Two Monetary Models with Alternating Markets

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Two Monetary Models with Alternating Markets

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Abstract

We present a thought-provoking study of two monetary models: the cash-in-advance and the Lagos and Wright (2005) models. We report that the different approach to modeling money—reduced-form vs. explicit role—neither induces fundamental theoretical nor quantitative differences in results. Given conformity of preferences, technologies and shocks, both models reduce to one difference equation. The equations do not coincide only if price distortions are differentially imposed across models. To illustrate, when cash prices are equally distorted in both models equally large welfare costs of inflation are obtained in each model. Our insight is that if results differ, then this is due to unequal assumptions about the pricing mechanism that governs cash transactions, not the explicit details about the role of money.

Keywords: cash-in-advance, matching, microfoundations, inflation.
JEL codes: E1, E4, E5

1 Introduction

The question “what’s the best approach to modeling money?” is one of those that economists have struggled with for a while and is yet unsettled. Three decades ago, some viewed the overlapping generations framework as the only satisfactory approach
to modeling money [5], while others saw merits from placing real balances in the utility function and noted that such a device could be used to unify several results in the literature [4,13]. These days, there is a lively debate about the framework proposed in [7], in relation to reduced-form models of money.

Advocates of the LW model underscore its appeal as a tool for theoretical analysis because unlike reduced-form models the role of money is made explicit [15, p.267]. This modeling approach contrasts with reduced-form models, such as those imposing cash-in-advance constraints [8–12]. Yet, one may also note key similarities with the cash-in-advance framework. In both models agents synchronously alternate between a centralized market (CM) and a decentralized market (DM); consumption utility depends on where the purchase is settled, in the DM or CM; and asset trading decisions (adjustments of money balances, in particular) are made before a random shock is observed; [10, p.10-11] and [7, pp.462-66]. It has also been argued that the explicit microfoundation of money can make a significant difference for quantitative results; in particular, it can generate higher welfare costs of inflation than reduced-form models [7, p.463-4].

These considerations have raised several questions among monetary economists. Are there differences in the main equilibrium equations associated with these two theoretical platforms? If so, what model features are responsible for such disparities? Finally, are these two frameworks generally incapable of producing similar quantitative results? We offer some answers by discussing what we found when we juxtaposed the models’ main equations and quantitative implications for the welfare cost of inflation. We proceed as follows. Section 2 lays out the cash-in-advance framework following [10], which has an explicit and transparent description of the physical environment. Section 3 reports the main mathematical relationships describing equilibrium allocations in the LW model and identifies the price distortion due to nonlinear pricing. Unlike the cash-in-advance model, in the LW model Nash bargaining determines prices in some transactions (which must be settled with the exchange of
money) but not others; hence, a price distortion may exist, depending on the seller’s bargaining power. Subsequently, the two frameworks are placed on equal footing in terms of preferences, technologies, and shocks. A way to introduce price distortions in the cash-in-advance model—without altering its fundamental structure—is illustrated, which involves a tax on cash revenues. At this point, the equations describing equilibrium allocations in the cash-in-advance model are derived.

Our analysis mainly focuses on stationary equilibrium because the literature based on the LW model has almost entirely focused on such equilibrium. We find that the equations characterizing stationary equilibrium in the LW model when sellers have no bargaining power coincide with the equations that characterize stationary competitive equilibrium in the cash-in-advance model. This also holds when sellers do have some bargaining power, when the price distortion from Nash bargaining is replicated in the other model. This is accomplished using a tax on cash revenues (equivalently, a sales tax on cash purchases) but other distortionary mechanisms could be explored. Such correspondence between equations immediately extends outside of steady-state, if sellers have no bargaining power and workers have isoelastic preferences; otherwise, a one-to-one mapping between the equations cannot be established outside of steady-state. Hence, there may exist dynamical equilibria which are not the same in the two models. Before concluding with Section 4 we propose a quantitative exercise, showing that the welfare costs of inflation in the cash-in-advance model match those in the LW model.

The main insight is thus that the two models (CIA, LW) reduce to a single difference equation. The equations correspond if the price distortion in one model is matched in the other model, in which case one cannot distinguish one model from the other based on their quantitative performance. The differences in the models’ main equations reduce to differences in the pricing mechanism imposed in decentralized markets. Hence, to the extent that the trading mechanism is not viewed as being an integral part of the model, or a primitive, then modeling money explicitly as opposed
to imposing cash-in-advance constraints neither induces theoretical nor quantitative differences in results. The price mechanism assumed to govern those transactions that must be settled with the exchange of cash is the source of differences. Overall, the analysis offers an important pedagogical lesson in the quest for the “best approach to modeling money.” On the one hand, it provides a unique perspective on the similarities in the performance of two models of money that are often perceived as being very different. On the other hand, it helps a reader to more deeply understand how to put to use such general equilibrium models; in particular, it suggests that one does not need to go through the heavier machinery of the LW model for many research questions.\footnote{We thank Christian Zimmerman for making this point in his NEP-DGE blog.}

2 A cash-in-advance model

This section discusses a standard general-equilibrium macroeconomic environment with incomplete markets. It is a compact version of the model in [10], where money is introduced by means of cash-in-advance constraints. The model adopts the convention that agents periodically alternate between centralized and decentralized markets, which is also found in the LW model.

Time is discrete and infinite, denoted $t = 0, 1, \ldots$ There is a constant population composed of a continuum of infinitely-lived agents, who are ex-ante homogeneous and expected utility maximizers. Preferences are defined over non-storable produced goods and labor. Each agent owns equal shares in a representative firm that produces goods using the concave technology $F$, which has labor as the only factor of production.

In a period, traders alternate synchronously between centralized and decentralized markets. Each period is divided into two subperiods, say, morning and afternoon. A decentralized market is open in the morning, while a centralized market is open in the afternoon. To introduce money, it is assumed that some of the morning trades must
be settled immediately with the exchange of money (= cash trades) while others can be settled in the afternoon (= credit trades). Goods purchased with cash are distinct from goods purchased on credit, called goods 1 and 2, respectively. Money is injected through lump-sum transfers by a central bank.

Let $s_t$ denote a shock realized at the start of $t$. The shock—which affects the households’ ability to consume and produce cash goods—is drawn from a time-invariant set. Let $\{s_t\}_{t=0}^{\infty}$ denote a path of shocks and let $S^t = (s_1, ..., s_t)$ denote a history of shocks (from the set of all possible histories), which is known prior to all period $t$ trading. Let $f^t(S^t)$ denote the density of the history $S^t$. Neither $F$ nor the money supply process depend on $S^t$.\footnote{A shock can also be added in the afternoon market, but since there are no such shocks in the LW model, that case is not studied here. The order of opening of the markets can also be inverted, without loss in generality.}

**Morning of $t$ (≡ decentralized market):** The shock $s_t$ is observed. Households and firms trade goods 1 and 2, and labor. Households hold $M_t(S^{t-1})$ money and buy $c_{1t}(S^t)$ goods in exchange for money (= cash goods), buy $c_{2t}(S^t)$ goods on credit (= credit goods) and supply $h_{t}(S^t)$ labor to the firm on credit. The firm demands $h_{Ft}(S^t)$ labor, buying it on credit, and supplies $F(h_{Ft}(S^t))$ goods. Credit trades are settled in the afternoon of $t$.

**Afternoon of $t$ (≡ centralized market):** Credit trades executed in the morning of $t$ are settled. Firms pay wages for work supplied in the morning and pay dividends out of morning profits. Households pay for credit goods bought in the morning. The central bank retires the old money supply $\bar{M}_{t-1}$ and issues a new money supply $\bar{M}_t$ through lump-sum money transfers $\Theta_t$ to households. Trade on a financial market also takes place: households trade state-contingent claims to money to be delivered in the afternoon of $t + 1$. Household exits the period holding $M_{t+1}(S^t)$ money.
2.1 Firm and households’ optimal choices

On date $t$, given history $S^t$, the constraint of the firm is

$$F(h_t^F(S^t)) = c_{1t}^F(S^t) + c_{2t}^F(S^t)$$

(1)

where $c_{1t}^F(S^t)$ and $c_{2t}^F(S^t)$ denote cash and credit goods. Because cash and credit goods are distinct, let $p_j(S^t)$ denote the nominal spot price of good $j = 1, 2$ and let $w_t(S^t)$ be the nominal spot wage on $t$. Nominal profits (net dollar inflows) on the morning of $t$ are

$$p_1(S^t)c_{1t}^F(S^t) + p_2(S^t)c_{2t}^F(S^t) - w_t(S^t)h_t^F(S^t),$$

(2)

which are distributed as dividends in the afternoon.

Since the firm sells for cash and for credit, payments accrue as follows: in the morning, it receives cash payments for cash-goods sales, and in the afternoon it receives payments for the morning’s credit sales. Let $q_t(S^t)$ denote the date−0 price of a claim to one dollar delivered in the afternoon of $t$, contingent on $S^t$ (= state-contingent nominal bond). The firm’s date−0 profit-maximization problem is: given state-contingent prices $q_t(S^t)$, choose sequences of output and labor ($c_{1t}^F(S^t), c_{2t}^F(S^t), h_t^F(S^t)$) to solve

$$\text{Maximize: } \sum_{t=0}^{\infty} \int q_t(S^t) \left( p_{1t}(S^t)c_{1t}^F(S^t) + p_{2t}(S^t)c_{2t}^F(S^t) - w_t(S^t)h_t^F(S^t) \right) dS^t$$

subject to:

$$c_{1t}^F(S^t) + c_{2t}^F(S^t) = F(h_t^F(S^t)).$$

(3)

Substituting for $c_{1t}^F(S^t)$ from the constraint, the FOCs for all $t, S^t$ are

$$h_t^F(S^t): \quad p_{1t}(S^t)F'(h_t^F(S^t)) - w_t(S^t) = 0$$

$$c_{2t}^F(S^t): \quad p_{1t}(S^t) - p_{2t}(S^t) = 0.$$

Consequently, for all $t, S^t$ we have $p_{1t}(S^t) = p_{2t}(S^t) = p_t(S^t)$ and

$$p_t(S^t)F'(h_t^F(S^t)) = w_t(S^t).$$

(4)
An agent who contracts on date 0 maximizes the expected utility

$$\sum_{t=0}^{\infty} \beta^t \int U(c_{1t}(S^t), c_{2t}(S^t), h_t(S^t)) f_t(S^t) dS^t$$

where we assume $U$ is a real-valued function, twice continuously differentiable in each argument, strictly increasing in $c_j$, decreasing in $h$, and concave. Maximization is subject to two constraints. One is the cash in advance constraint

$$p_{1t}(S^t)c_{1t}(S^t) \leq M_t(S^{t-1}) \quad \text{for all } t \text{ and } S^t,$$

where $M_t(S^{t-1})$ are money balances held at the start of $t$, brought in from the afternoon of $t - 1$, when the shock $s_t$ was not yet realized. Given this uncertainty, money may be held for the purpose of conducting transactions and for precautionary reasons.

The other constraint is the date-0 nominal intertemporal budget constraint:

$$\sum_{t=0}^{\infty} \int \left\{ q_t(S^t) \left[ p_{1t}(S^t)c_{1t}(S^t) + p_{2t}(S^t)c_{2t}(S^t) - w_t(S^t)h_t(S^t) - M_t(S^{t-1}) \
+ M_{t+1}(S^t) - \Theta_t \right] \right\} dS^t \leq \Pi + \tilde{M}$$

The date-0 sources of funds are $\tilde{M}$ initial money holdings (=initial liabilities of the central bank) and the firm’s nominal value $\Pi$. The left hand side is the date-0 present value of net expenditure. It is calculated by considering the price of money delivered in the afternoon of $t$, $q_t(S^t)$. There are two elements:

1. Morning net expenditure: $w_t(S^t)h_t(S^t)$ wages earned, paid in the afternoon; $M_t(S^{t-1}) - p_{1t}(S^t)c_{1t}(S^t)$ unspent balances available in the afternoon; $p_{2t}(S^t)c_{2t}(S^t)$ purchases of credit goods settled in the afternoon. These funds are available in the afternoon of $t$, where the date-0 value of one dollar is $q_t(S^t)$.

2. Afternoon net expenditures: the household receives $\Theta_t$ transfers and exits the period holding $M_{t+1}(S^t)$ money balances, so net expenditure is $M_{t+1}(S^t) - \Theta_t$, with date-0 value $q_t(S^t)$. 

Given that values can be history-dependent, we integrate over $S^t$. Consumers choose sequences of state-contingent consumption, labor and money holdings $c_1t(S^t)$, $c_2t(S^t)$, $h_t(S^t)$, and $M_{t+1}(S^t)$ to maximize the Lagrangian:

$$L := \sum_{t=0}^{\infty} \beta^t \int U(c_1t(S^t), c_2t(S^t), h_t(S^t)) f^t(S^t)dS^t + \lambda(\Pi + \tilde{M})$$

$$-\lambda \sum_{t=0}^{\infty} \int \{q_t(S^t)[p_{1t}(S^t)c_1t(S^t) + p_{2t}(S^t)c_2t(S^t) - w_t(S^t)h_t(S^t)]$$

$$-M_t(S^{t-1}) + M_{t+1}(S^t - \Theta_t]\right\} dS^t$$

$$+ \sum_{t=0}^{\infty} \int \mu_t(S^t)[M_t(S^{t-1}) - p_{1t}(S^t)c_1t(S^t)]dS^t,$$

where $\mu_t(S^t)$ is the Kühn-Tucker multiplier on the cash constraint on $t$, given $S^t$.

Omitting the arguments from $U$ and $f$ where understood, in an interior optimum the FOCs for all $t$ and $S^t$ are:

$$c_1t(S^t) : \quad \beta^tU_1 f^t(S^t) - \lambda p_{1t}(S^t)q_t(S^t) - \mu_t(S^t)p_{1t}(S^t) = 0$$

$$p_{1t}(S^t)c_1t(S^t) \leq M_t(S^{t-1})$$

$$c_2t(S^t) : \quad \beta^tU_2 f^t(S^t) - \lambda p_{2t}(S^t)q_t(S^t) = 0$$

$$h_t(S^t) : \quad \beta^tU_3 f^t(S^t) + \lambda w_t(S^t)q_t(S^t) = 0$$

$$M_{t+1}(S^t) : \quad -\lambda q_t(S^t) + \lambda \int q_{t+1}(S^{t+1})ds_{t+1} + \int \mu_{t+1}(S^{t+1})d\lambda s_{t+1} = 0.$$

Given $p_{2t}(S^t) = p_{1t}(S^t) = p(S^t)$ and (4) we get

$$\frac{U_3}{U_2} = F'(h_t(S^t); S^t) \quad \text{for all } t, S^t$$

$$\frac{U_1}{U_2} = \frac{\lambda q_t(S^t) + \mu_t(S^t)}{\lambda q_t(S^t)} \quad \text{for all } t, S^t. \quad (7)$$

**2.2 Risk-free rate and Central Bank constraint**

Fix $t$ and $S^t$. The (reciprocal of the) nominal risk-free interest rate on a bond sold in the afternoon of $t$ is $\frac{1}{1+r_t(S^t)}$. This is the price of a claim to money (bought on date 0) delivered in the afternoon of $t + 1$ conditional on $S^t$ (but not on $s_{t+1}$) divided by
the price of a claim to money delivered in the afternoon of $t$ conditional on $S^t$:

$$\frac{1}{1 + r_t(S^t)} := \frac{\int q_{t+1}(S^{t+1})ds_{t+1}}{q_t(S^t)} = \frac{\lambda \int q_{t+1}(S^{t+1})ds_{t+1}}{\lambda \int q_{t+1}(S^{t+1})ds_{t+1} + \int \mu_{t+1}(S^{t+1})ds_{t+1}},$$  

(8)

where the second step comes from the last line in (6).\(^3\)

From (7), the interest rate makes households indifferent between buying money or risk-free bonds in the afternoon of $t$. With cash the consumer can buy either cash- or credit-goods in $t + 1$; by holding bonds, he can only buy credit goods, as bonds mature in the afternoon of $t + 1$. So, the interest rate compensates consumers for the bond’s illiquidity, which is why $\mu_{t+1}$ appears in the denominator of (8). Substituting $q_t(S^t) = (1 + r_t(S^t)) \int q_{t+1}(S^{t+1})ds_{t+1}$ in the last line of (6) we get

$$(1 + r_t(S^t)) \int q_{t+1}(S^{t+1})ds_{t+1} = \int q_{t+1}(S^{t+1})ds_{t+1} + \frac{1}{\lambda} \int \mu_{t+1}(S^{t+1})ds_{t+1}.$$  

This is simply an indifference condition between buying an illiquid bond or holding money. The expected benefit from buying a risk-free bond in the afternoon of $t$ that pays one dollar in the afternoon of $t + 1$ is $(1 + r_t(S^t)) \int q_{t+1}(S^{t+1})ds_{t+1}$. Money has the lower expected value $\int q_{t+1}(S^{t+1})ds_{t+1}$, but provides the liquidity premium $\frac{1}{\lambda} \int \mu_{t+1}(S^{t+1})ds_{t+1}$ because, unlike the bond, a dollar worth of money can be spent in the morning of $t + 1$ to buy cash goods.

Let $\bar{M} \geq 0$ be the initial money supply. In the afternoon of $t$, the central bank issues $\bar{M}_{t+1}$ money, valued at $q_t(S^t)$ in date−0 prices, and retires it in the afternoon of $t + 1$, when the expected value of money is $\int q_{t+1}(S^{t+1})ds_{t+1}$. Money is injected via lump-sum transfers $\Theta_t$ valued at $q_t(S^t)$. The date−0 budget constraint is

$$\bar{M} = \sum_{t=0}^{\infty} \{ \bar{M}_{t+1} [q_t(S^t) - \int q_{t+1}(S^{t+1})ds_{t+1}] - \Theta_t q_t(S^t) \}dS^t.$$  

Equivalently, the flow constraint $\bar{M}_{t+1} - \bar{M}_t = \Theta_t$ for all $t$, $S^t$ identify monetary policy.

\(^3\)No-arbitrage requires that expenditures in period 0 are equivalent. The household can spend $q_t(S^t) \int \frac{1}{1 + r_t(S^t)}ds_{t+1}$ to buy $\frac{1}{1 + r_t(S^t)}$ delivered on $t$ conditional on $S^t$, and then reinvest on $t$ the receipts in a risk-free bond to get 1 good on date $t + 1$. Alternatively, the agent can spend $\int q_{t+1}(S^{t+1})ds_{t+1}$ on date 0 to have one unit on date $t + 1$, given $S^t$. 

9
3 Juxtaposing the two models

To compare the LW model and the cash-in-advance model, we utilize the feature that the LW model can be reduced to a single difference equation [7, p. 469].

3.1 The main equation in the LW model

Agents in [7] alternate between two markets: decentralized (DM) and centralized (CM). First, the DM opens and DM goods are traded and then the CM opens and CM goods are traded. CM markets are Walrasian; in the DM there is pairwise trade with Nash bargaining and an agent has equal probability $\delta \leq 1/2$ (using our notation—see also the Appendix) to buy with money or to sell for money, so the ratio of buyers to sellers is one (assume no barter). Preferences are additively separable with quasilinear labor disutility:

$$U(c_1, c_2, h_1, h_2) = u_1(c_1) - \eta(h_1) + u_2(c_2) - h_2,$$

where $h_1$ and $h_2$ denote labor effort in DM and CM, $c_1$ and $c_2$ denote consumption in DM and CM. It is assumed that $u_1, u_2, \eta$ are twice continuously differentiable, strictly increasing, $u_1$ and $u_2$ are concave, $\eta$ is convex and $u_1(0) = \eta(0) = 0$; furthermore, there exists $c^*_j \in \mathbb{R}_{++}$ for $j = 1, 2$ such that $u'_1(c^*_1) = \eta'(c^*_1)$ and $u'_2(c^*_2) = 1$ with $u_2(c^*_2) > c^*_2$.

We now discuss equilibrium in the LW model. From [7, p.469], on each $t$ equilibrium consumption of CM goods satisfies

$$u'_2(c_2) = 1.$$  

Let $\theta \in (0, 1]$ denote the buyer’s bargaining power. From [7, eq. (17)], in equilibrium

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4The equilibrium concept is a “blend of traditional Arrow-Debreu components describing aggregates as functions of time $t$ and recursive components describing individuals’ problems as functions of $t$ and individual state variables” [7, footnote 3].
\[ \frac{1}{p_{2t}} = \frac{\beta}{p_{2,t+1}} \left[ \delta u'_1(c_{1,t+1}) \frac{1}{z'(c_{1,t+1}; \theta)} + 1 - \delta \right], \]

with \( p_{2t} = \frac{M_t}{z(c_{1t}; \theta)} \) and, using [7, eq. (8)] and omitting the time subscript

\[ z(c_1; \theta) := \frac{\theta \eta(c_1) u'_1(c_1) + (1 - \theta) u_1(c_1) \eta'(c_1)}{\theta u'_1(c_1) + (1 - \theta) \eta'(c_1)} \]

Equations (10) and (11) determine equilibrium consumption in the LW model.

Consider a stationary equilibrium in which money grows at a constant rate \( \gamma \geq \beta \), and consumption and real money balances are constant. The inflation rate also equals \( \gamma \), \( r_t = r = \frac{\gamma}{\beta} - 1 \) and the LW model reduces to the equation

\[ \frac{u'_1(c_1)}{z'(c_1; \theta)} = 1 + r \frac{r}{\delta}. \]

The key observation is that the DM pricing schedule is nonlinear due to bargaining, so the marginal benefit from spending one more dollar is \( \frac{u'_1(c_1)}{z'(c_1; \theta)} \); instead, we would have \( \frac{u'_1(c_1)}{p_1/p_2} \) under linear pricing, with \( p_1/p_2 = \eta'(c_1) \leq z'(c_1; \theta) \).\(^5\) Such price distortion is measured by the ratio

\[ \psi(c_1, \theta) := \frac{\eta'(c_1)}{z'(c_1; \theta)}, \]

where \( \psi(c_1, 1) = 1 \) (no distortion) and \( \psi(c_1, \theta) < 1 \) for \( \theta < 1 \). Figure 1 illustrates that the price distortion depends on \( \theta \).

### 3.2 Model consistency

To present a meaningful comparison, preferences, technologies, and shocks in the cash-in-advance model must conform to those in the LW model. This section discusses how this logical coherence is achieved.

**Technologies:** Let \( F(h) = h \) as in the LW model. Since the marginal product of

\(^5\)If \( \theta = 1 \), then \( z' = \eta' \). If \( \theta < 1 \) we have \( z' > \eta' \). Indeed, \( u'_1 \geq \eta' \); hence, \( \theta u'_1 + (1 - \theta) \eta' < u'_1 \). From the definition of \( z(c_1; \theta) \) we have \( z' = \frac{u'_1}{\theta u'_1 + (1 - \theta) \eta'} \eta' + A \) where \( A > 0 \).
Figure 1: The price distortion in the LW model

Notes to Figure 1: The three curves correspond to $\psi(c_1; \theta)$ assuming—as in the calibration in [7]—that $\eta' = 1$, $u_1(c_1) = \frac{(c_1 + b)^{1-a} - b^{1-a}}{1-a}$, $a = 0.3$, $b = 0$, $\delta = 0.5$ and $r = 1.04 \gamma - 1$, with $\gamma = \beta$ (=Friedman rule), $\gamma = 1$ (=zero inflation) and $\gamma = 1.1$ (= 10% inflation).
labor is fixed and independent of $S^t$, it is convenient (and without loss in generality)
to interpret production of goods 1 and 2 as occurring in two batches. The firm chooses
$h_{jt}^F$ (= labor demanded to produce good $j = 1, 2$) and $c_{jt}^F$ (= supply) to solve

$$\text{Maximize: } \sum_{t=0}^\infty q_t(S^t)p_{1t}(S^t)c_{1t}^F + p_{2t}(S^t)c_{2t}^F - w_{1t}(S^t)h_{1t}^F - w_{2t}(S^t)h_{2t}^F.$$

subject to: $c_{2t}^F = h_{1t}^F$ and $c_{1t}^F = h_{1t}^F$.

Substituting the constraints, the FOCs are

$$p_{jt}(S^t) - w_{jt}(S^t) = 0 \quad \text{for all } t \text{ and } j = 1, 2. \quad (13)$$

Prices equal marginal cost and profits are zero, so $\Pi = 0$.

**Preferences and shocks:** Let $s_t$ be an i.i.d. shock such that in each $t$ a randomly
drawn portion $\delta \in (0, 1)$ of households desires good 1 and produces it. Hence,

$$f^t(S^t) = f^t(s_t; S^{t-1}) = f(s_t)f^{t-1}(S^{t-1}) \quad \text{for all } t \geq 0,$$

where $f$ denotes the distribution of the date-$t$ shock. Here $s_t = (s_{it})_{all \ i}$ where

$$s_{it} = \begin{cases} 1 & \text{with probability } \delta \\ 0 & \text{with probability } 1 - \delta \end{cases}$$

for all $t \geq 0$ and all agents $i$

where $s_{it} = 0$ means that household $i$ neither derives utility from consuming good 1 nor
can produce it. For any agent $i$, the marginal probabilities are thus $\int f(s_t)1_{\{s_{it} = 0\}}ds_t = 1 - \delta$ and $\int f(s_t)1_{\{s_{it} = 1\}}ds_t = \delta$.

Assume preferences (9), where $h_{jt}^i$ is labor supplied by household $i$ to produce
good $j = 1, 2$. For household $i$ on date $t$ we have:

$$U(c_{1t}, c_{2t}, h_{1t}, h_{2t}) = [u_1(c_{1t}^i) - \eta(h_{1t}^i)]1_{\{s_{it} = 1\}} + u_2(c_{2t}^i) - h_{2t}^i. \quad (14)$$

**Price distortion:** A parsimonious way to match the price distortion $\psi(c_1, \theta)$ is to
introduce a proportional tax either on sales or purchases involving cash goods. For
example, assume that a share $1 - \tau$ of revenue from cash-sales—taken as given—must
be rebated back to the firm’s owners, lump-sum. For mnemonic ease, we refer to $\tau$ as the parameter of a “cash-revenue tax.” The parameter $\tau$ distorts the relative price of cash and credit goods, without altering the model’s structure or equilibrium concept. In particular, the firm’s problem is unchanged: we must simply substitute $p_{1t}\tau c_{1t}^{\text{F}}$ for $p_{1t}c_{1t}^{\text{F}}$, so that the first order condition for cash goods becomes $p_{1t}\tau = w_{1t}$ and $\frac{w_{1t}}{w_{2t}} = \frac{1}{\tau}$. Because the buyer spends $p_{1t}c_{1t}$ and the seller receives $p_{1t}\tau c_{1t}$, we can interpret $p_{1t}c_{1t}(1-\tau)$ as a sales tax and $\frac{1}{\tau}-1$ as the sales tax rate on cash transactions. Viewed in this manner, introducing the tax parameter $\tau$ does not amount to adding an unrealistic feature to the model; in fact, sales taxes are commonplace at the state and local level in many countries.

### 3.3 The main result

The literature based on the LW model has almost entirely focused on stationary equilibrium (one exception is [6]). Consequently, we focus on stationary competitive equilibrium in the cash-in-advance model; later, we discuss what happens outside the steady state.

**Proposition 1.** Consider the cash-in-advance model with preferences, technologies, and shocks as in the LW model, and a cash-revenue tax with parameter $\tau$. If $\tau = \psi(c_1, \theta)$, then the equations characterizing stationary competitive equilibrium in the cash-in-advance model coincide with equations (10) and (12), which characterize stationary equilibrium in the LW model. The cash-in-advance model can generate the same welfare costs of inflation as the LW model.

To provide support for this finding we start by deriving the main equations of the cash-in-advance model. Consider a generic household $i$. On date 0, he can spend $q_t(S^t)$ to buy a claim to one unit of money delivered in the afternoon of $t$, contingent on the history $S^t$. Let $q_t$ be the price of money delivered on $t \text{ unconditional}$ on $S^t$ (a risk-free discount bond). No-arbitrage requires equal expenditures, i.e.,
\[ q_t = \int q_t(S^t) dS^t. \] It also implies\(^6\)

\[ q_t(S^t) = q_t f^t(S^t). \]

To keep the discussion focused, suppose \(\tau = 1\) (no tax, no price distortion). The problem of agent \(i\) is still given by \((5)\), where we substitute \(q_t(S^t) = q_t f^t(S^t)\), \(U\) from \((14)\), separate the labor choices for each production batch, and set \(\Pi = 0\) in the intertemporal budget constraint.\(^7\) Household \(i\) chooses sequences \(c_{1t}(S^t), c_{2t}(S^t), h_{1t}(S^t), h_{2t}(S^t)\) and \(M_{t+1}(S^t)\) to maximize:

\[
L^t := \sum_{t=0}^{\infty} \beta^t \int U(c_{1t}(S^t), c_{2t}(S^t), h_{1t}(S^t), h_{2t}(S^t)) f^t(S^t) dS^t + \lambda \tilde{M} - \lambda \sum_{t=0}^{\infty} \int q_t f^t(S^t) \left\{ [p_{1t}(S^t)c_{1t}(S^t) + p_{2t}(S^t)c_{2t}(S^t) - w_{1t}(S^t)h_{1t}(S^t)] - w_{2t}(S^t)h_{2t}(S^t) - M_t(S^{t-1}) + M_{t+1}(S^t) - \Theta_t \right\} dS^t + \sum_{t=0}^{\infty} \int \mu_t(\tilde{S}^t) [M_t(S^{t-1}) - p_{1t}(S^t)c_{1t}(S^t)] dS^t. \tag{15}
\]

The FOCs, for all \(t\) and \(S^t\), are

\[
c_{1t}(S^t) := \beta^t u_1'(c_{1t}(S^t)) f^t(S^t) - \lambda p_{1t}(S^t)q_t f^t(S^t) - \mu_t(S^t) p_{1t}(S^t) = 0 \quad \text{for} \quad s^i_t = 1
\]

\[
p_{1t}(S^t) c_{1t}(S^t) \leq M_t(S^{t-1}),
\]

\[
c_{2t}(S^t) := \beta^t u_2'(c_{2t}(S^t)) - \lambda p_{2t}(S^t)q_t = 0,
\]

\[
h_{1t}(S^t) := -\beta^t h'(h_{1t}(S^t)) + \lambda w_{1t}(S^t)q_t = 0 \quad \text{for} \quad s^i_t = 1,
\]

\[
h_{2t}(S^t) := -\beta^t + \lambda w_{2t}(S^t)q_t = 0,
\]

\[
M_{t+1}(S^t) := \lambda q_t f^t(S^t) = \lambda q_{t+1} f^t(S^t) + \int \mu_{t+1}(S^{t+1}) ds_{t+1}.
\tag{16}
\]

\(^6\)If \(q_t(S^t) < q_t f^t(S^t)\), then \(q_t(S_t^t) > q_t f^t(S_t^t)\) for some other state \(\tilde{S}^t\) since \(\int f^t(S^t) dS^t = 1\). In this case, the agent could make large profits with zero net investment by (i) purchasing claims that pay in state \(S^t\) at a cheap price \(q_t(S^t)\), while selling risk-free claims at price \(q_t\); and (ii) selling claims that pay in state \(S^t\) at a steep price \(q_t(S_t^t)\), while buying risk-free claims at price \(q_t\). Thus non-contingent claims would not be traded at price \(q_t\), which is a contradiction.

\(^7\)In competitive equilibrium the firm makes zero profits and since \(\tau = 1\) households get no rebate on cash purchases. Therefore, the value of holding the firm, \(\Pi\), must be zero.
The last line is derived using \( q_{t+1} f^{t+1}(S^{t+1}) = q_{t+1} f(s_{t+1}) f^t(S^t) \) and noticing that
\[
\int q_{t+1} f(s_{t+1}) f^t(S^t) ds_{t+1} = q_{t+1} f^t(S^t)
\]
because \( \int f(s_{t+1}) ds_{t+1} = 1 \) by definition.

From \(-\beta^t + \lambda w_{2t}(S^t) q_t = 0\) we have that \( w_{2t} \) is independent of \( S^t \) and therefore, using the firm’s optimality conditions, \( p_{2t} \) is independent of \( S^t \). Since \(-\beta^t + \lambda w_{2t} q_t = 0\) and \( w_{2t} = p_{2t} \) (from the firm’s problem), the optimal choice of credit goods in (16) satisfies \( \beta^t u'_2(c_2(S^t)) = \lambda p_{2t} q_t \); this implies
\[
u'_2(c_2(S^t)) = 1 \text{ for all } t, S^t,
\]
so \( c_2(S^t) = c_2 \) for all \( t, S^t \) and all agents \( i \). This coincides with (10).

Consider cash goods. Their consumption is heterogeneous because for if \( s^i_t = 0 \) for agent \( i \), then \( c^i_1(S^t) = 0 \); this also implies \( \mu_t(S^t) = 0 \) for agent \( i \) because this agent’s cash constraint does not bind. Now consider \( s^i_t = 1 \). We prove that if an agent desires to consume cash goods, then the quantity consumed is independent of the history of shocks \( S^t \) and of the identity of the agent, \( i \).

**Lemma 1.** Consider any agent \( i \) and let \( s^i_t = 1 \). In competitive equilibrium:

1. If \( \mu_t(S^t) = 0 \), then \( c_{1t}(S^t) = c_1 \) for all \( t, S^t \), with \( \frac{u'_1(c_1)}{\eta'(c_1)} = 1 \).

2. If \( \mu_t(S^t) > 0 \), then \( c_{1t}(S^t) = \frac{M_t}{p_{1t}} = c_{1t} \) for all \( t, S^t \), where \( c_{1t} \) satisfies
\[
\frac{\beta}{p_{2,t+1}} \left[ \delta u'_1(c_{1,t+1}) \frac{1}{\eta'(c_{1,t+1})} + 1 - \delta \right] - \frac{1}{p_{2t}} = 0 \text{ for all } t, \tag{17}
\]
with \( p_{2t} = \frac{M_t}{\eta'(c_{1t}) c_{1t}} \).

**Proof of Lemma 1.** See Appendix

On date \( t \), not everyone consumes cash goods (\( c_{1t} = 0 \) when \( s^i_t = 0 \)) but those who do consume an identical quantity \( c_{1t} \), independent of the history of shocks. Since \( U \) is linear in \( h_2 \), everyone saves the same amount of money \( M_t(S^{t-1}) = M_t \) on \( t - 1 \), there is a degenerate distribution of money, and prices are history-independent. If \( \mu_t = 0 \),
then $u_1' = \eta'$ and the agent consumes the efficient quantity $c_{1t} = c_1^\ast$. Otherwise, $u_1' > \eta'$ and $c_{1t} = \frac{M_t}{p_1t} < c_1^\ast$ (first and third equations in (16) with $p_{1t} = w_{1t}$).

Using the risk-free interest rate defined in (8), we have

$$\frac{1}{1 + r_t} = \int \frac{q_{t+1}(S_{t+1})dS_{t+1}}{q_t(S_t)} = \frac{q_{t+1}f_t(S_t)}{q_tf_t(S_t)} = \frac{\beta}{\pi_t}. $$

The second equality holds by substituting $q_t(S_t) = q_t f_t(S_t)$ and noting that $q_{t+1}f_{t+1}(S_{t+1}) = q_{t+1}f(s_{t+1})f_t(S_t)$ so that $\int q_{t+1}f(s_{t+1})f_t(S_t)ds_{t+1} = \int f(s_{t+1})ds_{t+1} = 1$. To perform the final step substitute $\frac{\beta u_2'(c_{2t})}{\lambda p_2t} = q_t$ from (16), use $u_2'(c_{2t}) = 1$, and define the gross inflation rate $\pi_t := \frac{p_{2t+1}}{p_{2t}}$.

Now let $M_{t+1} = \gamma M_t$ and consider stationary equilibrium with $\frac{M_{t+1}}{p_{2t+1}} = \frac{M_t}{p_{2t}} \frac{p_{2t+1}}{p_{2t}} = \gamma$ and $r_t = r = \frac{\gamma}{\beta} - 1$ for all $t$. Equation (17) yields

$$\frac{u_1'(c_1)}{\eta'(c_1)} = \frac{r}{\delta} + 1. $$

The only difference between (18) and (12) is given by the price distortion in the LW model. Due to linear pricing, the marginal benefit from spending one more dollar on cash goods is $\frac{u_1'(c_1)}{p_1/p_2}$ where $p_1/p_2 = \eta'(c_1)$ in equilibrium.

Now note that equation (18) coincides with (12) when $\theta = 1$, since $z' = \eta'$; intuitively, sellers are price-takers in both models.\(^{8}\) Otherwise, when $\theta < 1$, it does not because $z' > \eta'$, i.e., Nash bargaining induces a price distortion. This is evidence that the two frameworks’ differences, in terms of stationary equilibrium allocations, reduce to differences in assumptions about the pricing mechanism that governs those transactions that must be settled with the exchange of money. One wonders whether the distortion generated by the Nash bargaining solution can be reproduced by introducing a cash-revenue tax in the cash-in-advance model.

Re-introduce the cash-revenue tax parameter $\tau \leq 1$. The households’ problem is

\[^{8}\text{It must be also emphasized that the two equations coincide if DM goods are traded on competitive markets and the use of competitive pricing in the DM is very common within the LW model; consider for instance [1, 2, 14].}\]
The FOCs are in (16), so the model still reduces to the difference equation (17). However, in stationary equilibrium relative prices are \( \frac{p_1}{p_2} = \frac{\eta'(h_1)}{\eta'(c_1)} \), so we obtain

\[
\frac{u'_1(c_1)}{\eta'(c_1)/\tau} = 1 + \frac{r}{\delta}.
\]

This equation coincides with (12) if \( \tau = \psi(c_1, \theta) \), which is when the cash-revenue tax in equilibrium reproduces the price distortion induced by Nash bargaining. The lesson is that, in stationary equilibrium, differences in the frameworks’ main equations reduce to the price distortion due to bargaining. Such distortion can be replicated in the cash-in-advance model with an appropriate “tax” on revenues from cash transactions.

The result partially extends to non-stationary equilibrium.

**Corollary 1.** If \( \eta \) satisfies \( \frac{d \ln \eta(h)}{d \ln h} = \kappa > 0 \) and \( \theta = 1 \), then the equations characterizing non-stationary competitive equilibrium in the cash-in-advance model coincide with (10) and (11), which characterize non-stationary equilibrium in the LW model.

The result immediately follows from Lemma 1. Rewrite equation (17) as

\[
\frac{\eta'(c_1)c_{1t}}{M_t} = \beta \frac{\eta'(c_{1,t+1})c_{1,t+1}}{M_{t+1}} \left[ \frac{u'_1(c_{1,t+1})}{\eta'(c_{1,t+1})}\delta + 1 - \delta \right],
\]

and note that it coincides with (11) when \( \theta = 1 \) and \( \frac{d \ln \eta(h)}{d \ln h} = \kappa \), because \( p_{2t} = \frac{M_t}{\eta(c_{1t})} \) (since \( z(c_1; 1) = \eta(c_1) \)) and \( \eta'(c_1)c_1 = \kappa \eta(c_1) \). Both \( \eta \) linear and the common isoelastic formulation \( \eta(h) = \frac{h^x}{x} \) for \( x > 1 \) satisfy \( \frac{d \ln \eta(h)}{d \ln h} = \kappa \). The correspondence between the equations characterizing non-stationary allocations in the two models breaks down when \( \theta < 1 \). Again, the difference in allocations reduce to differences in assumptions about the pricing mechanism that governs those transactions that must be settled with the exchange of money.\(^{10}\) Hence, there may exist equilibria which are not the same in the two models.

\(^9\)The only difference is \( \Pi \) appears in the agent’s budget constraint—as it did in (5)—due to lump-sum rebates from the firm. In equilibrium we have \( \Pi = \sum_{t=0}^{\infty} \int q_t f'(S^t)T_t dS^t \) where the rebate \( T_t = p_{1t}(1-\tau)c_{1t}\delta \) on \( t \).

\(^{10}\)The equations characterizing non-stationary allocations coincide when DM goods are priced competitively.
3.4 Quantitative comparison

To evaluate possible quantitative differences between the cash-in-advance model and the LW model, we adopt the specification in [7, Table 1], which considers stationary equilibrium in the model calibrated to annual U.S. data.

Preferences over goods are defined by

\[ u_1(c_1) = \frac{(c_1 + b)^{1-a} - b^{1-a}}{1-a} \quad \text{and} \quad u_2(c_2) = B \log c_2, \]

for some \( a > 0, b \in (0, 1) \) and \( B > 0 \). Consumption \( c_2 \) satisfies (10), labor disutility satisfies \( \eta' = 1 \), so \( c_1 \) satisfies

\[ \frac{\gamma}{\beta} - 1 = \delta [\tau u'_1(c_1) - 1]. \tag{19} \]

Define ex-ante welfare

\[ W_\gamma := u_2(c_2) - c_2 + \delta [u_1(c_1(\gamma)) - c_1(\gamma)]. \]

Considering the compensating variation \( \Delta \), welfare at zero inflation is denoted

\[ W_1 := u_2(\Delta c_2) - c_2 + \delta [u_1(\Delta c_1(1)) - c_1(1)]. \]

The welfare cost of \( \gamma - 1 \) inflation is the value \( 1 - \Delta \) where \( \Delta \) satisfies \( W_1 - W_\gamma = 0 \).

In [7, p.475], \( \theta \) is calibrated to match the average price markup in U.S. data; the markup is \( \frac{z(c_1; \theta)}{c_1 \eta'(c_1)} \), i.e., the ratio of the DM good price \( p_1 \) to marginal cost.\(^{11}\) In our model the markup is \( \frac{w_1}{w_1} = \frac{1}{\tau} \equiv \frac{z(c_1; \theta)}{\eta'(c_1)} \) because we match the price distortion in the LW model by setting \( \tau = \psi(c_1; \theta) \) and use the calibrated value of \( \theta \) from the LW model. Hence, the markups in the two model generally do not coincide.

Table 1 compares results for the cash-in-advance and the LW model, in five different cases. Panel 1 shows that the cash-in-advance model can yield identical consump-

\(^{11}\) It varies with the bargaining power and it generally varies with \( c_1 \) (but not always; consider \( \eta(h) = \frac{h^x}{x !} \), \( x \geq 1 \) and \( \theta = 1 \)). In the calibration labor disutility is linear so the markup coincides with the relative price \( \frac{p_1}{p_2} \), which is \( \frac{z(c_1; \theta)}{c_1} \).
tion as in [7, Table 1]. Panel 2 reports average price markups, at each inflation rate; the average markups are comparable. Fixing the parameter \( \theta \), average markups increase with inflation in both models; if we interpret \( \tau - 1 \) as the sales tax rate on cash trades, then the model does not imply unreasonable average sales tax rates.\(^{12}\) Panel 3 shows that the cash-in-advance model can yield identical welfare cost of inflation as in the LW model.

<table>
<thead>
<tr>
<th>Parameter ( \equiv \alpha \sigma )</th>
<th>case 1</th>
<th>case 2</th>
<th>case 3</th>
<th>case 4</th>
<th>case 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta ) ( = .31 )</td>
<td>.5</td>
<td>.5</td>
<td>.5</td>
<td>.5</td>
<td>.5</td>
</tr>
<tr>
<td>( a(\equiv \eta) ) ( = .27 )</td>
<td>.16</td>
<td>.30</td>
<td>.30</td>
<td>.30</td>
<td>.30</td>
</tr>
<tr>
<td>( B ) ( = 2.13 )</td>
<td>1.97</td>
<td>1.91</td>
<td>1.78</td>
<td>1.78</td>
<td>1.78</td>
</tr>
<tr>
<td>( \theta ) ( = 1 )</td>
<td>1</td>
<td>.5</td>
<td>.343</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Inflation Panel 1: Equilibrium \( c_1 \)

<table>
<thead>
<tr>
<th>( \gamma - 1 )</th>
<th>Panel 2: Average markup</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>.243</td>
</tr>
<tr>
<td>0</td>
<td>.638</td>
</tr>
<tr>
<td>( \beta^{-1} - 1 )</td>
<td>1</td>
</tr>
</tbody>
</table>

Panel 3: Welfare cost of 10% inflation

<table>
<thead>
<tr>
<th>( \gamma - 1 )</th>
<th>Panel 3: Welfare cost of 10% inflation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.014</td>
</tr>
<tr>
<td>( \beta^{-1} - 1 )</td>
<td>0</td>
</tr>
</tbody>
</table>

Notes to Table 1: The comparison involves the calibration in [7, Table 1]. The Parameters column reports our notation (the corresponding notation from [7], when different from ours, is reported in parentheses). In both models \( c_2 = B \) in equilibrium and \( \beta^{-1} = 1.04 \). The inflation rate is \( \gamma - 1 \). When numbers are different in the two models we report them as the pair \{LW, cash-in-advance\}.

\(^{12}\) The share of DM output in the LW model is easily constructed, given that in the calibrated model everyone is matched in the DM (\( \alpha = 1 \) in the LW model). DM output is \( \delta c_1 \) and CM output is \( c_2 = B \), in the calibrated model. Hence, total output is \( Y = \delta c_1 + B \) and the DM output share is \( \frac{\delta}{\delta + 1} \) (it increases as inflation falls because real money balances increase); this also gives us the share of cash goods to total goods in the cash-in-advance model. This share is used to calculate average markups. In the calibration, when \( \theta = 0.5 \) we have \( \tau = \psi(c_1; \theta) = .719, .846, .928 \) for, respectively, \( \gamma = .1, 0, \frac{1 - \beta}{\beta} \), the corresponding average sales tax rates are: \(.025, .037, .034 \). Instead, when \( \theta = 0.343 \), we have \( \tau = \psi(c_1; \theta) = .511, .672, .802 \); the corresponding average sales tax rates are: \(.014, .019, .013 \). As inflation decreases the markup in cash trades, \( \frac{1}{\tau} \), falls; yet, the average markup increases because the share of cash goods to total output rises.
In a nutshell, the cash-in-advance model can replicate the same, large welfare cost of inflation found in the LW model, once price distortions are accounted for (cases 3-4). This suggests that the difference in the assumed pricing mechanisms is primarily what lies behind the dissimilarities in quantitative results between the two models, and not the explicit microfoundation for money in the LW model as opposed to the reduced-form approach of the other model.

4 Final comments

We have examined two monetary models characterized by periodic interactions in centralized and decentralized markets: the cash-in-advance model, and the model in [7]. Prices are linear in the former but are non-linear in the latter when trades must be settled with the exchange of cash, due to Nash bargaining. Our analysis indicates that this is the one difference that matters.

When the models are placed on equal footing in terms of preferences, technologies and shocks, both models reduce to a single equation describing stationary equilibrium. The equations coincide when sellers have no bargaining power. Otherwise, the equations differ in just one element—the price distortion from bargaining. Yet, such distortion can be replicated in the cash-in-advance model using a proportional tax. For simplicity, we have considered a tax on cash revenues, in which case allocations and welfare costs of inflation are comparable in stationary equilibrium.

Our findings neither rely on altering the market structure of the LW model, nor the equilibrium concept or the basic structure of the cash-in-advance model. The analysis should neither be taken to imply that nothing can be done with one model, which could not be done with the other, nor that the models are identical. In fact, our analysis has emphasized the central role played by assumptions about the pricing mechanisms presumed to govern cash-based trades in the two models.
References


Appendix

Proof of Lemma 1

Consider an equilibrium with history-independent prices \( p_{1t}(S^t) = p_{1t} \) and \( w_{1t}(S^t) = w_{1t} \), as in [7].\(^{13}\) To prove the first part of the Lemma let \( s^i_t = 1 \) and \( \mu_t(S^t) = 0 \). From the first and third expressions in (16) we have

\[
\beta^t u'_1(c_{1t}(S^t)) = \lambda p_{1t} q_t = \lambda w_{1t} q_t = \beta^t \eta'(h_{1t}(S^t)), \quad \text{for all } t, S^t,
\]

From market clearing \( h^F_{1t}(S^t) = \delta h_{1t}(S^t) = \delta c_{1t}(S^t) = c^F_{1t}(S^t) \).\(^{14}\) Hence, \( u'_1(c_{1t}(S^t)) = 1 \) for all \( t, S^t \). That is \( c_{1t}(S^t) = c_1 \) for all \( t \) and all agents \( i \) such that \( s^i_t = 1 \).

To prove the second part of the Lemma let \( s^i_t = 1 \) and \( \mu_t(S^t) > 0 \). Update by one period the first expression in the FOCs (16) to get

\[
\frac{\beta^{t+1}}{p_{1, t+1}} u'_1(c_{1, t+1}(S^{t+1})) f(s_{t+1}) f'(S^t) = \lambda q_{t+1} f(s_{t+1}) f'(S^t) + \mu_{t+1}(S^{t+1}), \quad \text{if } s^i_{t+1} = 1
\]

where we substituted \( f^{t+1}(S^{t+1}) = f(s_{t+1}) f'(S^t) \). Now substitute \( c_{1, t+1}(S^{t+1}) = \frac{M_{t+1}(S^{t+1})}{p_{1, t+1}} \) since \( \mu_{t+1}(S^{t+1}) > 0 \). The expression above has the status of an equality only if \( s^i_{t+1} = 1 \). In that case, we can integrate both sides with respect to \( s_{t+1} \), conditional on \( s^i_{t+1} = 1 \). For the left-hand-side we get

\[
\frac{\beta^{t+1}}{p_{1, t+1}} \int_1 \begin{cases} 1_{\{s^i_{t+1} = 1\}} \end{cases} u'_1(c_{1, t+1}(S^{t+1})) f(s_{t+1}) f'(S^t) d_{s_{t+1}}
\]

\[
= \frac{\beta^{t+1}}{p_{1, t+1}} u'_1 \left( \frac{M_{t+1}(S^t)}{p_{1, t+1}} \right) \int_1 \begin{cases} 1_{\{s^i_{t+1} = 1\}} \end{cases} f(s_{t+1}) f'(S^t) d_{s_{t+1}}
\]

\[
= \frac{\beta^{t+1}}{p_{1, t+1}} u'_1 \left( \frac{M_{t+1}(S^t)}{p_{1, t+1}} \right) f'(S^t) \int_1 \begin{cases} 1_{\{s^i_{t+1} = 1\}} \end{cases} f(s_{t+1}) d_{s_{t+1}}
\]

\[
= \frac{\beta^{t+1}}{p_{1, t+1}} u'_1 \left( \frac{M_{t+1}(S^t)}{p_{1, t+1}} \right) f'(S^t) \delta
\]

\(^{13}\)Prices and wages will not depend on the history \( S^t \) here if the distribution of money holdings is degenerate at the start of each period \( t \), which we will prove to be the case.

\(^{14}\)Under linear labor disutility, households are indifferent to how much labor \( h_1 \) they supply at the given wage \( w_1 \). In that case, we consider symmetric choices, i.e., every household supplies the same labor effort. This is as in [7].
For the right-hand-side we get
\[
\int \mathbf{1}_{\{s_{t+1} = 0\}} \left[ \lambda q_{t+1} f(s_{t+1}) f'(S^t) + \mu_{t+1}(S^{t+1}) \right] ds_{t+1} = \lambda q_{t+1} f'(S^t) + \int \mu_{t+1}(S^{t+1}) ds_{t+1} - \Phi = \lambda q_{t} f'(S^t) - \Phi,
\]
where the last step follows from the last line in (16) and
\[
\Phi := \int \mathbf{1}_{\{s_{t+1} = 0\}} \left[ \lambda q_{t+1} f(s_{t+1}) f'(S^t) + \mu_{t+1}(S^{t+1}) \right] ds_{t+1} = \lambda q_{t+1} f'(S^t) f(s_{t+1}) + \mu_{t+1}(S^{t+1}),
\]
\[
= \int \mathbf{1}_{\{s_{t+1} = 0\}} \lambda q_{t+1} f(s_{t+1}) ds_{t+1},
\]
\[
= \lambda q_{t+1} f'(S^t)(1 - \delta),
\]
\[
= \beta^{t+1} u_2'(c_{2,t+1}) f'(S^t)(1 - \delta),
\]
from (16).

Equating the expectations of both sides from (20) and (21) we have
\[
\frac{\beta^{t+1}}{p_{1,t+1}} u_1' \left( \frac{M_{t+1}(S^t)}{p_{1,t+1}} \right) \delta = \lambda q_{t} - \frac{\Phi}{f'(S^t)}.
\]
Substituting \( \Phi \) in the equation above we get
\[
\frac{\beta^{t+1}}{p_{1,t+1}} u_1' \left( \frac{M_{t+1}(S^t)}{p_{1,t+1}} \right) \delta = \lambda q_{t} - \frac{\beta^{t+1} u_2'(c_{2,t+1})}{p_{2,t+1}} (1 - \delta),
\]
or equivalently, since \( u_2'(c_{2,t+1}) = 1 \) for all \( t+1 \) and \( S^{t+1} \), we have
\[
\beta^{t+1} \left[ u_1' \left( \frac{M_{t+1}(S^t)}{p_{1,t+1}} \right) \frac{\delta}{p_{1,t+1}} + \frac{1 - \delta}{p_{2,t+1}} \right] = \lambda q_{t}.
\]
This implies that if \( s_{t+1} = 1 \), then \( c_{1,t+1}(S^{t+1}) = \frac{M_{t+1}(S^t)}{p_{1,t+1}} = \frac{M_{t+1}}{p_{1,t+1}} = c_{1,t+1} \) for all \( t \) and \( S^t \) and for all agents \( i \), because \( q_t \) is independent of \( S^t \). The distribution of money is degenerate because there are no wealth effects due to the linear disutility from producing credit goods. Households equally reach the same cash holdings by adjusting their labor supply \( h_2^i \). By market clearing, \( h_{2t}^F = \int h_{2t}^i di = c_2t \) where \( h_{2t}^i \)
satisfies the agents’ budget constraint.

Now substitute $\lambda q_t = \frac{\beta t u'_2(c_2)}{p_{2t}} = \frac{\beta t}{p_{2t}}$ from (16) and write the equation above as (17). Finally, from the firm’s problem, we have $\eta'(h_{1t}) = \frac{u_{1t}}{u_{2t}} = \frac{p_{1t}}{p_{2t}}$.

**Comparing notations in [7] and in our model**

In [7], $U(X)$ is the utility received from consuming $X$ CM goods ($u_2(c_2)$ in our notation). The technology to produce CM goods is linear and the disutility from labor is linear. In the DM, a portion $\alpha \sigma$ ($\delta$ in our notation) of agents desires to consume (but cannot produce) and an identical portion can produce but does not consume; $u(q)$ is the utility received from consuming $q$ DM goods ($u_1(c_1)$ in our notation); $c$ is the disutility from labor in the DM ($\eta$ in our notation); the nominal price is $\frac{d}{q}$ per unit of consumption ($p_1$ in our notation); the real price is $\frac{\phi d}{q}$, where $\phi$ is $\frac{1}{p_2}$ in our notation. With binding cash constraints $d = M$ and $\frac{\phi M}{q}$ where $M$ is the agent’s money holdings. We also have $\phi M \equiv z(q)$ where $0 < \theta \leq 1$ is the buyer’s bargaining power. The nominal interest rate is $i$ ($r$ in our notation).
1 A generalized version of the model

Here we study equilibrium when the model is generalized to account for the possibility of aggregate shocks and when the timing of opening of the two markets is reversed.

In a period, traders alternate synchronously between centralized and decentralized markets as in [10, p-21-22]. Each period is divided into two subperiods, say, morning and afternoon. A centralized market (CM) is open in the morning, while a decentralized market (DM) is open in the afternoon; trades in the second market are subject to a cash-in-advance constraint, as follows. To introduce money, it is assumed that some of the afternoon trades must be settled immediately with the exchange of money (= cash trades) while others can be settled in the subsequent market interaction (= credit trades), in the morning of the following period.

Let $s_t$ denote a shock at time $t$, where the realization (also denoted $s_t$) is public knowledge in period $t$. The shock is drawn from a time-invariant set. Let $\{s_t\}_{t=0}^{\infty}$ denote a path of shocks and for $t \geq 1$, $S^t = (s_1, ..., s_t) \in S^t$ a history of shocks, where $S^t$ is the set of all possible histories. Let $f^t(S^t)$ be the joint density of $S^t$ for $t \geq 1$. Letting $s_t = (s_{1t}, s_{2t})$, we have $s_{1t}$ as a shock in the CM and $s_{2t}$ as a shock.
in the DM of \( t \). For example, \( s_{1t} \) may be a shock to the money supply and \( s_{2t} \) may be a shock to TFP or to preferences. It is assumed that CM trades are carried out before \( s_{2t} \) is known. Therefore, \( S^{t} = (s_{1}, \ldots, s_{t}) \) is known in the DM of \( t \), but only \( S^{t-1} = (s_{1}, \ldots, s_{t-1}) \) and \( s_{1t} \) are known in the CM of \( t \). Hence, we use \( f_{1}^{t}(S^{t-1}, s_{1t}) \) to denote the density of \((S^{t-1}, s_{1t})\) and \( f_{2}(s_{2t}; S^{t-1}, s_{1t}) = f^{t}(S^{t})/f_{1}^{t}(S^{t-1}, s_{1t}) \) for the density of \( s_{2t} \) conditional on \((S^{t-1}, s_{1t})\).

Note that this maintains the key assumption that the random shock in the DM is not observed when money balances are chosen in the CM because \( s_{2t} \) is not known at the start of \( t \) (in the CM) but only in the middle of the period (in the DM).

Events on date \( t \) occur as follows.

**Morning of \( t \):** The CM is open. The shock \( s_{1t} \) is observed but not \( s_{2t} \). Credit trades executed on \( t - 1 \) are settled. Firms pay wages for work supplied on \( t - 1 \) and pay dividends out of profits made on \( t - 1 \). Households pay for credit goods bought on \( t - 1 \). The central bank retires the old money supply \( \tilde{M}_{t-1}(S^{t-2}, s_{1t-1}) \) and issues a new money supply \( \tilde{M}_{t}(S^{t-1}, s_{1t}) \) through lump-sum money transfers \( \theta_{t}(S^{t-1}, s_{1t}) \) to households. Trade on a financial market also takes place. In the financial market households trade state-contingent claims to money to be delivered in the CM of date \( t + 1 \). As a consequence of the above activities, the household exits the CM holding \( M_{t}(S^{t-1}, s_{1t}) \) money.

**Afternoon of \( t \):** The DM is open. Household and firms trade goods 1 and 2, and labor. Households buy \( c_{1t}(S^{t}) \) goods in exchange for money (= cash goods), buy \( c_{2t}(S^{t}) \) goods on credit (= credit goods) and supply \( h_{t}(S^{t}) \) labor to the firm on credit. The firm demands \( h_{t}^{F}(S^{t}) \) labor, buying it on credit, and supplies \( F(h_{t}^{F}(S^{t}); S^{t}) \) goods. Credit trades are settled in the CM of \( t + 1 \).
1.1 Firm’s optimal choices

On date $t$, given history $S^t$, the constraint of the firm is

$$F(h_t^F(S^t); S^t) = c_{1t}^F(S^t) + c_{2t}^F(S^t) \quad (1)$$

where $c_{1t}^F(S^t)$, $c_{2t}^F(S^t)$ and $h_t^F(S^t)$ are chosen in the DM of $t$, hence depend on $S^t$. Because cash and credit goods are distinct, for full generality let $p_{1t}(S^t)$ and $p_{2t}(S^t)$ denote the nominal spot price of goods 1 and 2 in the DM of $t$, and let $w_t(S^t)$ be the nominal spot wage in the DM of $t$. These nominal prices are contingent on the history of shocks $S^t$. Nominal profits (net dollar inflows) on $t$, given $S^t$, are

$$p_{1t}(S^t)c_{1t}^F(S^t) + p_{2t}(S^t)c_{2t}^F(S^t) - w_t(S^t)h_t^F(S^t), \quad (2)$$

which are distributed as dividends in the CM of $t + 1$.

Since the firm sells for cash and for credit in the DM of $t$, payments accrue as follows: in the CM of $t + 1$ it receives payments for credit sales in the DM of $t$; in the DM of $t$, it receives cash payments for contemporaneous cash-goods sales, which are carried into $t + 1$ as “overnight” balances. Now note that in the CM of $t + 1$ only the history $S^t$ and the shock $s_{1,t+1}$ are known (but not $s_{2,t+2}$). Let $q_{t+1}(S^t, s_{1,t+1})$ denote the date−0 price of a claim to one dollar delivered in the CM of $t + 1$, contingent on $(S^t, s_{1,t+1})$. The date−0 value of a dollar earned by the firm in the DM of $t$ is

$$\int q_{t+1}(S^t, s_{1,t+1})ds_{1,t+1},$$

because the dollar is paid out only on $t + 1$; this is simply the price of a claim to one dollar unconditionally delivered in the CM of $t + 1$.$^{15}$

In the CM of date 0 (= at the start of the economy) the firm chooses sequences

$$^{15}$$Equivalently, $\int q_{t+1}(S^t, s_{1,t+1})ds_{1,t+1}$ is the date-0 price of a state-contingent bond that delivers one dollar in the CM of date $t + 1$, conditional on $S^t$. 


of output and labor to solve:

Maximize: \[
\sum_{t=0}^{\infty} \int \int \int q_{t+1}(S^t, s_{1,t+1}) ds_{1,t+1} \{ p_{1t}(S^t) c_{1t}^F(S^t) \\
+ p_{2t}(S^t) c_{2t}^F(S^t) - w_t(S^t) h_t^F(S^t) \} dS^t ds_{20}
\]

subject to: \[
c_{1t}^F(S^t) + c_{2t}^F(S^t) = F(h_t^F(S^t); S^t) \text{ on all } t, S^t.
\]

Substituting for \(c_{1t}^F(S^t)\) from the constraint, the FOCs are

\[
h_t^F(S^t) : \quad p_{1t}(S^t) F'(h_t^F(S^t); S^t) - w_t(S^t) = 0 \quad \text{for all } t, S^t, s_{20}
\]

\[
c_{2t}^F(S^t) : \quad p_{1t}(S^t) - p_{2t}(S^t) = 0 \quad \text{for all } t, S^t, s_{20}.
\]

Consequently, \(p_{1t}(S^t) = p_{2t}(S^t) = p_t(S^t)\) and

\[
p_t(S^t) F'(h_t^F(S^t); S^t) = w_t(S^t) \quad \text{for all } t, S^t, s_{20}.
\]

### 1.2 Households’ optimal choices

Households choose consumption of cash and credit goods, and labor effort in the DM of \(t\) after observing the shock \(s_t = (s_{1t}, s_{2t})\). Consumption on \(t\) is thus conditional on the history \(S^t = (s_1, s_2, \ldots, s_t)\).

Below we solve the problem faced by an agent who contracts on date 0.\(^{16}\) On date 0 households maximize expected utility

\[
\sum_{t=0}^{\infty} \beta^t \int \int U(c_{1t}(S^t), c_{2t}(S^t), h_t(S^t)) f_t^t(S^t) dS^t f_0^t(s_{20}) ds_{20}
\]

subject to a cash in advance constraint for good 1

\[
p_t(S^t) c_{1t}(S^t) \leq M_t(S^{t-1}, s_{1t}) \quad \text{for all } t \text{ and } S^t,
\]

where we have used \(p_{1t}(S^t) = p_{2t}(S^t) = p_t(S^t)\). Here \(M_t(S^{t-1}, s_{1t})\) denotes money holdings in the DM of \(t\), which are bought in the CM of \(t\), when \(s_{2t}\) is not yet known. Given this uncertainty, money may be held for transactions purposes and for

\(^{16}\)The formulation with sequential contracting is available from the authors upon request.
precautionary reasons.

The households date−0 budget constraint in nominal prices is

\[
\sum_{t=0}^{\infty} \int \int \left\{ \int q_{t+1}(S^t, s_{1,t+1}) ds_{1,t+1} \left[ p_t(S^t) \left( c_{1t}(S^t) + c_{2t}(S^t) \right) - M_t(S^{t-1}, s_{1t}) \right] 
- w_t(S^t) h_t(S^t) \right\} ds^t ds_20 \leq \Pi + \bar{M}
\]

The left hand side is the date−0 present value of net expenditure. The right hand side lists the date−0 sources of funds: \( \bar{M} \) initial money holdings (=initial liabilities of the central bank) and the nominal value of the firm, \( \Pi \). The date−0 present value of net expenditure is calculated using the price of money delivered at the start of \( t \), i.e., in the CM. Net expenditure consists of two elements:

1. **Net expenditure in the CM of \( t \):** the agent exits the CM holding \( M_t(S^{t-1}, s_{1t}) \) money balances that have date−0 value \( q_t(S^{t-1}, s_{1t}) \). Since the agent receives the transfer \( \theta_t(S^{t-1}, s_{1t}) \) in the CM of \( t \), the agent spends \( M_t(S^{t-1}, s_{1t}) - \theta_t(S^{t-1}, s_{1t}) \) in the CM of \( t \).

2. **Net expenditures in the DM of \( t \):** the agent earns \( w_t(S^t) h_t(S^t) \) wages during \( t \), which are paid on \( t + 1 \); \( M_t(S^{t-1}, s_{1t}) - p_t(S^t) c_{1t}(S^t) \) money balances are not spent on \( t \) and are carried over to \( t + 1 \); \( p_t(S^t) c_{2t}(S^t) \) is the expenditure on credit goods on \( t \), paid on \( t + 1 \). These funds are available in the CM of \( t + 1 \); hence, their date-0 value is \( \int q_{t+1}(S^t, s_{1,t+1}) ds_{1,t+1} \).

The date-0 value of net expenditure depends on the initial shock \( s_{20} \) and the history of shocks \( S^t \); hence the double integral.

Consumers choose \( c_{1t}(S^t), c_{2t}(S^t), h_t(S^t) \), and \( M_t(S^{t-1}, s_{1t}) \) for all \( t \) and \( S^t \) to
maximize the Lagrangian:

\[ L := \sum_{t=0}^{\infty} \beta^t \int U(c_{1t}(S^t), c_{2t}(S^t), h_t(S^t)) f' (S^t) dS^t f_2^0 (s_{2t}) ds_{2t} + \lambda (\Pi + \bar{M}) \]

- \lambda \sum_{t=0}^{\infty} \int \{ \int q_{t+1}(S^t, s_{1,t+1}) ds_{1,t+1} [p_t(S^t) (c_{1t}(S^t) + c_{2t}(S^t)) - M_t(S^{t-1}, s_{1t})] - w_t(S^t) h_t(S^t) \} dS^t ds_{2t}

\[ + \sum_{t=0}^{\infty} \int \mu_t(S^t) [M_t(S^{t-1}, s_{1t}) - p_t(S^t) c_{1t}(S^t)] dS^t ds_{2t} \]  \hspace{1cm} (5)

where \( \mu_t(S^t) \) is the Kühn-Tucker multiplier on the cash in advance constraint on \( t \), given history \( S^t \).

Omitting the arguments from \( U \) and \( f \) where understood, taking the partial of \( L \) relative to \( c_{1t}(S^t), c_{2t}(S^t), h_t(S^t) \) and \( M_t(S^{t-1}, s_{1t}) \), in an interior optimum we have respectively:

\[ \beta^t U_1 f' f_2^0 - \lambda p_t(S^t) \int q_{t+1}(S^t, s_{1,t+1}) ds_{1,t+1} - \mu_t(S^t) p_t(S^t) = 0 \]

with \( p_t(S^t) c_{1t}(S^t) \leq M_t(S^{t-1}, s_{1t}) \)

\[ \beta^t U_2 f' f_2^0 - \lambda p_t(S^t) \int q_{t+1}(S^t, s_{1,t+1}) ds_{1,t+1} = 0 \]  \hspace{1cm} (6)

\[ \beta^t U_3 f' f_2^0 + \lambda w_t(S^t) \int q_{t+1}(S^t, s_{1,t+1}) ds_{1,t+1} = 0 \]

\[ \int \{ \lambda [\int q_{t+1}(S^t, s_{1,t+1}) ds_{1,t+1} - q_t(S^{t-1}, s_{1t})] + \mu_t(S^t) \} ds_{2t} = 0. \]

All but the last expression are valid for all \( t, s_{2t}, S^t \), while the last expression is valid for all \( t, s_{20}, s_{1t}, S^{t-1} \). The last expression holds because \( M_t(S^{t-1}, s_{1t}) > 0 \).

Noticing that \( \int q_t(S^{t-1}, s_{1t}) ds_{2t} = q_t(S^{t-1}, s_{1t}) \) we rewrite the last equation in (6) as

\[ \lambda \int q_{t+1}(S^t, s_{1,t+1}) ds_{1,t+1} ds_{2t} - \lambda q_t(S^{t-1}, s_{1t}) + \int \mu_t(S^t) ds_{2t} = 0 \]

In a representative agent setting, given market clearing \( h_t = h_t^F \), from (4) and (6) we get

\[ \frac{U_3}{U_2} = F'(h_t(S^t); S^t) \]  \hspace{1cm} for all \( t, s_{20}, S^t \).
The marginal rate of substitution between goods 1 and 2 is
\[
\frac{U_1}{U_2} = \frac{\lambda \int q_{t+1}(S^t, s_{1,t+1})ds_{1,t+1} + \mu_t(S^t)}{\lambda \int q_{t+1}(S^t, s_{1,t+1})ds_{1,t+1}} \tag{7}
\]
for all \(t, s_{20}, S^t\).

As a means of comparison, we later report the main equations for the complete-markets model without money.

### 1.3 Monetary policy

Let \(\bar{M} \geq 0\) denote initial money balances, on date 0. In the CM of each period \(t\), the central bank issues a new money supply \(\bar{M}_t(S^{t-1}, s_{1t})\) using lump-sum transfers \(\theta_t(S^{t-1}, s_{1t})\). Hence, the money supply process can be history-dependent, but is not dependent on \(s_{2t}\), because that shock is not observed until the DM opens in \(t\). In the CM of \(t+1\), the central bank retires the money supply issued on \(t\), and issues a new money supply. The date-0 value of assets held by the central bank in the CM of \(t\) is \(\bar{M}_t(S^{t-1}, s_{1t})q_t(S^{t-1}, s_{1t})\). The liabilities include the lump-sum transfers \(\theta_t(S^{t-1}, s_{1t})q_t(S^{t-1}, s_{1t})\) and the cost of retiring the money supply on \(t+1\), which is \(\bar{M}_t(S^{t-1}, s_{1t}) \int \int q_{t+1}(S^t, s_{1,t+1})ds_{1,t+1}ds_{2t}\), given the uncertainty on \(s_{2t}\) and \(s_{1,t+1}\).

The intertemporal budget constraint of the central bank is thus
\[
\bar{M} = \sum_{t=0}^{\infty} \int \{ \bar{M}_t(S^{t-1}, s_{1t}) \left[q_t(S^{t-1}, s_{1t}) - \int q_{t+1}(S^t, s_{1,t+1})ds_{1,t+1}ds_{2t}\right] \\
- \theta_t(S^{t-1}, s_{1t})q_t(S^{t-1}, s_{1t}) \} d(S^{t-1}, s_{1,t}),
\]
which can be rewritten as a set of flow constraints (see later section)
\[
\bar{M}_t(S^{t-1}, s_{1t}) - \bar{M}_{t-1}(S^{t-2}, s_{1,t-1}) = \theta_t(S^{t-1}, s_{1t}). \tag{8}
\]
That is, each money injection equals the lump-sum-transfer in that period.

To see how to derive the flow constraint in (8) note that, on any date \(\tau > 0\), there
are $\bar{M}_{t-1}$ outstanding liabilities, hence we have

$$q_t(S^{t-1}, s_{1t}) \bar{M}_{t-1}(S^{t-2}, s_{1,t-1}) = \sum_{\tau=t}^{\infty} \int \{ \bar{M}_\tau(S^{\tau-1}, s_{1\tau})[q_\tau(S^{\tau-1}, s_{1\tau})$$

$$- \int \int q_{\tau+1}(S^{\tau}, s_{1,\tau+1}) ds_{1,\tau+1} ds_{2\tau}]$$

$$- \theta_\tau(S^{\tau-1}, s_{1\tau}) q_\tau(S^{\tau-1}, s_{1\tau}) \} d(S^{\tau-1}, s_{1\tau}),$$

and updating this expression we have

$$q_{t+1}(S^t, s_{1,t+1}) \bar{M}_t(S^{t-1}, s_{1t}) = \sum_{\tau=t+1}^{\infty} \int \{ \bar{M}_\tau(S^{\tau-1}, s_{1\tau})[q_\tau(S^{\tau-1}, s_{1\tau})$$

$$- \int \int q_{\tau+1}(S^{\tau}, s_{1,\tau+1}) ds_{1,\tau+1} ds_{2\tau}]$$

$$- \theta_\tau(S^{\tau-1}, s_{1\tau}) q_\tau(S^{\tau-1}, s_{1\tau}) \} d(S^{\tau-1}, s_{1\tau}).$$

To derive a flow constraint for the central bank we now fix $\bar{M}_t(S^{t-1}, s_{1t})$, which means that the shocks $(S^{t-1}, s_{1t})$ are known. Take the difference between (9) and (10). The right hand side of (9) minus the right hand side of (10) is

$$\bar{M}_t(S^{t-1}, s_{1t})[q_t(S^{t-1}, s_{1t}) - \int \int q_{t+1}(S^t, s_{1,t+1}) ds_{1,t+1} ds_{2t}] - \Theta_t(S^{t-1}, s_{1t}) q_t(S^{t-1}, s_{1t}).$$

The difference of the left hand sides is

$$q_t(S^{t-1}, s_{1t}) \bar{M}_{t-1}(S^{t-2}, s_{1,t-1}) - q_{t+1}(S^t, s_{1,t+1}) \bar{M}_t(S^{t-1}, s_{1t}).$$

Equating the differences in LHS and RHS, and integrating both sides with respect to $s_{1,t+1}$ and $s_{2t}$, we obtain the flow constraint of the Central Bank

$$q_t(S^{t-1}, s_{1t}) \left[ \bar{M}_t(S^{t-1}, s_{1t}) - \bar{M}_{t-1}(S^{t-2}, s_{1,t-1}) \right] = q_t(S^{t-1}, s_{1t}) \Theta_t(S^{t-1}, s_{1t}).$$

That is, each money injection equals the lump-sum-transfer in that period.
2 The model with complete markets

Here we study the model just presented, when no cash in advance constraint is imposed. In a representative agent setting, given a history of shocks $S^t$ and $h_t(S^t)$ labor, feasibility implies

$$F(h_t(S^t); S^t) = c_t(S^t) \text{ for all } t, S^t, \quad (11)$$

where $c_t(S^t) \geq 0$ denotes the amount of the good consumed on $t$, given $S^t$.

Suppose a complete market of state- and date-contingent claims opens on date $t = 0$. Let $\pi_t(S^t)$ denote the period-0 price of a claim to one good delivered on $t \geq 1$ contingent on history $S^t$. The spot price of the good on date 0 is $\pi_0(s_0) = 1$ for all $s_0$, i.e., it is the numeraire good. Let $\pi_{ht}(S^t)$ denote state-contingent wages, i.e., the price of a claim to a unit of labor on date $t \geq 1$ conditional on history $S^t$. Let $h_t(S^t) \in [0, L]$ denote the amount of labor supplied on $t$. The date-0 value of the consumption plan $\{c_t(S^t)\}_{all \ t, S^t}$ is thus $\sum_{t=0}^{\infty} \int_{S^t} [\pi_t(S^t)c_t(S^t) - \pi_{ht}(S^t)h_t(S^t)]dS^t$ and the value of the labor plan $\{h_t(S^t)\}_{all \ t, S^t}$ is $\sum_{t=0}^{\infty} \int_{S^t} \pi_{ht}(S^t)h_t(S^t)dS^t$.

Let $c^F_t(S^t)$ and $h^F_t(S^t)$ denote the firm’s choices of output and labor on date $t$, contingent on history $S^t$. On date 0 the firm chooses a plan $\{c^F_t(S^t), h^F_t(S^t)\}_{all \ t, S^t}$ given the state-contingent prices $\{\pi_t(S^t), \pi_{ht}(S^t)\}_{all \ t, S^t}$ to solve

$$\begin{align*}
\text{maximize:} & \quad \sum_{t=0}^{\infty} \int_{S^t} [\pi_t(S^t)c^F_t(S^t) - \pi_{ht}(S^t)h^F_t(S^t)]dS^t \\
\text{subject to:} & \quad F(h^F_t(S^t); S^t) = c^F_t(S^t) \text{ for all } t, S^t. \quad (12)
\end{align*}$$

Since there is no capital, this is equivalent to solving a series of static maximization problems. Substituting for the constraint, the optimal choice $h^F_t(S^t)$ must satisfy

$$\pi_t(S^t)F'(h^F_t(S^t); S^t) - \pi_{ht}(S^t) = 0 \quad \text{for all } t, S^t. \quad (13)$$

Let $U(c_t(S^t), h_t(S^t))$ denote the period utility of an agent who, on date $t$, consumes $c_t(S^t)$ goods and supplies $h_t(S^t)$ labor. $U$ satisfies the expected utility property and is monotone in both arguments, strictly increasing in the first and decreasing in the
second. On date 0, agents choose consumption and labor plans given state-contingent prices to solve

maximize: \[ \sum_{t=0}^{\infty} \beta^t \int U(c_t(S^t), h_t(S^t)) f^t(S^t) dS^t \]
subject to: \[ \sum_{t=0}^{\infty} \int \{ \pi_t(S^t)c_t(S^t) - \pi_{ht}(S^t)h_t(S^t) \} dS^t = A, \]
where \( A \geq 0 \) denotes the maximized value of the firm’s objective function (the value of the firm on date 0). The constraint implies that the net present value of planned expenditure (consumption minus labor income) must equal the value of the firm on date 0. Let \( \lambda \) be the Lagrange multiplier on the constraint and let \( U_i = \frac{\partial U(x)}{\partial x_i} \) for \( x = (x_i)_{i \in I} \). The first order conditions in an interior optimum are

\[
\begin{align*}
   c_t(S^t) : & \quad \beta^t U_c(c_t(S^t), h_t(S^t)) f^t(S^t) - \lambda \pi_t(S^t) = 0, \quad \text{for all } t, S^t \\
   h_t(S^t) : & \quad \beta^t U_h(c_t(S^t), h_t(S^t)) f^t(S^t) - \lambda \pi_{ht}(S^t) = 0, \quad \text{for all } t, S^t.
\end{align*}
\]

The equations in (13) and (14), and the relevant constraints implicitly define the set of stochastic processes for quantities and prices that support a competitive equilibrium. Equilibrium relative prices satisfy

\[ \frac{\pi_{ht}(S^t)}{\pi_t(S^t)} = \frac{U_h(c_t(S^t), h_t(S^t))}{U_c(c_t(S^t), h_t(S^t))} \quad \text{for all } t, S^t. \]

If there is no heterogeneity in labor plans, then labor market clearing requires \( h^F_t(S^t) = h_t(S^t) \). From (13), it follows that equilibrium quantities must satisfy

\[ \frac{U_h(c_t(S^t), h_t(S^t))}{U_c(c_t(S^t), h_t(S^t))} = F'(h_t(S^t); S^t) \quad \text{for all } t, S^t. \]

Using (14), the relative price for claims to goods delivered on date \( \tau > t \) is

\[ \frac{\pi_{\tau}(S^\tau)}{\pi_t(S^t)} = \beta^{\tau-t} \frac{U_c(c_\tau(S^\tau), h_\tau(S^\tau)) f_\tau(S^\tau)}{U_c(c_t(S^t), h_t(S^t)) f^t(S^t)} \quad \text{for all } t, \tau, S^t, S^\tau. \]

Given a sequence of shocks \( \{s_t\}_{t=0}^\infty \), equation (15) and feasibility (11) define the equilibrium allocation \( \{c_t(S^t), h_t(S^t)\}_{t=0}^\infty \).