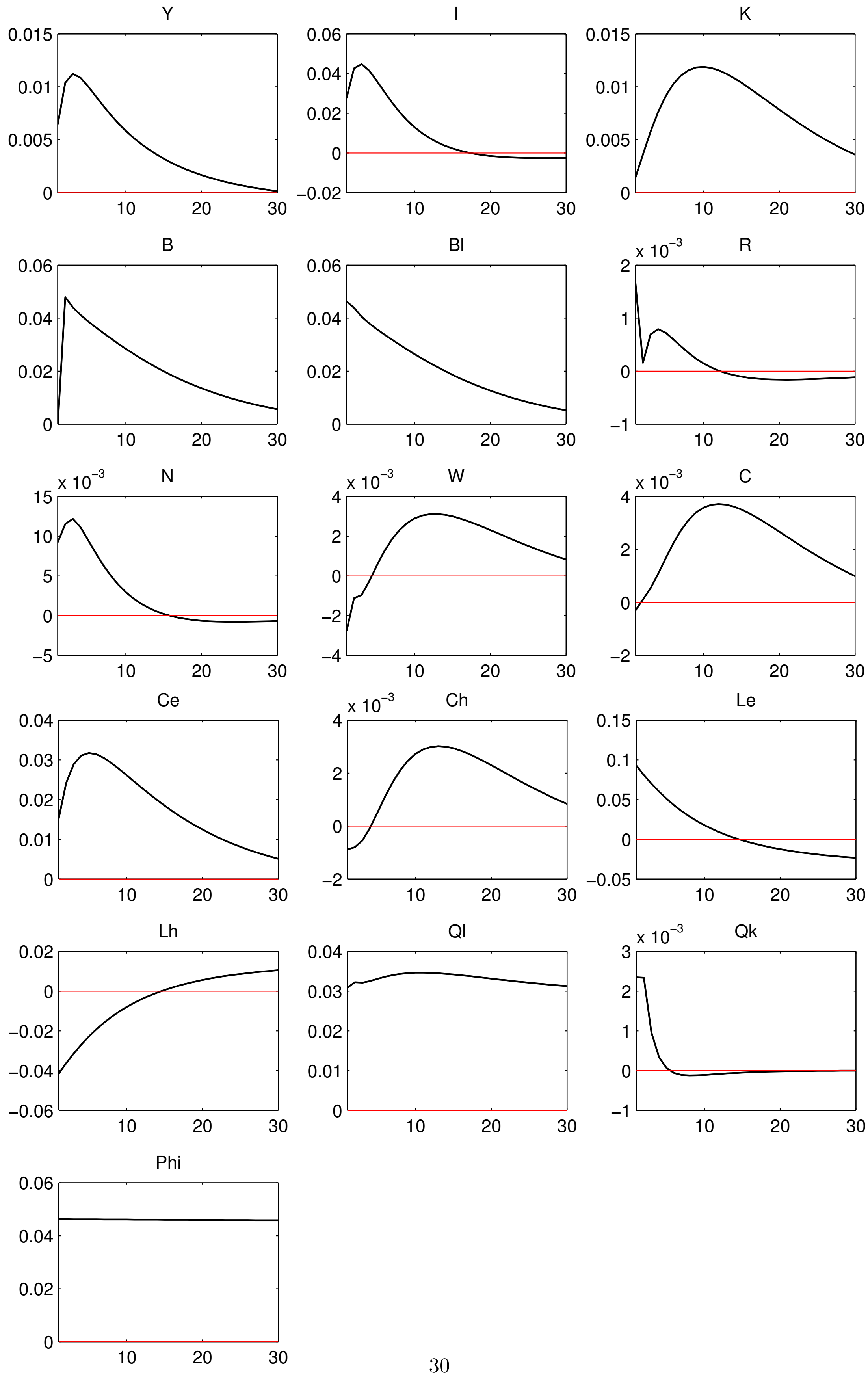


Table 2: Moments under positive land demand shock in fixed-rate - determinate - economy ($\rho_\varphi = 0.9997$, $\sigma_\varphi = 0.0462$)

Variable (log)	S.D. relative to output (%)	CORR with output	ACF1	ACF2	ACF3	ACF4	ACF5
Y	100	1	0.9815	0.9424	0.8974	0.8527	0.8113
I	253.5463	0.8403	0.9478	0.8394	0.7156	0.5954	0.4881
K	129.4317	0.8322	0.9958	0.9841	0.966	0.9427	0.9156
B	324.038	0.8729	0.9412	0.8815	0.8258	0.7725	0.7239
R	5.6797	0.4798	0.5321	0.6048	0.5555	0.4464	0.3431
N	65.2197	0.6661	0.9299	0.8089	0.666	0.5279	0.4066
w	74.5977	0.7581	0.9933	0.988	0.9798	0.9702	0.9603
C	77.5277	0.825	0.9987	0.9955	0.9904	0.9837	0.9758
C_e	274.0718	0.8797	0.9856	0.9537	0.9118	0.8647	0.8159
C_h	74.0065	0.7748	0.9984	0.9948	0.9894	0.9828	0.9755
L_e	2445.2123	0.8033	0.9954	0.9911	0.9871	0.9833	0.9801
L_h	1095.9218	-0.8033	0.9954	0.9911	0.9871	0.9833	0.9801
q_l	2277.1502	-0.6026	0.9994	0.9989	0.9983	0.9977	0.9971
q_k	8.3326	0.3343	0.6876	0.2773	0.0763	-0.0239	-0.0594
φ	3549.4778	-0.616	0.9995	0.999	0.9985	0.998	0.9976

the collateral channel through land price is reversed at impact. As a consequence, the entrepreneur's borrowing constraint tightens because of land reallocation to the lender, which further contributes to the recession over and above the reduction in the stock of productive land. Second, we shut down all fundamental shocks and feed the model with a sunspot shock only. Figure 11 reports the IRFs to a positive sunspot shock on investment and Table 4 reports moment statistics. The main features of Figure 11 are that indeterminacy generates persistence endogenously, since the shock has zero autocorrelation, a higher level of volatility compared to Figure 9 (given the same standard deviations for both shocks' innovations). Land price, credit, consumption and labor are procyclical while and the interest rate is countercyclical, which accords with evidence reported in Section 2. Comparing Figures 9 and 11 suggests that sunspot shocks to investment could potentially be as quantitatively important as land demand shocks in explaining booms and busts in the credit market and real production activities. This is what we examine next in the estimation section of the paper.

4.4 Bayesian Estimation of Determinate and Sunspot Models with Hybrid Interest Rate

This section addresses the following questions: have sunspot shocks contributed to US business cycles and as importantly as fundamental shocks? Our estimation results, reported below, unambiguously yield "yes" and "yes" as the answers. For comparison purpose, we use Bayesian techniques and all estimation results reported below are based on the dataset made available by Liu et al. (2013) through the Econometrica website referenced in the published version of that paper.

Our estimation strategy is as follows. It is obvious that the determinate and sunspot models are both unrealistic in the sense that the firm sector as a whole is expected to use a combination of fixed-rate and

Figure 10: IRFs to a positive land demand shock in the variable-rate - indeterminate - economy ($\rho_\varphi = 0.9997$, $\sigma_\varphi = 0.0462$)

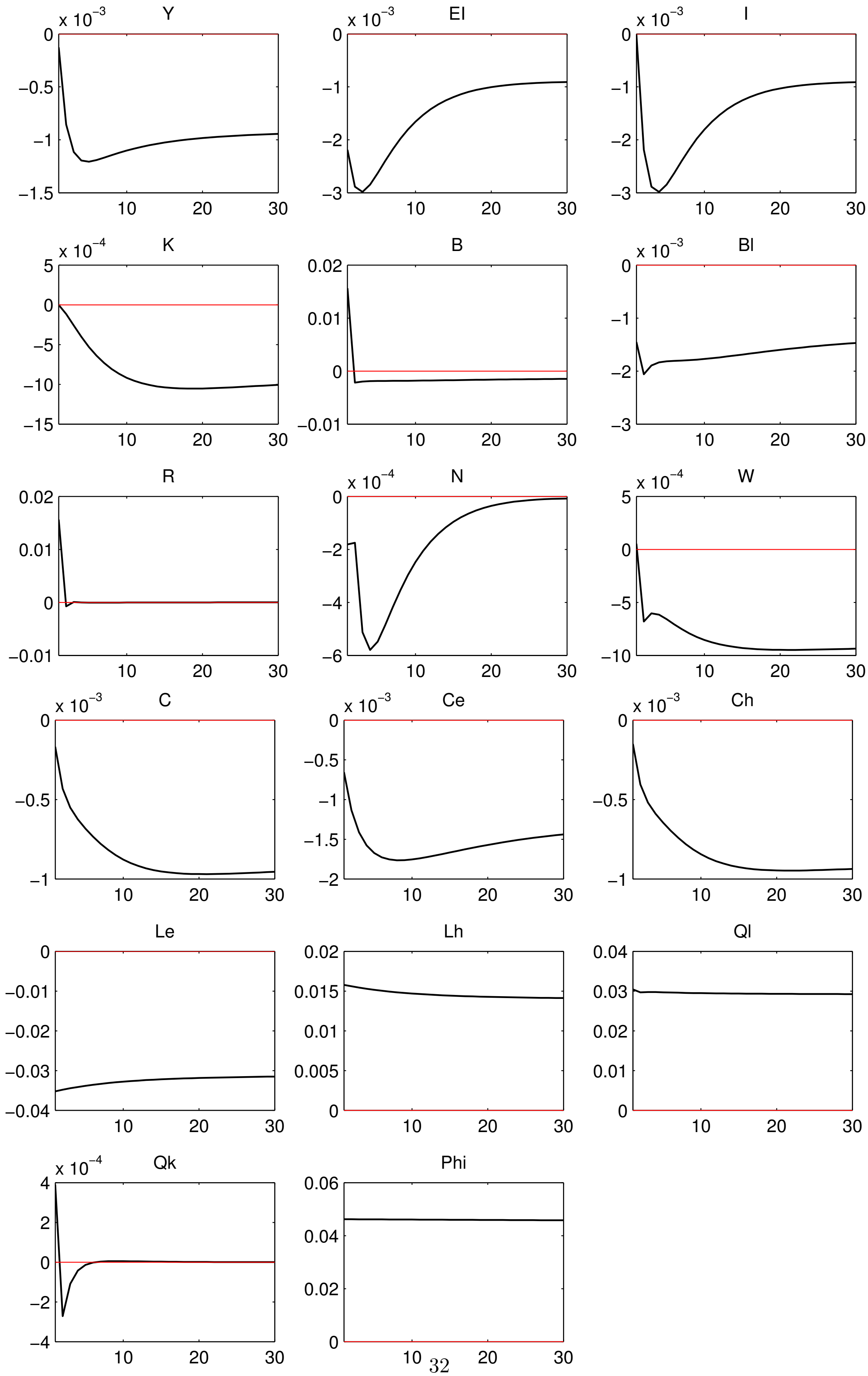


Figure 11: IRFs to a positive sunspot shock on investment in the variable-rate - indeterminate - economy (zero autocorrelation, $\sigma_{sun} = 0.0462$)

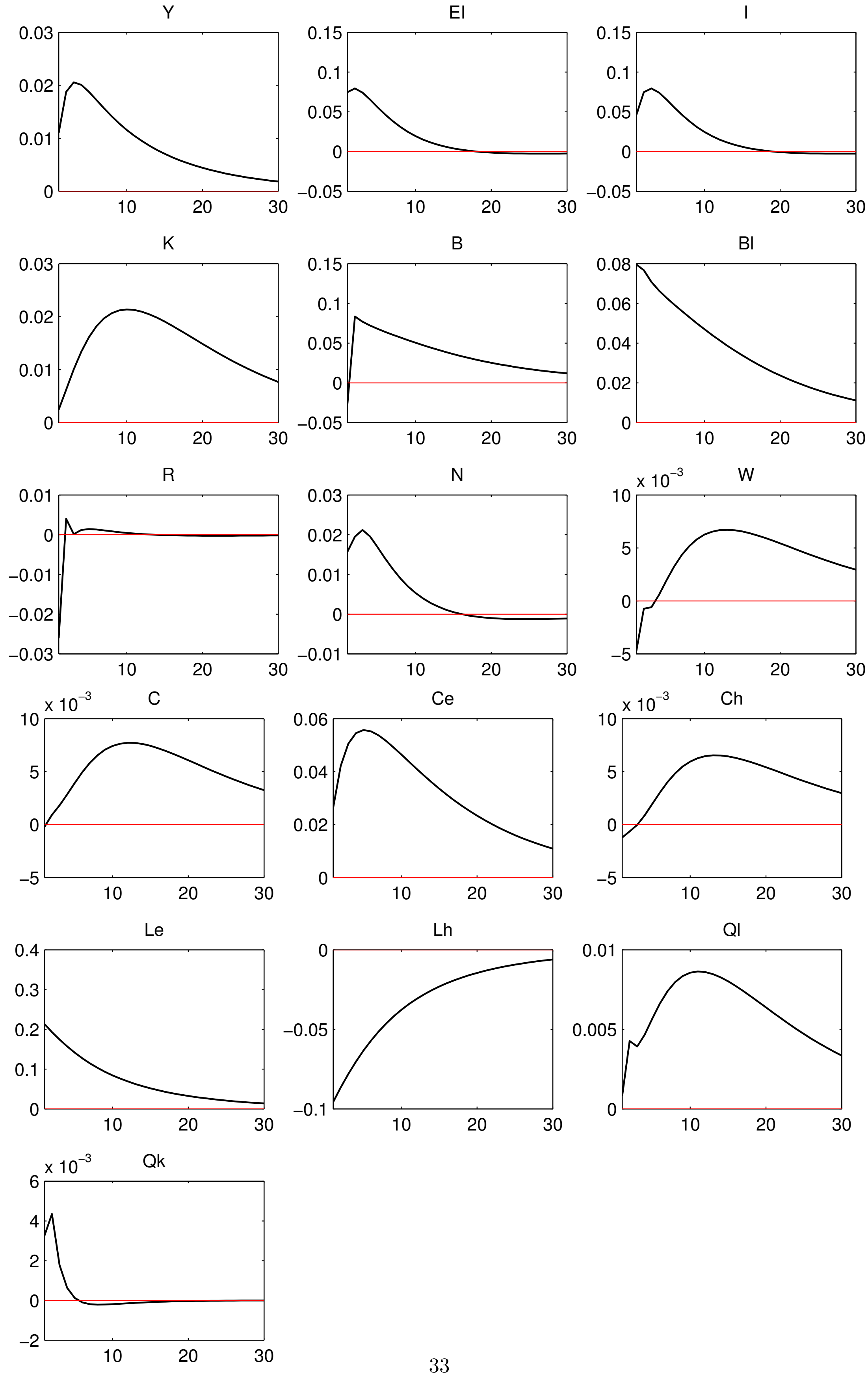


Table 3: Moments under positive land demand shock in variable-rate - indeterminate - economy ($\rho_\varphi = 0.9997$, $\sigma_\varphi = 0.0462$)

Variable (log)	S.D. relative to output (%)	CORR with output	ACF1	ACF2	ACF3	ACF4	ACF5
Y	100	1	0.9996	0.9989	0.9981	0.9973	0.9965
$\mathbb{E}[I]$	102.3068	0.9893	0.9971	0.9922	0.9869	0.9817	0.9772
I	102.3062	0.9911	0.9971	0.9922	0.9869	0.9817	0.9772
K	100.3067	0.9983	0.9998	0.9997	0.9994	0.9991	0.9987
B	155.077	0.9447	0.8829	0.8813	0.8809	0.8771	0.8774
R	50.1402	0.0015	-0.0213	0.012	0.0074	-0.0217	-0.0138
N	4.6024	0.444	0.9559	0.888	0.7862	0.6809	0.5848
w	98.0431	0.9991	0.9996	0.9994	0.9991	0.9989	0.9986
C	99.6501	0.999	0.9998	0.9997	0.9994	0.9991	0.9988
C_e	143.0388	0.9998	0.9997	0.9992	0.9986	0.9978	0.997
C_h	98.0162	0.9989	0.9998	0.9997	0.9994	0.9992	0.9989
L_e	3393.9973	0.9994	0.9993	0.9988	0.9982	0.9976	0.9971
L_h	1521.1585	-0.9994	0.9993	0.9988	0.9982	0.9976	0.9971
q_l	3167.7913	-0.9994	0.9994	0.999	0.9985	0.998	0.9975
q_k	1.5576	0.022	-0.2802	-0.1251	-0.045	-0.0447	-0.0191
φ	4955.9201	-0.9994	0.9995	0.999	0.9985	0.998	0.9976

Table 4: Moments under positive sunspot shock to investment in variable-rate - indeterminate - economy (zero autocorrelation, $\sigma_{sun} = 0.0462$)

Variable (log)	S.D. relative to output (%)	CORR with output	ACF1	ACF2	ACF3	ACF4	ACF5
Y	100	1	0.9676	0.8971	0.8156	0.7346	0.6596
I	321.2725	0.9236	0.9461	0.8306	0.6981	0.5696	0.4549
K	148.1905	0.7984	0.9946	0.9793	0.9556	0.9251	0.8896
B_l	404.3678	0.9759	0.9381	0.874	0.8143	0.7573	0.7053
R	44.8755	-0.0634	-0.1147	0.0159	-0.0199	-0.0592	-0.0497
N	84.4207	0.8808	0.9324	0.8154	0.6739	0.536	0.4143
w	47.4896	0.54	0.9683	0.9489	0.9148	0.8723	0.827
C	54.141	0.6887	0.9952	0.9821	0.9613	0.9339	0.9012
C_e	338.5766	0.9638	0.9842	0.949	0.9028	0.851	0.7973
C_h	45.8382	0.5778	0.9932	0.9766	0.9512	0.9189	0.8815
L_e	870.8564	0.961	0.9022	0.8086	0.7222	0.6426	0.5746
L_h	390.3099	-0.961	0.9022	0.8086	0.7222	0.6426	0.5746
$\mathbb{E}[q_l]$	60.6178	0.8274	0.9902	0.977	0.9576	0.9317	0.9004
q_l	60.6089	0.7801	0.9921	0.9788	0.9599	0.9346	0.9038
q_k	10.3338	0.4514	0.7103	0.2839	0.0761	-0.026	-0.0623

variable-rate loans at any point in time.[21]

We therefore estimate hybrid versions of the model with a fixed fraction - say, $\omega \in (0,1)$ - of variable-rate-rate loans. It is not difficult to show numerically that such an hybrid model has a determinate steady-state if and only if $\omega < 0.5$ while the steady state is indeterminate if and only if $\omega > 0.5$, just as in the simple model of Section 3.3 in which it can be proved analytically (see Appendix 6.1). Of course, the versions simulated in Section 4.3 correspond to extreme cases, such that either $\omega = 0$ (see Section 4.1) or $\omega = 1$ (see Section 4.2). We therefore estimate the determinate model under the prior that $\omega \in (0, 0.5)$ and the indeterminate model using the prior $\omega \in (0.5, 1)$.

The simulation results reported in Section 4.3 already contain some information that can be used to form some guess about what estimating the model should deliver. First, because sunspot shocks generate procyclical land price, which is line with the data, the contribution of sunspot shocks is expected to be significant. In contrast, because land demand shocks have opposite effects in the indeterminate economy, it could well be that their contribution appears to be reduced in the variable-rate economy. These observations turn out to accord with the estimation results that we report next.

Estimated Parameters:

As a benchmark, we first estimate the determinate model, which is essentially Liu et al.'s (2013) with an additional parameter, that is, ω, the share of variable-rate loans. Table 5 reports the estimated parameters, which are very close to the ones found by Liu et al.'s (2013, Tables 1 and 2). Most importantly, in Table 5 the posterior mode of ω is 0.5, which suggests that the estimation procedure pushes towards high values for the share of variable-rate loans and finally settles on the highest possible value conditional of the model being determinate, which we know in theory to be precisely 0.5. This already suggests that data might favor the indeterminate model with $\omega > 0.5$. Table 6 reports the estimated parameters obtained from the indeterminate model. Because the share ω is not properly identified in the sunspot economy, we choose to set its value to 0.7, using the time series constructed by Vickery (2008) as an help to guide our calibration.[22] However, we have checked that our estimation results remain similar under different values for ω.

Comparing Tables 5 and 6 reveals some differences. Most notably, in the sunspot model the patience shock has a very low autocorrelation (that is, the mode of ρ_a is 0.0215) and a standard deviation σ_a that is two orders of magnitude smaller compared to the determinacy regime, where it actually has the most volatile innovations by far. In light of these changes, one expects patience shocks to become negligible in the indeterminate economy, which is confirmed in the variance decomposition that we discuss next.

[21]To our knowledge, there exists no comprehensive measure of how prevalent variable-rate loans to US firms are. Historically, floating-rate debt was introduced in the US in 1974 (see Allen and Gale, 1994, page 19). Since then it has been increasingly used by companies to borrow funds, with a pronounced acceleration in the 1980s and 1990s when non-bank investors like mutual funds and insurance companies massively entered the market as buyers, followed by collateralized loan obligations structures and hedge funds in the 2000s. Modern forms of floating-rate loans are investment-grade corporate floaters and sub-investment-grade bank loans (also referred to as senior secured loans, leveraged loans, or syndicated loans), which are both classified as senior collateralized debt in the borrowing firm's capital structure. Although this is only indicative, we notice that, at least since 2000, the market size for floating-rate loans has been exceeding that of high-yield (usually fixed-rate and unsecured) bonds. As of December 2014, the former exceeds \$1.9 trillion while the latter represents slightly more than \$1.3 trillion (source: Crédit Suisse and Loan Pricing Corporation). In periods of low yields, bank loans are particularly attractive to investors and recommended by many portfolio management firms. See for example http://www.loomissayles.com/internet/internetdata.nsf/id/8yaj9c/$file/bankloans-lookingbeyondinterestrateexpectations.pdf.

[22]It is not difficult to see that our hybrid economy's moments depend on ω only in the determinacy regime. In a nutshell, this happens because ω affects the set of unstable eigenvalues and the linear saddle-path solution used to solve the determinate model, and this is why ω is identified when estimating the determinate model. In contrast, as long as it is larger than 0.5, ω does not matter in the indeterminate regime and, therefore, is not identified in that case.

Table 5: Estimated parameters of determinate model with hybrid interest rate

parameters	prior			posterior			
	distribution	mean	s.d.	mode	mean	low	high
ω	beta	0.167	0.1179	0.5	0.4975	0.4956	0.5
γ_h	beta	0.333	0.2357	0.5175	0.5217	0.4673	0.5777
γ_e	beta	0.333	0.2357	0.7688	0.7241	0.5641	0.879
Ω	gamma	2	2	0.1581	0.1692	0.129	0.2108
$100(g_\gamma - 1)$	gamma	0.618	0.453	0.4174	0.3987	0.2846	0.5081
$100(\lambda_q - 1)$	gamma	0.618	0.453	1.2188	1.2147	1.078	1.3453
ρ_a	beta	0.333	0.2357	0.9098	0.9004	0.8667	0.9367
ρ_z	beta	0.333	0.2357	0.3897	0.3965	0.2836	0.5115
ρ_{ν_z}	beta	0.333	0.2357	0.2751	0.3049	0.0923	0.5167
ρ_q	beta	0.333	0.2357	0.5378	0.5349	0.4287	0.6351
ρ_{ν_q}	beta	0.333	0.2357	0.3096	0.3493	0.0555	0.6162
ρ_φ	beta	0.333	0.2357	0.9997	0.9994	0.9988	1
ρ_ψ	beta	0.333	0.2357	0.9877	0.987	0.978	0.9968
ρ_θ	beta	0.333	0.2357	0.9809	0.982	0.9757	0.9884
σ_a	inverse gamma	0.01	∞	0.095	0.1358	0.0552	0.2231
σ_z	inverse gamma	0.01	∞	0.0047	0.0047	0.0038	0.0056
σ_{ν_z}	inverse gamma	0.01	∞	0.0038	0.0039	0.0032	0.0046
σ_q	inverse gamma	0.01	∞	0.0043	0.0044	0.0036	0.0052
σ_{ν_q}	inverse gamma	0.01	∞	0.0027	0.0028	0.0021	0.0035
σ_φ	inverse gamma	0.01	∞	0.0459	0.0488	0.0419	0.0561
σ_ψ	inverse gamma	0.01	∞	0.0076	0.0078	0.0068	0.0087
σ_θ	inverse gamma	0.01	∞	0.0117	0.0119	0.0106	0.0131

Variance Decomposition:

Our metric to assess the importance of each shock at business-cycle frequency is the conditional variance decomposition at various horizons (quarters), as in Liu et al. (2013). In fact, Table 7 (see also Figure 12) shows that the variance decomposition that obtains in our hybrid version of the determinate model delivers results that are very close to those in Liu et al. (2013). More precisely, the land demand shock contributes in a significant manner to explain all aggregates, some more importantly than others. For instance, the movements of land price, investment, labor and output are largely due to disturbances to the household's utility from land in Table 7. In that sense, the estimation of ω does not add anything to the analysis and all in all Table 7 replicates Liu et al. (2013)'s findings.

However, the variance decomposition that arises in the sunspot economy and that is reported in Table 8 and Figure 13 tells an altogether very different story. While patience shocks play no role, as already anticipated, the most striking result is that sunspot shocks are an important driver of output, investment, credit and labor. In particular, after four quarters, sunspot shocks explain around 32% of output volatility, 31% of investment, 15% of credit and 20% of labor.

On the other hand, land demand shocks turn out to explain only the land price and to a little bit more extent the interest rate, as in Liu et al. (2013). For all other aggregates, however, the contribution of land demand shocks is very modest (less than 5% for all except land price and interest rate and less than 3% for most). This

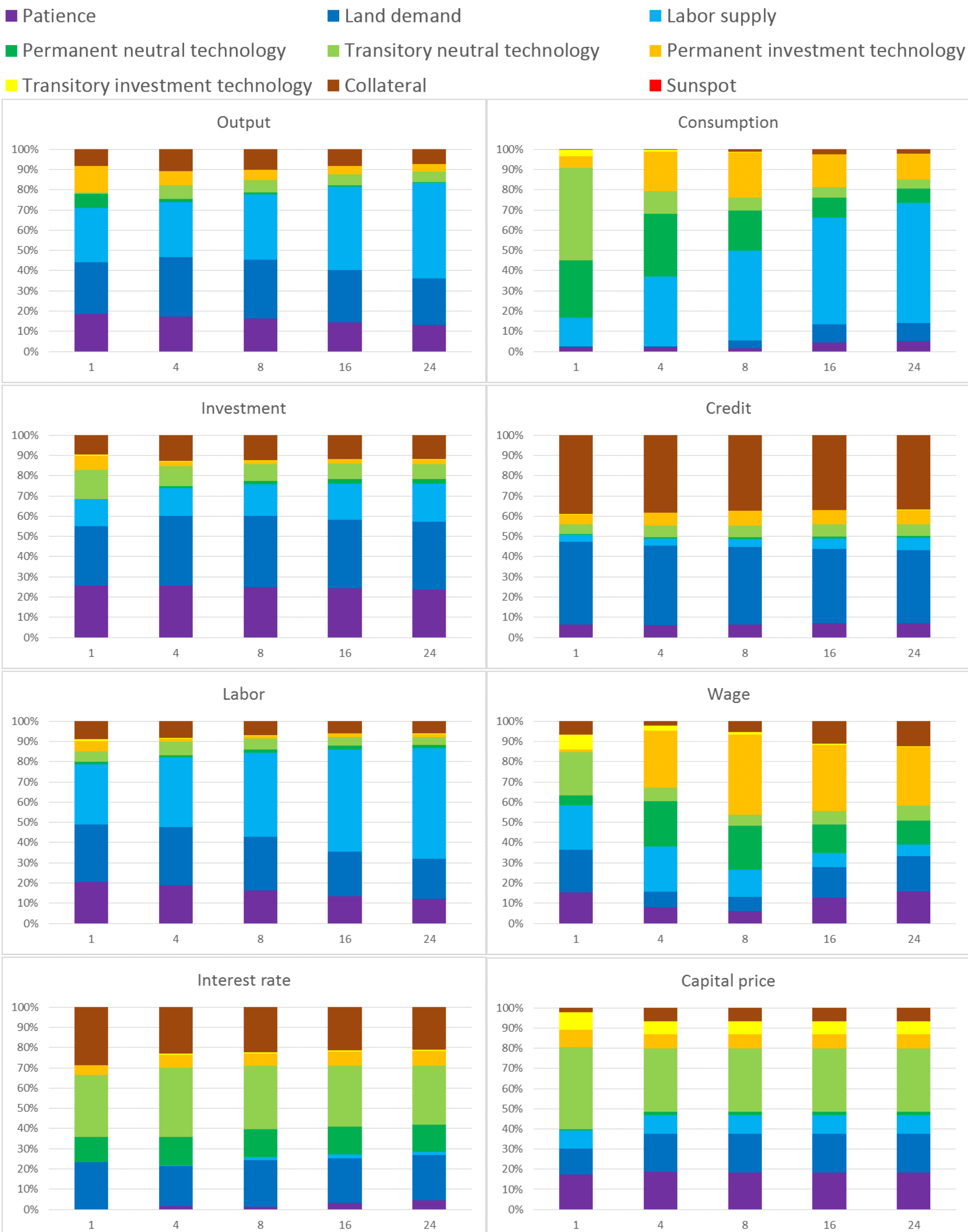


Figure 12: Variance decomposition of fundamental shocks in determinate model with hybrid interest rate

Table 6: Estimated parameters of indeterminate model with hybrid rate

parameters	prior			posterior			
	distribution	mean	s.d.	mode	mean	low	high
γ_h	beta	0.333	0.2357	0.4481	0.4492	0.3904	0.506
γ_e	beta	0.333	0.2357	0.9962	0.9109	0.7401	0.9999
Ω	gamma	2	2	0.0808	0.0839	0.063	0.1037
$100(g_\gamma - 1)$	gamma	0.618	0.453	0.2574	0.1887	0.0305	0.3413
$100(\lambda_q - 1)$	gamma	0.618	0.453	1.206	1.1902	1.0618	1.3082
ρ_a	beta	0.333	0.2357	0.0215	0.2681	0	0.5535
ρ_z	beta	0.333	0.2357	0.339	0.3452	0.2424	0.4452
ρ_{ν_z}	beta	0.333	0.2357	0	0.0521	0	0.1145
ρ_q	beta	0.333	0.2357	0.3202	0.3473	0.2551	0.4469
ρ_{ν_q}	beta	0.333	0.2357	0.3182	0.2484	0.0015	0.4651
ρ_φ	beta	0.333	0.2357	0.9992	0.9986	0.9975	0.9999
ρ_ψ	beta	0.333	0.2357	0.998	0.9955	0.9911	0.9997
ρ_θ	beta	0.333	0.2357	0.9895	0.9852	0.9788	0.9923
σ_a	inverse gamma	0.01	∞	0.0033	0.008	0.0012	0.0159
σ_z	inverse gamma	0.01	∞	0.0049	0.005	0.0043	0.0058
σ_{ν_z}	inverse gamma	0.01	∞	0.0031	0.0033	0.0028	0.0039
σ_q	inverse gamma	0.01	∞	0.0054	0.0053	0.0045	0.0061
σ_{ν_q}	inverse gamma	0.01	∞	0.0018	0.002	0.0013	0.0026
σ_φ	inverse gamma	0.01	∞	0.0468	0.05	0.0412	0.0583
σ_ψ	inverse gamma	0.01	∞	0.0074	0.0075	0.0066	0.0084
σ_θ	inverse gamma	0.01	∞	0.017	0.0169	0.0148	0.019
σ_{sun}	inverse gamma	0.01	∞	0.0099	0.01	0.0075	0.0126

is where the sunspot model departs from the determinate model: while land demand shocks alone explain land price in both, their importance in explaining the volatilities of other variables vanishes in the sunspot economy. Other major differences between Tables 7 and 8 are that productivity and investment-specific shocks become much more important in the sunspot model, in particular to account for movements in output and investment, in line with earlier results in the business-cycle literature (e.g. Greenwood, Hercowitz, Huffman, 1997, Justiniano, Primiceri, Tambalotti, 2011). In contrast, the contributions of each fundamental shock to consumption movements are not very different in each regime and sunspot shocks have only a modest effect on consumption.

A natural question to ask at this stage is which model does better fit the data. To that aim, Table 9 reports the marginal data density, using Geweke's criterion. The four models for which the data density is reported are Liu et al. (2013)'s original version with $\omega = 0$ (second column in Table 9), the hybrid version with ω estimated to have a mode about 0.5 (third column), the pure sunspot model with $\omega = 1$ (fourth column) and its hybrid version with $\omega = 0.7$ (fifth column). Table 9 shows that while the hybrid determinate model is preferred to the pure determinate model, they are both overwhelmingly rejected against the sunspot models: if the prior distribution over models is agnostic, the posterior probability of the determinate models is 0. On the other hand, the data does not strongly discriminate between the pure and the hybrid sunspot models, since the posterior probability of the hybrid sunspot model is about 0.58 with identical priors across all models.

Table 7: Variance decomposition of fundamental shocks in determinate model with hybrid interest rate

Horizon	ε_a	ε_z	ε_{ν_z}	ε_q	ε_{ν_q}	ε_φ	ε_ψ	ε_θ
Output (Y)								
1	18.6	7.09	0.66	13.11	0	25.69	26.77	8.09
4	17.15	1.64	6.71	6.95	0.02	29.53	27.15	10.86
8	16.29	0.93	6.17	5.18	0.03	29.15	32.22	10.04
16	14.42	0.67	5.52	4.27	0.04	25.97	41	8.11
24	13.03	0.59	5.04	3.86	0.03	23.23	47.08	7.13
Consumption (C)								
1	2.35	28.47	45.79	5.7	3.33	0.29	13.87	0.18
4	2.09	31.23	11.26	19.47	0.92	0.43	34.45	0.15
8	1.58	19.73	6.48	22.41	0.39	3.96	44.31	1.15
16	4.4	10.07	5.05	16.29	0.17	8.94	52.76	2.32
24	5.12	7.25	4.42	12.75	0.12	8.95	59.38	2.01
Investment (I)								
1	25.73	0.15	14.34	7.04	0.56	29.28	13.36	9.55
4	25.52	0.88	9.85	2.4	0.16	34.63	13.72	12.85
8	25.1	1.71	8.23	1.76	0.16	35.09	15.51	12.44
16	24.16	2.09	7.74	2.06	0.17	34.04	18.04	11.7
24	23.7	2.11	7.58	2.15	0.17	33.43	19.04	11.83
Credit (B^l)								
1	6.43	0.38	4.75	4.91	0.3	40.87	3.39	38.98
4	6.21	0.79	5.53	6.55	0.15	39.28	3.36	38.14
8	6.52	0.89	5.88	7.24	0.1	38.18	3.86	37.33
16	7	0.84	5.98	7.21	0.08	36.94	5.08	36.88
24	7.13	0.8	5.95	7.05	0.08	36.03	6.17	36.8
Labor (N)								
1	20.58	1.43	4.89	5.28	0.82	28.43	29.62	8.95
4	18.87	0.84	6.78	1.49	0.24	28.67	34.73	8.38
8	16.28	1.6	5.34	1.41	0.2	26.45	41.77	6.95
16	13.35	1.73	4.35	1.72	0.18	22.1	50.69	5.88
24	12.06	1.59	3.91	1.67	0.16	19.83	54.74	6.04
Wage (w)								
1	15.33	4.82	21.49	1.12	7.34	21.18	22.06	6.66
4	7.96	22.35	6.9	28.22	2.43	7.58	22.41	2.16
8	6.15	21.72	5.63	39.5	1.21	7.03	13.31	5.44
16	12.79	13.97	6.81	32.62	0.62	15.19	6.85	11.16
24	15.66	11.67	7.33	29.18	0.5	17.45	5.93	12.28
Interest rate (R)								
1	0.04	12.44	30.87	4.56	0.1	23.26	0.01	28.72
4	1.53	13.9	34.34	6.51	0.66	19.83	0.43	22.8
8	1.4	13.62	31.42	6.13	0.61	22.91	1.64	22.26
16	3.05	13.82	29.88	7.2	0.6	22.36	1.77	21.34
24	4.55	13.47	29.01	7.38	0.58	22.18	1.82	21.02
Land price (q_l)								
1	5.33	0.15	0.19	0.18	0.25	90.87	3.02	0.01
4	4.3	0.6	0.09	1.07	0.08	91.09	2.69	0.07
8	3.83	0.59	0.19	1.37	0.04	90.88	2.93	0.18
16	3.06	0.37	0.25	1.06	0.02	91.62	3.37	0.25
24	2.4	0.26	0.22	0.8	0.01	92.57	3.53	0.2
Capital price (q_k)								
1	17.47	0.69	40.63	8.72	8.38	12.62	9.19	2.31
4	18.5	1.6	31.48	7.01	6.27	19.21	9.18	6.75
8	18.44	1.64	31.39	7.2	6.27	19.16	9.14	6.75
16	18.47	1.64	31.33	7.19	6.26	19.21	9.13	6.77
24	18.47	1.64	31.33	7.19	6.26	19.21	9.13	6.77

Table 8: Variance decomposition of fundamental and investment sunspot shocks in indeterminate model with hybrid interest rate

Horizon	ε_a	ε_z	ε_{ν_z}	ε_q	ε_{ν_q}	ε_φ	ε_ψ	ε_θ	ε_{sun}
				Output (Y)					
1	0.00	27.69	23.68	9.94	1.26	0.05	20.25	0.00	17.14
4	0.00	7.09	32.73	2.81	6.41	4.55	14.11	0.18	32.13
8	0.00	3.55	29.97	2.25	6.48	5.22	23.79	2.31	26.43
16	0.00	1.74	18.75	1.27	4.16	4.71	45.2	8.09	16.08
24	0.00	1.09	12.26	0.8	2.73	4.03	57.5	11.12	10.48
				Consumption (C)					
1	0.00	33.35	28.52	11.97	1.51	0.06	24.39	0.00	0.2
4	0.00	33.18	6.67	20.57	0.37	0.55	38.55	0.01	0.1
8	0.00	27.11	4.26	19.71	0.31	1.22	46.46	0.02	0.9
16	0.00	17.97	4.81	13.29	0.68	2.39	57.26	0.81	2.78
24	0.00	12.34	4.39	9.08	0.72	2.86	65.18	2.57	2.86
				Investment (I)					
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100.00
4	0.00	11.99	29.24	17.14	5.48	2.16	2.99	0.2	30.81
8	0.00	14.29	27.74	20.98	5.92	2.24	1.98	2.28	24.58
16	0.00	13.59	23.35	19.76	5.09	2.23	7.05	8.79	20.14
24	0.00	11.98	20.24	17.32	4.41	2.09	13.72	12.77	17.46
				Credit (B^l)					
1	0.00	0.63	23.26	0.07	6.35	1.07	19.18	35.01	14.41
4	0.00	1.75	20.65	0.66	5.29	1.54	19.6	35.21	15.3
8	0.00	1.99	20.78	0.77	5.14	1.95	18.28	34.9	16.2
16	0.00	1.87	21.76	0.69	5.3	2.82	15.86	34.43	17.27
24	0.00	1.78	22.01	0.65	5.35	3.64	15.02	34.04	17.51
				Labor (N)					
1	0.00	5.5	12.89	1.08	0.2	0.1	43.45	0.00	36.78
4	0.00	8.98	18.39	18.36	3.49	0.8	29.19	0.39	20.4
8	0.00	9.51	13.59	16.9	2.96	0.66	41.78	2.13	12.48
16	0.00	6.04	7.32	10.25	1.62	0.38	63.18	4.64	6.58
24	0.00	4.23	5.07	7.13	1.12	0.26	72.47	5.16	4.56
				Wage (w)					
1	0.00	23.69	4.99	46.47	6.45	0.02	9.95	0.00	8.43
4	0.00	32.91	0.7	43.65	0.81	1.34	19.07	0.47	1.06
8	0.00	30.1	1.31	37.11	0.57	2.28	26.06	0.56	2.02
16	0.00	22.78	5.08	26.86	1.27	4.54	33.56	0.5	5.4
24	0.00	19.51	6.73	22.79	1.63	6.59	34.38	1.71	6.66
				Interest rate (R)					
1	0.00	0.12	9.69	4.58	0.55	39.41	11.77	30.5	3.37
4	0.00	0.37	11.48	4.49	0.55	38.38	11.7	29.66	3.36
8	0.00	0.44	11.5	4.57	0.56	38.28	11.68	29.58	3.39
16	0.00	0.48	11.49	4.61	0.56	38.24	11.67	29.56	3.38
24	0.00	0.49	11.49	4.61	0.56	38.23	11.68	29.56	3.39
				Land price (q_l)					
1	0.00	0.24	0.51	0.4	0.26	95.49	2.51	0.58	0.01
4	0.00	0.86	0.22	1.03	0.1	95.41	1.79	0.51	0.08
8	0.00	0.78	0.23	0.88	0.07	95.65	1.67	0.53	0.18
16	0.00	0.48	0.29	0.51	0.08	95.83	1.87	0.69	0.25
24	0.00	0.33	0.26	0.35	0.07	95.63	2.29	0.86	0.22
				Capital price (q_k)					
1	0.00	1.06	73.52	2.02	14.19	2.17	4.07	0.04	2.94
4	0.00	4.66	65.08	6.27	11.11	2.33	5.08	0.29	5.18
8	0.00	4.69	64.99	6.32	11.1	2.32	5.09	0.3	5.19
16	0.00	4.69	64.96	6.33	11.09	2.32	5.1	0.31	5.2
24	0.00	4.69	64.96	6.33	11.09	2.32	5.1	0.31	5.2

Remark: the numbers in the ε_a column are smaller than 10^{-2}.

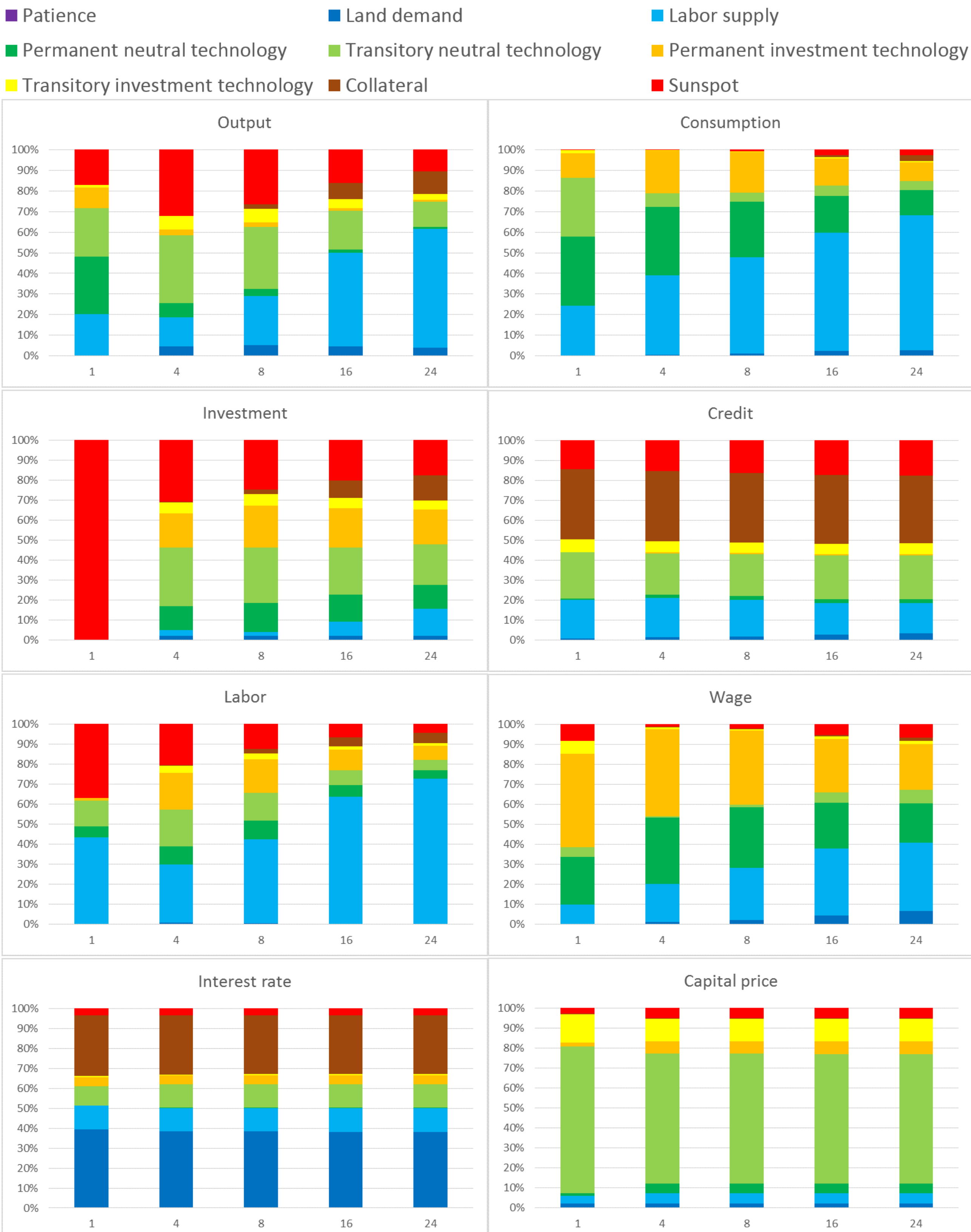


Figure 13: Variance decomposition of fundamental and investment sunspot shocks in indeterminate model with hybrid interest rate

Table 9: Model Fit

	LWZ Model	Hybrid LWZ Model	Sunspot Model	Hybrid Sunspot Model
Log marginal data density	2354.757502	2359.849251	2468.285258	2468.598982
Model posterior probability	10^{-50}	10^{-48}	0.42	0.58

LWZ Model: Liu et al. (2013) with $\omega = 0$. Hybrid LWZ Model: augmented Liu et al. (2013) with ω estimated. Sunspot Model: indeterminate model with $\omega = 1$. Hybrid Sunspot Model: indeterminate model with $\omega = 0.7$. Log marginal data density is Geweke's for four estimated models

5 Conclusion

The contribution of this paper is twofold. On the theory side, we have shown that indeterminacy and sunspot equilibria arise in standard versions of DSGE models with endogenous collateral constraints, provided that loans have a variable-rate component. The empirical part of the paper has given content to the claim that, far from being only a theoretical curiosity, the model with sunspot equilibria accords with data. In particular, while the indeterminate model with variable-rate collateralized loans predicts that the borrowing cost is countercyclical, in line with data, such a model also replicates the S-shaped pattern of dynamic correlations between the interest rate and aggregate variables that is also present in data. In contrast, the interest rate is procyclical in the determinate model with predetermined-rate loans and its lagged values correlate positively with contemporaneous aggregates, which is at odds with quarterly data.

We conjecture that our set of results could be of interest to understand the business-cycle consequences of household's debt and housing investment, in view of the fact that variable-rate loans (that is, adjustable rate mortgages) have been an important source of funding up to the 2007-08 crisis. The main mechanism that we emphasize in this paper could in particular have first-order effects on the monetary transmission channel, when embedded in particular in the setting developed by Kydland, Rupert and Sustek (2015), Garriga, Kydland and Sustek (2013). In relation to this, it is obvious that real interest rate movements arise also in the context of nominal debt contracts when inflation is not perfectly stabilized and this is a second reason why embedding the mechanism of this paper in a framework with monetary policy, whether conventional or not, is worth pursuing. Our results also complement the recent analysis of Justiniano, Primiceri, and Tambalotti (2015b), who analyze the 2000s US trend in housing and credit markets in a very similar model and show that falling interest rates must be part of the story. We have further shown that countercyclical borrowing cost is an important driver of fluctuations at business-cycle frequency in output, investment and other aggregate variables. In our model, however, countercyclical interest rate results from self-fulfilling swings in investment that move both credit supply and credit demand endogenously. In addition, because collateralized lending with variable rates is standard practice in interbank credit markets, our results point at a potentially empirically relevant force that could explain sudden freezes in those markets that have been under the spotlight after the last financial crisis (see e.g. Gorton and Metrick, 2012). In particular, sunspot shocks could well be an important driver of banking crisis that reinforce fundamental shocks (see Boissay, Collard and Smets, 2015, for an analysis of the latter). Of course, some unrealistic aspects of the settings that we have used and estimated in this paper need to be fixed. At the top of the list, there is need for further work to incorporate debt maturity into standard macroeconomic models. This avenue is presumably one way to make the interest rate in the model as persistent as it is in the data. In addition, our models feature no policy instruments that could potentially either prevent ex-ante, or fight against the consequences of sunspot-driven market gyrations. We believe this calls for further research.

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6 Appendix

6.1 Global Sunspot with Fixed and Variable-Rate Loans

This section shows that global sunspot equilibria exist in the simple model of Section 3.3 provided that the proportion of variable-rate loans in the economy is larger than 0.5. Suppose that a constant fraction $\omega \in [0,1]$ of total loans has a variable interest rate while a fraction $1-\omega$ of total loans has a fixed rate. This means that the interest rate paid in period t is now $\mathcal{R}_t \equiv \omega R_t + (1-\omega)R_{t-1}$ and it follows that the first order condition (10) now reads $\mathbb{E}_t \mathcal{R}_{t+1} = \tilde{\beta}^{-1}$. Two cases occur depending on the value for ω. When $\omega < 0.5$, then the latter equality $\omega \mathbb{E}_t R_{t+1} + (1-\omega)R_t = \tilde{\beta}^{-1}$ can be solved forward for $R_t = \tilde{\beta}^{-1}$ so that the interest rate is constant and the economy stays in steady state for all t, exactly as in the case with $\omega = 0$. In other words, the steady state solution for the interest rate is determinate. When $\omega > 0.5$, however, this is no longer true and the steady state interest rate is indeterminate: $\omega \mathbb{E}_t R_{t+1} + (1-\omega)R_t = \tilde{\beta}^{-1}$ cannot be solved forward and there exist sunspot equilibria such that $R_{t+1} = (\tilde{\beta}\omega)^{-1} - (1-\omega)\omega^{-1}R_t + \varepsilon_{t+1}$, where the sunspot innovation ε is i.i.d. with zero mean.

In addition, the intuition developed in Section 3.3 still applies to the case with $\omega > 0.5$. While the expression for credit demand $B^d_{t+1} = \tilde{\beta} Q L_{t+1}$ does not change, credit supply is now $B^s_{t+1} = QL_{t+1} - \beta L_t[1 + \omega Q(1-\tilde{\beta}R_t)]$, which of course collapses to (22) in Section 3.3 when $\omega = 1$. The situation depicted in Figure 8 therefore applies just the same if $\omega > 0.5$: if the borrower expects a lower interest rate in period t, she invests more so that L_{t+1} goes up and the expectation of a falling interest rate is self-fulfilling because credit supply shifts rightward by more than credit demand.

6.2 Linearized Version of Basic Model

The purpose of this appendix is to report the linearized version of the equations describing the competitive equilibrium with borrowing constraints in Section 3. In all equations below, x_t denotes the deviation of variable X_t from its steady-state X level in percentage terms. For example, $k_t \equiv (K_t - K)/K$, where K is the steady-state capital stock. The linearized versions of (1), (3)-(5) and (7)-(14) are respectively:

$$\frac{\tilde{C}}{Y}\tilde{c}_t + \frac{Q\tilde{L}}{Y}(\tilde{l}_{t+1} - \tilde{l}_t) + \frac{B}{Y}b_{t+1} = \frac{RB}{Y}(r_t + b_t) \tag{56}$$

$$y_t = a_t + \alpha k_t + \gamma l_t \tag{57}$$

$$\tilde{l}_t = -\frac{L}{\tilde{L}}l_t \tag{58}$$

$$\frac{C}{Y}c_t + \frac{K}{Y}(k_{t+1} - (1-\delta)k_t) + \frac{QL}{Y}(l_{t+1} - l_t) + \frac{RB}{Y}(r_t + b_t) = \frac{B}{Y}b_{t+1} + y_t \tag{59}$$

$$\mathbb{E}_t(r_{t+1}) + b_{t+1} = \mathbb{E}_t(q_{t+1}) + l_{t+1} \tag{60}$$

$$\sigma_L \tilde{c}_t = -\tilde{\lambda}_t \tag{61}$$

$$q_t + \tilde{\lambda}_t = \tilde{\beta}\mathbb{E}_t(q_{t+1} + \tilde{\lambda}_{t+1}) - (1-\tilde{\beta})\sigma_W \tilde{l}_{t+1} \tag{62}$$

$$\tilde{\lambda}_t = \mathbb{E}_t(\tilde{\lambda}_{t+1} + r_{t+1}) \tag{63}$$

$$\sigma_B c_t = -\lambda_t \tag{64}$$

$$q_t + \lambda_t = \beta\mathbb{E}_t(q_{t+1} + \lambda_{t+1}) + \frac{\beta\gamma Y}{QL}(\mathbb{E}_t(\lambda_{t+1} + y_{t+1}) - l_{t+1}) + \frac{\theta\Phi}{\Lambda}(\mathbb{E}_t(q_{t+1}) + \phi_t) \tag{65}$$

$$\lambda_t = \beta(1-\delta)\mathbb{E}_t(\lambda_{t+1}) + \frac{\alpha\beta Y}{K}(\mathbb{E}_t(\lambda_{t+1} + y_{t+1}) - k_{t+1}) \tag{66}$$

$$\lambda_t = \beta R\mathbb{E}_t(\lambda_{t+1} + r_{t+1}) + \frac{R\Phi}{\Lambda}(\mathbb{E}_t(r_{t+1}) + \phi_t) \tag{67}$$

6.3 Linearized Version of Extended Model

The purpose of this appendix is to report the stationary and linearized versions of the equations describing the competitive equilibrium with borrowing constraints in Section 4.

6.3.1 Fixed-Rate Model

This is the model described in Section 4.1.

Stationary equilibrium:

Since there is technological progress, a steady state is defined in terms of detrended variables. Define $\tilde{X}_{1t} = \frac{X_{1t}}{\Gamma_t}$ where $\Gamma_t = (Z_t Q_t^{(1-\phi)\alpha})^{\frac{1}{1-(1-\phi)\alpha}}$, $X_1 \in \{Y, C_h, C_e, B, w, q_l\}$, define $\tilde{X}_{2t} = X_{2t}\Gamma_t$ where $X_2 \in \{\mu_e, \mu_b\}$, define $\tilde{X}_{3t} = \frac{X_{3t}}{Q_t \Gamma_t}$ where $X_3 \in \{I, K\}$ and define $\tilde{\mu}_{ht} = \frac{\mu_{ht}\Gamma_t}{A_t}$, $\tilde{q}_{kt} = q_{kt}Q_t$. The first-order and market clearing conditions in detrended variables are then:

$$\tilde{\mu}_{ht} = \frac{1}{\tilde{C}_{ht} - \gamma_h \tilde{C}_{ht-1}\Gamma_{t-1}/\Gamma_t} - \mathbb{E}_t\left[\frac{\beta\gamma_h}{\tilde{C}_{ht+1}\Gamma_{t+1}/\Gamma_t - \gamma_h \tilde{C}_{ht}}(1+\lambda_{at+1})\right] \tag{68}$$

$$\tilde{w}_t = \frac{\psi_t}{\tilde{\mu}_{ht}} \tag{69}$$

$$\tilde{q}_{lt} = \beta\mathbb{E}_t\left[\frac{\tilde{\mu}_{ht+1}}{\tilde{\mu}_{ht}}(1+\lambda_{at+1})\tilde{q}_{lt+1}\right] + \frac{\varphi_t}{\tilde{\mu}_{ht}L_{ht}} \tag{70}$$

$$1 = \beta\mathbb{E}_t\left[\frac{\tilde{\mu}_{ht+1}}{\tilde{\mu}_{ht}}\frac{\Gamma_t}{\Gamma_{t+1}}(1+\lambda_{at+1})\right]R_t \tag{71}$$

$$\tilde{\mu}_{et} = \frac{1}{\tilde{C}_{et} - \gamma_e \tilde{C}_{et-1}\Gamma_{t-1}/\Gamma_t} - \mathbb{E}_t\left[\frac{\beta\gamma_e}{\tilde{C}_{et+1}\Gamma_{t+1}/\Gamma_t - \gamma_e \tilde{C}_{et}}\right] \tag{72}$$

$$\tilde{w}_t = (1-\alpha)\tilde{Y}_t/N_{et} \tag{73}$$

$$\begin{aligned} 1 =& \tilde{q}_{kt}\left(1 - \frac{\Omega}{2}\left(\frac{\tilde{I}_t}{\tilde{I}_{t-1}}\frac{Q_t\Gamma_t}{Q_{t-1}\Gamma_{t-1}} - \bar{\lambda}_I\right)^2 - \Omega\left(\frac{\tilde{I}_t}{\tilde{I}_{t-1}}\frac{Q_t\Gamma_t}{Q_{t-1}\Gamma_{t-1}} - \bar{\lambda}_I\right)\frac{\tilde{I}_t}{\tilde{I}_{t-1}}\frac{Q_t\Gamma_t}{Q_{t-1}\Gamma_{t-1}}\right) \\ &+ \beta\Omega\mathbb{E}_t\left[\frac{\tilde{\mu}_{et+1}}{\tilde{\mu}_{et}}\frac{Q_t\Gamma_t}{Q_{t+1}\Gamma_{t+1}}\tilde{q}_{kt+1}\left(\frac{\tilde{I}_{t+1}}{\tilde{I}_t}\frac{Q_{t+1}\Gamma_{t+1}}{Q_t\Gamma_t} - \bar{\lambda}_I\right)\left(\frac{\tilde{I}_{t+1}}{\tilde{I}_t}\frac{Q_{t+1}\Gamma_{t+1}}{Q_t\Gamma_t}\right)^2\right] \end{aligned} \tag{74}$$

$$\tilde{q}_{kt} = \mathbb{E}_t\left[\beta\frac{\tilde{\mu}_{et+1}}{\tilde{\mu}_{et}}\left(\alpha(1-\phi)\frac{\tilde{Y}_{t+1}}{\tilde{K}_t} + \tilde{q}_{kt+1}\frac{Q_t\Gamma_t}{Q_{t+1}\Gamma_{t+1}}(1-\delta)\right) + \frac{\tilde{\mu}_{bt}}{\tilde{\mu}_{et}}\theta_t\tilde{q}_{kt+1}\frac{Q_t}{Q_{t+1}}\right] \tag{75}$$

$$\tilde{q}_{lt} = \mathbb{E}_t\left[\beta\frac{\tilde{\mu}_{et+1}}{\tilde{\mu}_{et}}\left(\alpha\phi\frac{\tilde{Y}_{t+1}}{\tilde{L}_{et}} + \tilde{q}_{lt+1}\right) + \frac{\tilde{\mu}_{bt}}{\tilde{\mu}_{et}}\theta_t\tilde{q}_{lt+1}\frac{\Gamma_{t+1}}{\Gamma_t}\right] \tag{76}$$

$$1 = \mathbb{E}_t\left[\beta\frac{\tilde{\mu}_{et+1}}{\tilde{\mu}_{et}}\frac{\Gamma_t}{\Gamma_{t+1}} + \frac{\tilde{\mu}_{bt}}{\tilde{\mu}_{et}}\right]R_t \tag{77}$$

$$\tilde{Y}_t = \left(\frac{Q_t Z_t}{Q_{t-1}Z_{t-1}}\right)^{\frac{-(1-\phi)\alpha}{1-(1-\phi)\alpha}} L_{et-1}^{\phi\alpha}\tilde{K}_{t-1}^{(1-\phi)\alpha}N_{et}^{1-\alpha} \tag{78}$$

$$\tilde{K}_t = (1-\delta)\tilde{K}_{t-1}\frac{Q_{t-1}\Gamma_{t-1}}{Q_t\Gamma_t} + \left(1 - \frac{\Omega}{2}\left(\frac{\tilde{I}_t}{\tilde{I}_{t-1}}\frac{Q_t\Gamma_t}{Q_{t-1}\Gamma_{t-1}} - \bar{\lambda}_I\right)^2\right)\tilde{I}_t \tag{79}$$

$$\tilde{Y}_t = \tilde{C}_{ht} + \tilde{C}_{et} + \tilde{I}_t \tag{80}$$

$$\bar{L} = L_{ht} + L_{et} \tag{81}$$

$$\alpha\tilde{Y}_t = \tilde{C}_{et} + \tilde{I}_t + \tilde{q}_{lt}(L_{et} - L_{et-1}) + \tilde{B}_{t-1}\frac{\Gamma_{t-1}}{\Gamma_t} - \frac{\tilde{B}_t}{R_t} \tag{82}$$

$$\tilde{B}_t = \theta_t \mathbb{E}_t\left[\tilde{q}_{lt+1}\frac{\Gamma_{t+1}}{\Gamma_t}L_{et} + \tilde{q}_{kt+1}\tilde{K}_t\frac{Q_t}{Q_{t+1}}\right] \tag{83}$$

For simplicity we can define

$$g_{zt} \equiv \frac{Z_t}{Z_{t-1}} = \frac{Z_t^p v_{zt}}{Z_{t-1}^p v_{zt-1}} = \lambda_{zt}\frac{v_{zt}}{v_{zt-1}} \tag{84}$$

$$g_{qt} \equiv \frac{Q_t}{Q_{t-1}} = \frac{Q_t^p v_{qt}}{Q_{t-1}^p v_{qt-1}} = \lambda_q\frac{v_{qt}}{v_{qt-1}} \tag{85}$$

$$g_{\gamma t} \equiv \frac{\Gamma_t}{\Gamma_{t-1}} = \left(g_{zt}g_{qt}^{(1-\phi)\alpha}\right)^{\frac{1}{1-(1-\phi)\alpha}} \tag{86}$$

Calibration Strategy:

We follow the calibration strategy used by Liu et al. (2013). First we have

$$\frac{1}{R} = \frac{\beta(1+\lambda_a)}{g_\gamma} \Leftrightarrow \lambda_a = \frac{g_\gamma}{\beta R} - 1 \tag{87}$$

$$\frac{\tilde{\mu}_b}{\tilde{\mu}_e} = \frac{\beta\lambda_a}{g_\gamma} \tag{88}$$

then we derive

$$\frac{\tilde{q}_l L_e}{\tilde{Y}} = \frac{\beta\alpha\phi}{1-\beta-\beta\lambda_a\theta} \Leftrightarrow \phi = \frac{1-\beta-\theta\beta\lambda_a}{\beta\alpha}\frac{\tilde{q}_l L_e}{\tilde{Y}} \tag{89}$$

On the other hand, define

$$\lambda_k = g_\gamma\lambda_q \tag{90}$$

it follows that the investment-capital ratio is

$$\frac{\tilde{I}}{\tilde{K}} = 1 - \frac{1-\delta}{\lambda_k} \Leftrightarrow \delta = 1 - \lambda_k\left(1 - \frac{\tilde{I}}{\tilde{K}}\right) \tag{91}$$

and the capital-output ratio is

$$\frac{\tilde{K}}{\tilde{Y}} = \frac{\beta\alpha(1-\phi)}{1-\frac{\beta}{\lambda_k}(\lambda_a\theta+1-\delta)} = \frac{\beta\alpha(1-\frac{1-\beta-\theta\beta\lambda_a}{\beta\alpha}\frac{\tilde{q}_l L_e}{\tilde{Y}})}{1-\frac{\beta}{\lambda_k}(\lambda_a\theta+1-\delta)} = \frac{\beta\left(\alpha+(1+\theta\lambda_a)\frac{\tilde{q}_l L_e}{\tilde{Y}}\right) - \frac{\tilde{q}_l L_e}{\tilde{Y}}}{1-\frac{\beta}{\lambda_k}(\lambda_a\theta+1-\delta)} \tag{92}$$

which gives the discount factor

$$\beta = \frac{\frac{\tilde{K}}{\tilde{Y}} + \frac{\tilde{q}_l L_e}{\tilde{Y}}}{\alpha + \frac{\tilde{q}_l L_e}{\tilde{Y}}(1+\theta\lambda_a) + \frac{\tilde{K}}{\tilde{Y}}\frac{1}{\lambda_k}(\lambda_a\theta+1-\delta)} \tag{93}$$

and the investment-output ratio

$$\frac{\tilde{I}}{\tilde{Y}} = \frac{\tilde{I}}{\tilde{K}}\frac{\tilde{K}}{\tilde{Y}} = \frac{\beta\alpha(1-\phi)(\lambda_k-(1-\delta))}{\lambda_k-\beta(\lambda_a\theta+1-\delta)} \tag{94}$$

Besides, the credit-to-output ratio is

$$\frac{\tilde{B}}{\tilde{Y}} = \theta\left(g_\gamma\frac{\tilde{q}_l L_e}{\tilde{Y}} + \frac{1}{\lambda_q}\frac{\tilde{K}}{\tilde{Y}}\right) \tag{95}$$

which gives the entrepreneur's consumption as a fraction of output

$$\frac{\tilde{C}_e}{\tilde{Y}} = \alpha - \frac{\tilde{I}}{\tilde{Y}} - \frac{1-\beta(1+\lambda_a)}{g_\gamma}\frac{\tilde{B}}{\tilde{Y}} \tag{96}$$

and the household's consumption-to-output ratio as well

$$\frac{\tilde{C}_h}{\tilde{Y}} = 1 - \frac{\tilde{C}_e}{\tilde{Y}} - \frac{\tilde{I}}{\tilde{Y}} \tag{97}$$

In addition

$$\frac{\tilde{q}_l L_h}{\tilde{C}_h} = \frac{\varphi(g_\gamma-\gamma_h)}{g_\gamma(1-g_\gamma/R)(1-\gamma_h/R)} \Leftrightarrow \varphi = \frac{\frac{\tilde{q}_l L_h}{\tilde{Y}}}{\frac{\tilde{C}_h}{\tilde{Y}}}\frac{g_\gamma(1-g_\gamma/R)(1-\gamma_h/R)}{(g_\gamma-\gamma_h)} \tag{98}$$

$$\frac{L_h}{L_e} = \frac{\varphi(g_\gamma-\gamma_h)(1-\beta-\beta\lambda_a\theta)}{\beta\alpha\phi g_\gamma(1-g_\gamma/R)(1-\gamma_h/R)}\frac{\tilde{C}_h}{\tilde{Y}} \tag{99}$$

and the steady-state quantity of labor is

$$N = \frac{(1-\alpha)g_\gamma(1-\gamma_h/R)}{\psi(g_\gamma-\gamma_h)}\frac{\tilde{Y}}{\tilde{C}_h} \Leftrightarrow \psi = \frac{(1-\alpha)g_\gamma(1-\gamma_h/R)}{N(g_\gamma-\gamma_h)}\frac{\tilde{Y}}{\tilde{C}_h} \tag{100}$$

Linearization:

Defining the following constant

$$\Omega_h = (g_\gamma - \beta(1+\lambda_a)\gamma_h)(g_\gamma-\gamma_h) \tag{101}$$

$$\Omega_e = (g_\gamma - \beta\gamma_e)(g_\gamma-\gamma_h) \tag{102}$$

then we dynamic linear system follows

$$\hat{\mu}_{ht}\Omega_h = -(g_\gamma^2+\beta\gamma_h^2(1+\lambda_a))\hat{C}_{ht} + g_\gamma\gamma_h(\hat{C}_{ht-1}-\hat{g}_{\gamma t}) - \beta\lambda_a\gamma_h(g_\gamma-\gamma_h)\hat{\lambda}_{at+1} + \beta(1+\lambda_a)g_\gamma\gamma_h(\hat{C}_{ht+1}+\hat{g}_{\gamma t+1}) \tag{103}$$

$$\hat{w}_t + \hat{\mu}_{ht} = \hat{\psi}_t \tag{104}$$

$$\hat{q}_{lt} + \hat{\mu}_{ht} = \beta(1+\lambda_a)\mathbb{E}_t[\hat{\mu}_{ht+1}+\hat{q}_{lt+1}] + (1-\beta(1+\lambda_a))(\hat{\varphi}_t - \hat{L}_{ht}) + \beta\lambda_a\mathbb{E}_t[\hat{\lambda}_{at+1}] \tag{105}$$

$$\hat{\mu}_{ht} - \hat{R}_t = \mathbb{E}_t\left[\hat{\mu}_{ht+1} + \frac{\lambda_a}{1+\lambda_a}\hat{\lambda}_{at+1} - \hat{g}_{\gamma t+1}\right] \tag{106}$$

$$\Omega_e\hat{\mu}_{et} = -(g_\gamma^2+\beta\gamma_e^2)\hat{C}_{et} + g_\gamma\gamma_e(\hat{C}_{et-1}-\hat{g}_{\gamma t}) + \beta g_\gamma\gamma_e\mathbb{E}_t[\hat{C}_{et+1}+\hat{g}_{\gamma t+1}] \tag{107}$$

$$\hat{w}_t = \hat{Y}_t - \hat{N}_t \tag{108}$$

$$\hat{q}_{kt} = (1+\beta)\Omega\lambda_k^2\hat{I}_t - \Omega\lambda_k^2\hat{I}_{t-1} + \Omega\lambda_k^2(\hat{g}_{\gamma t}+\hat{g}_{qt}) - \beta\Omega\lambda_k^2\mathbb{E}_t[\hat{I}_{t+1}+\hat{g}_{\gamma t+1}+\hat{g}_{qt+1}] \tag{109}$$

$$\begin{aligned}\hat{q}_{kt}+\hat{\mu}_{et} =& \frac{\tilde{\mu}_b}{\tilde{\mu}_e}\frac{\theta}{\lambda_q}(\hat{\mu}_{bt}+\hat{\theta}_t) + \frac{\beta(1-\delta)}{\lambda_k}\mathbb{E}_t[\hat{q}_{kt+1}-\hat{g}_{qt+1}-\hat{g}_{\gamma t+1}] \\ &+ \left(1-\frac{\tilde{\mu}_b}{\tilde{\mu}_e}\frac{\theta}{\lambda_q}\right)\mathbb{E}_t[\hat{\mu}_{et+1}] + \frac{\tilde{\mu}_b}{\tilde{\mu}_e}\frac{\theta}{\lambda_q}\mathbb{E}_t[\hat{q}_{kt+1}-\hat{g}_{qt+1}] + \beta\alpha(1-\phi)\frac{\tilde{Y}}{\tilde{K}}\mathbb{E}_t[\hat{Y}_{t+1}-\hat{K}_t]\end{aligned} \tag{110}$$

$$\begin{aligned}\hat{q}_{lt}+\hat{\mu}_{et} =& \frac{\tilde{\mu}_b}{\tilde{\mu}_e}g_\gamma\theta(\hat{\theta}_t+\hat{\mu}_{bt}) + \left(1-\frac{\tilde{\mu}_b}{\tilde{\mu}_e}g_\gamma\theta\right)\mathbb{E}_t[\hat{\mu}_{et+1}] + \frac{\tilde{\mu}_b}{\tilde{\mu}_e}g_\gamma\theta\mathbb{E}_t[\hat{q}_{lt+1}+\hat{g}_{\gamma t+1}] \\ &+ \beta\mathbb{E}_t[\hat{q}_{lt+1}] + (1-\beta-\beta\lambda_a\theta)\mathbb{E}_t[\hat{Y}_{t+1}-\hat{L}_e]\end{aligned} \tag{111}$$

$$\hat{\mu}_{et} - \hat{R}_t = \frac{1}{1+\lambda_a}(\mathbb{E}_t[\hat{\mu}_{et+1}-\hat{g}_{\gamma t+1}] + \lambda_a\hat{\mu}_{bt}) \tag{112}$$

$$\hat{Y}_t = \alpha\phi\hat{L}_{et-1} + \alpha(1-\phi)\hat{K}_{t-1} + (1-\alpha)\hat{N}_t - \frac{(1-\phi)\alpha}{1-(1-\phi)\alpha}(\hat{g}_{zt}+\hat{g}_{qt}) \tag{113}$$

$$\hat{K}_t = \frac{1-\delta}{\lambda_k}(\hat{K}_{t-1}-\hat{g}_{\gamma t}-\hat{g}_{qt}) + \left(1-\frac{1-\delta}{\lambda_k}\right)\hat{I}_t \tag{114}$$

$$\hat{Y}_t = \frac{\tilde{C}_h}{\tilde{Y}}\hat{C}_{ht} + \frac{C_e}{\tilde{Y}}\hat{C}_{et} + \frac{\tilde{I}}{\tilde{Y}}\hat{I}_t \tag{115}$$

$$0 = \frac{L_h}{\bar{L}}\hat{L}_{ht} + \frac{L_e}{\bar{L}}\hat{L}_{et} \tag{116}$$

$$\alpha\hat{Y}_t = \frac{\tilde{C}_e}{\tilde{Y}}\hat{C}_{et} + \frac{\tilde{I}}{\tilde{Y}}\hat{I}_t + \frac{\tilde{q}_l L_e}{\tilde{Y}}(\hat{L}_{et}-\hat{L}_{et-1}) + \frac{1}{g_\gamma}\frac{\tilde{B}}{\tilde{Y}}(\hat{B}_{t-1}-\hat{g}_{\gamma t}) - \frac{1}{R}\frac{\tilde{B}}{\tilde{Y}}(\hat{B}_t-\hat{R}_t) \tag{117}$$

$$\hat{B}_t = \hat{\theta}_t + g_\gamma\theta\frac{\tilde{q}_l L_e}{\tilde{B}}\mathbb{E}_t[\hat{q}_{lt+1}+\hat{L}_{et}+\hat{g}_{\gamma t+1}] + \left(1-g_\gamma\theta\frac{\tilde{q}_l L_e}{\tilde{B}}\right)\mathbb{E}_t[\hat{q}_{kt+1}+\hat{K}_t+\hat{g}_{qt+1}] \tag{118}$$

$$\hat{g}_{zt} = \hat{\lambda}_{zt} + \hat{\nu}_{zt} - \hat{\upsilon}_{zt-1} \tag{119}$$

$$\hat{g}_{qt} = \hat{\lambda}_{qt} + \hat{\nu}_{qt} - \hat{\upsilon}_{zqt-1} \tag{120}$$

$$\hat{g}_{\gamma t} = \frac{1}{1-(1-\phi)\alpha}\hat{g}_{zt} + \frac{(1-\phi)\alpha}{1-(1-\phi)\alpha}\hat{g}_{qt} \tag{121}$$

$$\hat{\lambda}_{zt} = \rho_z\hat{\lambda}_{zt-1} + \hat{\varepsilon}_{zt} \tag{122}$$

$$\hat{\nu}_{zt} = \rho_{\nu_z}\hat{\nu}_{zt-1} + \hat{\varepsilon}_{\nu_z t} \tag{123}$$

$$\hat{\lambda}_{qt} = \rho_q\hat{\lambda}_{qt-1} + \hat{\varepsilon}_{qt} \tag{124}$$

$$\hat{\nu}_{qt} = \rho_{\nu_q}\hat{\nu}_{qt-1} + \hat{\varepsilon}_{\nu_q t} \tag{125}$$

$$\hat{\lambda}_{at} = \rho_a\hat{\lambda}_{at-1} + \hat{\varepsilon}_{at} \tag{126}$$

$$\hat{\varphi}_t = \rho_\varphi \hat{\varphi}_{t-1} + \hat{\varepsilon}_{\varphi t} \tag{127}$$

$$\hat{\psi}_t = \rho_\psi \hat{\psi}_{t-1} + \hat{\varepsilon}_{\psi t} \tag{128}$$

$$\hat{\theta}_t = \rho_\theta \hat{\theta}_{t-1} + \hat{\varepsilon}_{\theta t} \tag{129}$$

Following Sims (2001), the above linear system can be written in the following state-space form:

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Gamma_2 \varepsilon_t + \Gamma_3 \eta_t \tag{130}$$

where X_t is a 39 dimensional vector containing all the endogenous variables and the forward looking variables, ε_t is a 8 dimensional vector containing the 8 exogenous shocks, and η_t is a 11 dimensional vector containing 11 endogenous expectation errors. In specific, we have

$$X_t = (X_{1t}', \mathbb{E}_t[X_{2t+1}]', X_{3t}')' \tag{131}$$

where

$$X_{1t} = (\hat{\mu}_{ht}, \hat{w}_t, \hat{q}_{lt}, \hat{R}_t, \hat{\mu}_{et}, \hat{\mu}_{bt}, \hat{N}_t, \hat{I}_t, \hat{Y}_t, \hat{C}_{ht}, \hat{C}_{et}, \hat{q}_{kt}, \hat{L}_{ht}, \hat{L}_{et}, \hat{K}_t, \hat{B}_t, \hat{g}_{\gamma t}, \hat{g}_{zt}, \hat{g}_{qt}, \hat{C}_t)_{20\times 1}' \tag{132}$$

$$X_{2t+1} = (\hat{\mu}_{ht+1}, \hat{q}_{lt+1}, \hat{\mu}_{et+1}, \hat{I}_{t+1}, \hat{Y}_{t+1}, \hat{C}_{ht+1}, \hat{C}_{et+1}, \hat{q}_{kt+1}, \hat{g}_{\gamma t+1}, \hat{g}_{qt+1}, \hat{\lambda}_{at+1})_{11\times 1}' \tag{133}$$

$$X_{3t} = (\hat{\theta}_t, \hat{\psi}_t, \hat{\varphi}_t, \hat{\nu}_{qt}, \hat{\nu}_{zt}, \hat{\lambda}_{zt}, \hat{\lambda}_{at}, \hat{\lambda}_{qt})_{8\times 1}' \tag{134}$$

$$\varepsilon_t = (\hat{\varepsilon}_{zt}, \hat{\varepsilon}_{\nu_z t}, \hat{\varepsilon}_{qt}, \hat{\varepsilon}_{\nu_q t}, \hat{\varepsilon}_{at}, \hat{\varepsilon}_{\varphi t}, \hat{\varepsilon}_{\psi t}, \hat{\varepsilon}_{\theta t})_{8\times 1}' \tag{135}$$

$$\eta_t = X_{2t} - \mathbb{E}_{t-1}[X_{2t}] \tag{136}$$

6.3.2 Variable-Rate Model

This is the model described in Section 4.2. Equations are identical to those in Appendix 6.3.1 except for the following changes:

(106)→

$$\hat{\mu}_{ht} - \mathbb{E}_t[\hat{R}_{t+1}] = \mathbb{E}_t\left[\hat{\mu}_{ht+1} + \frac{\lambda_a}{1+\lambda_a}\hat{\lambda}_{at+1} - \hat{g}_{\gamma t+1}\right] \tag{137}$$

(112)→

$$\hat{\mu}_{et} - \mathbb{E}_t[\hat{R}_{t+1}] = \frac{1}{1+\lambda_a}(\mathbb{E}_t[\hat{\mu}_{et+1} - \hat{g}_{\gamma t+1}] + \lambda_a \hat{\mu}_{bt}) \tag{138}$$

(117)→

$$\alpha \hat{Y}_t = \frac{\tilde{C}_e}{\tilde{Y}}\hat{C}_{et} + \frac{\tilde{I}}{\tilde{Y}}\hat{I}_t + \frac{\tilde{q}_l L_e}{\tilde{Y}}(\hat{L}_{et} - \hat{L}_{et-1}) + \frac{1}{g_\gamma}\frac{\tilde{B}}{\tilde{Y}}(\hat{R}_t + \hat{B}_{t-1}^l - \hat{g}_{\gamma t}) - \frac{1}{R}\frac{\tilde{B}}{\tilde{Y}}\hat{B}_t^l \tag{139}$$

(118)→

$$\hat{R}_{t+1} + \hat{B}_t^l = \hat{\theta}_t + g_\gamma \theta \frac{\tilde{q}_l L_e}{\tilde{B}}\mathbb{E}_t[\hat{q}_{lt+1} + \hat{L}_{et} + \hat{g}_{\gamma t+1}] + \left(1 - g_\gamma \theta \frac{\tilde{q}_l L_e}{\tilde{B}}\right)\mathbb{E}_t[\hat{q}_{kt+1} + \hat{K}_t + \hat{g}_{qt+1}] \tag{140}$$

In state-space form the linearized system is now:

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Gamma_2 \varepsilon_t + \Gamma_3 \eta_t \tag{141}$$

where X_t is a 39 dimensional vector containing all the endogenous variables and the forward looking variables, ε_t is a 9 dimensional vector containing the 9 innovations (including the sunspot), and η_t is a 12 dimensional vector containing 12 endogenous expectation errors. In specific, we have

$$X_t = (X_{1t}', \mathbb{E}_t[X_{2t+1}]', X_{3t}')' \tag{142}$$

where

$$X_{1t} = (\hat{\mu}_{ht}, \hat{w}_t, \hat{q}_{lt}, \hat{\mu}_{et}, \hat{\mu}_{bt}, \hat{N}_t, \hat{I}_t, \hat{Y}_t, \hat{C}_{ht}, \hat{C}_{et}, \hat{q}_{kt}, \hat{L}_{ht}, \hat{L}_{et}, \hat{K}_t, \hat{B}_t, \hat{g}_{\gamma t}, \hat{g}_{zt}, \hat{g}_{qt}, \hat{C}_t)_{19\times 1}' \tag{143}$$

$$X_{2t+1} = (\hat{\mu}_{ht+1}, \hat{q}_{lt+1}, \hat{R}_{t+1}, \hat{\mu}_{et+1}, \hat{I}_{t+1}, \hat{Y}_{t+1}, \hat{C}_{ht+1}, \hat{C}_{et+1}, \hat{q}_{kt+1}, \hat{g}_{\gamma t+1}, \hat{g}_{qt+1}, \hat{\lambda}_{at+1})_{12\times 1}' \tag{144}$$

$$X_{3t} = (\hat{\theta}_t, \hat{\psi}_t, \hat{\varphi}_t, \hat{\nu}_{qt}, \hat{\nu}_{zt}, \hat{\lambda}_{zt}, \hat{\lambda}_{at}, \hat{\lambda}_{qt})_{8\times 1}' \tag{145}$$

$$\varepsilon_t = (\hat{\varepsilon}_{zt}, \hat{\varepsilon}_{\nu_z t}, \hat{\varepsilon}_{qt}, \hat{\varepsilon}_{\nu_q t}, \hat{\varepsilon}_{at}, \hat{\varepsilon}_{\varphi t}, \hat{\varepsilon}_{\psi t}, \hat{\varepsilon}_{\theta t}, \hat{\varepsilon}_{st})_{9\times 1}' \tag{146}$$

$$\eta_t = X_{2t} - \mathbb{E}_{t-1}[X_{2t}] \tag{147}$$