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Abstract

This article examines monetary policy when it is constrained by the zero lower bound (ZLB) on the nominal interest rate. Our analysis uses a nonlinear New Keynesian model with technology and discount factor shocks that accounts for the expectational effects of falling to and remaining at the ZLB. Specifically, we investigate why technology shocks may have unconventional effects at the ZLB, what factors affect the likelihood of hitting the ZLB, and the tradeoffs a central bank faces under a dual mandate. We initially focus on the New Keynesian model without capital (Model 1) but then study the model with capital (Model 2). The advantage of including capital is that it introduces another mechanism for intertemporal substitution that strengthens the expectational effects of the ZLB. Three main findings emerge: (1) In Model 1, the output gap specification in the Taylor rule may reverse the effects of technology shocks at the ZLB; (2) When the central bank targets the steady-state output gap in Model 2, a positive technology shock at the ZLB leads to more pronounced unconventional dynamics than in Model 1; (3) In Model 1, the constrained linear solution provides a decent approximation of the nonlinear solution, but meaningful differences exist between the solutions in Model 2.

Keywords: Monetary Policy; Zero Lower Bound; Nonlinear Solution Method; Capital

JEL Classifications: E31; E42; E58; E61

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1 INTRODUCTION

In the aftermath of the financial crisis, aggregate demand fell sharply. The Fed quickly responded by lowering its policy rate to its zero lower bound (ZLB) by the end of 2008. Five years after the crisis began, the Fed’s target interest rate remains near zero and the economy is below potential. Figure 1 shows the U.S. and Japanese interbank lending rate and employment-to-population percentage from 1992-2013. The U.S. policy rate (solid line) has varied between 6.5% and 0% since 1992 and has been held below 25 basis points since the end of 2008. During this period, policymakers shifted their focus from inflation to the real economy, since the inflation rate has been at or below the Fed’s inflation target. The Bank of Japan sharply lowered its policy rate in 1992 (dashed line), reaching 50 basis points in 1995. Since then it has remained between 0 and 50 basis points, while the employment-to-population percentage steadily fell from 62% to about 57.5%. The Japanese economy slightly rebounded in the mid-2000s, but following the financial crisis, the policy rate was cut and the employment-to-population percentage fell further.

Over the last two decades, the Japanese economy has endured anemic growth in real GDP and slight deflation. Their experience generated a significant amount of research on the effects of the Bank of Japan’s zero interest rate policy [Braun and Waki (2006); Eggertsson and Woodford (2003); Hoshi and Kashyap (2000); Ito and Mishkin (2006); Krugman (1998); Posen (1998)]. Many arguments for avoiding the ZLB are motivated, in part, by the recent Japanese experience.

This article examines monetary policy when it is constrained by the zero lower bound (ZLB) on the nominal interest rate. Our analysis uses a nonlinear New Keynesian model with technology and discount factor shocks that accounts for the expectational effects of falling to and remaining at the ZLB. Discount factor shocks are a proxy for changes in demand that occurred during the Great Recession, while technology shocks account for changes in supply. Specifically, we investigate why technology shocks may have unconventional effects at the ZLB, what factors affect the likelihood of hitting the ZLB, and the tradeoffs a central bank faces under a dual mandate.

We initially focus on the New Keynesian model without capital but then study the model with capital to draw comparisons. In the model without capital, positive technology shocks may have unconventional effects at the ZLB. We find that whether or not those unconventional effects occur
depends on which output gap the central bank targets in its policy rule. When the central bank aggressively targets the steady-state output gap (the deviation of output from its steady state), positive technology shocks can cause output to decline at the ZLB. Those unconventional dynamics, however, nearly disappear when the central bank targets the potential output gap (the deviation of output from its potential), where potential output equals the output produced in our model with flexible prices. In this case, large technology shocks do not reduce output at the ZLB.\footnote{Wieland (2014) uses structural VAR evidence to argue that these unconventional dynamics did not occur following the 2011 earthquake/tsunami in Japan or the recent oil supply shocks. Braun and Waki (2006) show that technology shocks yield unconventional dynamics at the ZLB in a log-linearized model with capital accumulation where monetary policy responds to a steady-state output gap. Using a nonlinear model with capital, Braun and Körber (2011) show that these unconventional dynamics may disappear if the expected duration at the ZLB is short enough.}

We focus primarily on the case where the central bank targets the steady-state output gap because policymakers, in the short-to-medium term, assume potential output grows at a relatively constant rate [Basu and Fernald (2009)]. Potential output measures are often revised following incoming information about shocks, but the revisions occur well after the temporary economic effects from sticky prices have dissipated. Furthermore, Orphanides and van Norden (2002) and Orphanides (2003a,b) document that historically neither the Fed nor standard statistical methods have been able to detect changes in potential output until well after they have occurred.

Much of the literature on the ZLB considers models without capital.\footnote{A notable exception is Christiano (2004), which generalizes Eggertsson and Woodford (2003) to include capital.} Capital, however, provides households with another margin to smooth consumption, which strengthens the expectational effects of the ZLB and impacts the model’s dynamics. Arbitrage implies the real interest rate equals the expected future real rental rate of capital. The large decline in demand when the ZLB binds leads to a sharp reduction in the rental rate of capital. Thus, the household places increasing weight on a lower rental rate as the policy rate approaches the ZLB, which pushes down the real interest rate. We also include capital adjustment costs to dampen the volatility of investment. Specifically, investment is less attractive as a consumption smoothing mechanism, which causes a greater reduction in consumption and a larger increase in the real interest rate at the ZLB. When the central bank targets the steady-state output gap in our model with capital, a positive technology shock at the ZLB produces more pronounced unconventional dynamics than in the model without capital.

We also evaluate how alternative versions of the Taylor rule affect the likelihood of encountering the ZLB and the efficacy of stabilization policy. A policy rule based on a dual mandate is more likely to cause ZLB events when the central bank places greater emphasis on the steady-state output gap in our model without capital. The opposite result occurs when the central bank emphasizes the potential output gap.\footnote{Several papers discuss optimal policy with a ZLB constraint and provide analysis of the welfare losses at the ZLB [Eggertsson and Woodford (2003); Günter et al. (2004); Jung et al. (2005); Nakov (2008); Werning (2011)]. For example, Adam and Billi (2006) find that it is optimal to reduce the nominal interest rate more aggressively in response to adverse shocks than in models without a ZLB constraint, despite the welfare consequences that occur at the ZLB.} When our two models are compared with only discount factor shocks, we find that a large weight on the steady-state output gap in the policy rule decreases the frequency of ZLB events in our model without capital but increases the frequency in our model with capital.

Any analysis of the ZLB is complicated by the kink in the monetary policy rule that occurs at a zero nominal interest rate. The literature has used a variety of techniques to address this problem. Many papers separate the problem into pre- and post-ZLB periods [e.g., Braun and Körber (2011); Braun and Waki (2006); Christiano et al. (2011); Eggertsson and Woodford (2003); Erceg and Lindé (2014); Gertler and Karadi (2011)]. With this approach, a specific sequence of shocks pushes...
the nominal interest rate to zero. Each period, there is then some positive probability the nominal interest rate exits the ZLB. Once this happens, it cannot return to the ZLB. These simplifying assumptions are made for computational tractability. The drawback is that if a shock causes the ZLB to bind in one period, the same shock would not cause the ZLB to bind in a future period.

Most studies of the ZLB linearize all of their equations with an exception of the monetary policy rule around their non-stochastic steady states. Such a procedure, however, can generate approximation errors. Braun et al. (2012) and Fernández-Villaverde et al. (2012) provide examples of the mistakes resulting from linearized models without capital evaluated at the ZLB. Braun et al. (2012) also argue that linearized models often lead to incorrect inferences about existence and uniqueness of the equilibrium and the local dynamics of the model. Our findings indicate the constrained linear model provides a good approximation of the nonlinear model without capital, but the errors are much larger in a model with capital. Thus, the simulated moments and model predictions are very different in the linearized model with capital than in the nonlinear model.

Our paper avoids the problems from simplifying the model by obtaining the nonlinear solution to standard New Keynesian models that include an occasionally binding ZLB constraint on the nominal interest rate. Rather than focus on specific sequences of shocks, we calculate the solution for all combinations of discount factor and technology shocks and provide a thorough explanation of how dynamics change across the entire state space. Our solution method emphasizes accuracy in order to capture important expectational effects of going to and returning from the ZLB.

The paper proceeds as follows. Section 2 outlines our models with and without capital. Section 3 describes the calibration and solution method, and sections 4 through 6 present the results. These sections report the model solutions across all technology and discount factor shocks, the dynamics at the ZLB, and the likelihood of hitting the ZLB. We also explain how the monetary policy rule impacts these results and provide a comparison between the New Keynesian models with and without capital. Lastly, we present new evidence that the solutions to the constrained linear and nonlinear models are significantly different in the model with capital. Section 7 concludes.

2 Economic Models

This section presents two New Keynesian models with Rotemberg (1982) price adjustment costs. Both models assume stochastic processes for the discount factor and technology, but they differ in their treatment of capital. That is, Model 1 does not include capital while Model 2 does.

2.1 Model 1: Baseline A representative household chooses \( \{c_t, n_t, b_t\}_{t=0}^{\infty} \) to maximize expected lifetime utility, given by, \( E_0 \sum_{t=0}^{\infty} \tilde{\beta}_t \left[ \log c_t - \chi n_t^{1+\eta}/(1 + \eta) \right] \), where \( 1/\eta \) is the Frisch elasticity of labor supply, \( c_t \) is consumption of the final good, \( n_t \) is labor hours, \( E_0 \) is an expectation operator conditional on information available in period 0, \( \tilde{\beta}_0 \equiv 1 \), and \( \tilde{\beta}_t = \prod_{j=1}^{t} \beta_j \) for \( t > 0 \). \( \beta_j \) is a time-varying subjective discount factor that evolves according to

\[
\beta_j = \tilde{\beta} (\beta_{j-1}/\tilde{\beta})^{\rho_\beta} \exp(\varepsilon_j),
\]

\footnote{Braun and Waki (2010) show that the approximation error in a perfect-foresight version of a linear model with capital where monetary policy does not respond to an output gap overstates the government spending multiplier.}

\footnote{Several recent papers study the ZLB using nonlinear solution methods. Fernández-Villaverde et al. (2012) calculates the probabilities ZLB events. Wolman (2005) shows the real effects of the ZLB depend on the policy rule and nominal rigidities. Gust et al. (2013) estimates the extent to which the ZLB constrained the central bank. Aruoba and Schorfheide (2013) and Mertens and Ravn (2014) show how the ZLB affects fiscal multipliers and Basu and Bundick (2012) and Nakata (2012) show the ZLB magnifies the effect of uncertainty on aggregate demand.}
where $\bar{\beta}$ is the steady-state discount factor, $0 \leq \rho_\beta < 1$, and $\epsilon_j \sim N(0, \sigma^2_\epsilon)$. These choices are constrained by $c_t + b_t = w_t n_t + r_t - b_{t-1}/\pi_t + d_t$, where $\pi_t = p_t/p_{t-1}$ is the gross inflation rate, $w_t$ is the real wage rate, $b_t$ is a 1-period real bond, $r_t$ is the gross nominal interest rate, and $d_t$ are profits from intermediate firms. The optimality conditions to the household’s problem imply

$$w_t = \chi n_t^\beta c_t,$$

$$1 = r_t E_t [\beta_{t+1} (c_t/c_{t+1})/\pi_{t+1}].$$

The production sector consists of monopolistically competitive intermediate goods firms who produce a continuum of differentiated inputs and a representative final goods firm. Each firm $f \in [0, 1]$ in the intermediate goods sector produces a differentiated good, $y_t(f)$, with identical technologies given by $y_t(f) = z_t n_t(f)$, where $n_t(f)$ is the level of employment used by firm $f$. $z_t$ represents the level of technology, which is common across firms and follows

$$z_t = \bar{z} (z_{t-1}/\bar{z})^{\rho_t} \exp(v_t),$$

where $\bar{z}$ is steady-state technology, $0 \leq \rho_z < 1$, and $v_t \sim N(0, \sigma^2_v)$. Each intermediate firm chooses its labor supply to minimize its operating costs, $w_t n_t(f)$, subject to its production function. The final goods firm purchases $y_t(f)$ units from each intermediate goods firm to produce the final good, $y_t \equiv \left[\int_{0}^{1} y_t(f) (\theta-1)/\theta \, df\right]^{\theta/(\theta-1)}$ according to a Dixit and Stiglitz (1977) aggregator, where $\theta > 1$ measures the elasticity of substitution between the intermediate goods. The optimality condition to the firm’s profit maximization problem then yields the demand function for intermediate inputs given by $y_t(f) = \left(p_t(f)/p_t\right)^{-\theta} y_t$, where $p_t = \left[\int_{0}^{1} p_t(f)^{1-\theta} \, df\right]^{1/(1-\theta)}$ is the price of the final good.

Following Rotemberg (1982), each firm faces a cost to adjusting its price, $adj_t(f)$, which emphasizes the negative effect that price changes can have on customer-firm relationships. Using the functional form in Ireland (1997), $adj_t(f) = \varphi [p_t(f)/(\bar{\pi} p_{t-1}(f)) - 1] y_t/2$, the real profits of firm $f$ are $d_t(f) = \left[p_t(f)/p_t y_t(f) - (w_t n_t(f) + adj_t(f))\right]$, where $\varphi \geq 0$ controls the size of the adjustment costs and $\bar{\pi}$ is the steady-state gross inflation rate. Each intermediate goods firm chooses its price, $p_t(f)$, to maximize the expected discounted present value of real profits $E_t \sum_{k=t}^{\infty} \lambda_{t,k} d_k(f)$, where $\lambda_{t,t} \equiv 1$, $\lambda_{t,t+1} = \beta_{t+1} (c_t/c_{t+1})$ is the stochastic pricing kernel between periods $t$ and $t + 1$ and $\lambda_{t,k} \equiv \prod_{j=t+1}^{k} \lambda_{j-1,j}$. In a symmetric equilibrium, all intermediate goods firms make the same decisions and the optimality condition reduces to

$$\varphi \left(\frac{\bar{\pi}_t}{\bar{\pi}} - 1\right) \frac{\bar{\pi}_t}{\bar{\pi}} = (1 - \theta) + \theta \Psi_t + \varphi E_t \left[\lambda_{t,t+1} \left(\frac{\bar{\pi}_{t+1}}{\bar{\pi}} - 1\right) \frac{\bar{\pi}_{t+1}}{\bar{\pi}} \frac{y_{t+1}}{y_t}\right],$$

where $\Psi_t = w_t/z_t$ is the real marginal cost. In the absence of price adjustment costs (i.e., $\varphi = 0$), $\Psi_t = (\theta - 1)/\theta$, which is the inverse of a firm’s markup of price over marginal cost.

Each period, the central bank sets the gross nominal interest rate according to

$$r_t = \max \{1, \bar{r} (\pi_t/\pi^*)^{\phi_\pi} (y_t/y_t^*)^{\phi_y}\},$$

where $\pi^* = \bar{\pi}$ is the inflation rate target and $\phi_\pi$ and $\phi_y$ are the policy responses to inflation and output.\(^6\) In this paper, the output gap is either defined as the deviation of output from its steady

\(^6\)Although the policy rate cannot fall below zero, the same dynamics would occur if it is a small but positive value. The key point is that a lower bound exists, which prevents the Fed from adjusting the policy rate to adverse shocks.
are calibrated at a quarterly frequency and the parameters are given in Model 2 adds capital accumulation to Model 1. The quantities \( \{c_t, n_t, b_t, y_t\}_{t=0}^{\infty} \), prices \( \{w_t, r_t, \pi_t\}_{t=0}^{\infty} \), and exogenous variables \( \{\beta_t, z_t\}_{t=0}^{\infty} \) that satisfy the household’s and firm’s optimality conditions [(2), (3), (5)], the production function \( y_t = z_t n_t \), the monetary policy rule [(6)], the stochastic processes [(1), (4)], the bond market clearing condition \( b_t = 0 \), and the budget constraint.

2.2 Model 2: Basic with Capital

Model 2 adds capital accumulation to Model 1. The household chooses sequences \( \{c_t, i_t, n_t, b_t\}_{t=0}^{\infty} \) to maximize the preferences in Model 1 subject to

\[
c_t + i_t + \Phi(i_t/k_{t-1})k_{t-1} + b_t = w_t n_t + r_t^F k_{t-1} + r_{t-1} b_{t-1}/\pi_t + d_t, \tag{7}
\]

where \( i_t \) is investment, \( k_t \) is the capital stock, \( r_t^F \) is the real capital rental rate, and \( \Phi(\cdot) \) is a positive, increasing, and convex function that measures the cost of adjusting the capital stock. We assume \( \Phi(x) = \phi(x - \delta)^2/2 \), where \( \phi \) controls the size of the adjustment cost. While other papers utilize alternative specifications of capital/investment adjustment costs, we use this specification because it does not add another state variable to our model, which allows us to present the complete model solution. Optimality yields an equation for Tobin’s \( q \) and a consumption Euler equation, given by,

\[
q_t = 1 + \phi(i_t/k_{t-1} - \delta), \tag{9}
\]

\[
q_t = E_t \left[ \beta_{t+1} c_t/c_{t+1} \left( r_t^F k_{t-1}^{\delta} - \frac{\phi}{2} \left( \frac{i_{t+1}}{k_{t+1}} - \delta \right)^2 + \phi \left( \frac{i_{t+1}}{k_{t+1}} - \delta \right) \frac{i_{t+1}}{k_{t+1}} + (1 - \delta) q_{t+1} \right) \right]. \tag{10}
\]

Each firm \( f \in [0, 1] \) in the intermediate goods sector produces a differentiated good, \( y_t(f) \), with identical technologies given by

\[
y_t(f) = z_t k_t^{-1}(f)^{\alpha} n_t(f)^{1-\alpha}, \tag{11}
\]

where \( k_t(f) \) and \( n_t(f) \) are the levels of capital and employment used by firm \( i \). Every intermediate firm then chooses its inputs to minimize its operating costs, \( r_t^F k_{t-1}(f) + w_t n_t(f) \), subject to its production function, which yields a consolidated optimality condition, given by,

\[
\alpha w_t n_t = (1 - \alpha) r_t^F k_{t-1}. \tag{11}
\]

The firm pricing equation [(5)] remains unchanged, except that \( \Psi_t = u_t^{1-\alpha}(r_t^F)^{\alpha}/[z_t(1 - \alpha)^{1-\alpha} \alpha^\alpha] \).

The resource constraint includes the output lost due to both price and capital adjustment costs and is given by

\[
c_t + i_t + \Phi(i_t/k_{t-1})k_{t-1} = y_t^{adj}. \tag{9}
\]

A competitive equilibrium consists of sequences of quantities \( \{c_t, n_t, i_t, k_t, b_t, y_t\}_{t=0}^{\infty} \), prices \( \{w_t, r_t^F, r_t, \pi_t, q_t\}_{t=0}^{\infty} \), and exogenous variables \( \{\beta_t, z_t\}_{t=0}^{\infty} \) that satisfy the household’s and firm’s optimality conditions [(2), (3), (5), (9), (10), (11)], the production function \( y_t = z_t k_t^{-1}(f)^{\alpha} n_t^{1-\alpha} \), the monetary policy rule [(6)], the stochastic processes [(1), (4)], the capital law of motion [(8)], bond market clearing \( b_t = 0 \), and the resource constraint.

3 Calibration and Solution Technique

The models in section 2 are calibrated at a quarterly frequency and the parameters are given in table 1. The real interest rate is set to 2% annually, which implies a steady-state quarterly discount
factor, $\bar{\beta}$, equal to 0.995. The Frisch elasticity of labor supply, $1/\eta$, is set to 3, which is consistent with Peterman (2012). The leisure preference parameter, $\chi$, is calibrated so that steady-state labor equals 1/3 of the available time. Capital’s share of output, $\alpha$, is set to 0.33 and the quarterly depreciation rate, $\delta$, equals 2.5%. The capital adjustment cost parameter, $\phi$, is set to 5.6, which follows Eberly (1997) and Erceg and Levin (2003). The elasticity of substitution between intermediate goods, $\theta$, is set to 6, which corresponds to an average markup of price over marginal cost equal to 20%. The price adjustment cost parameter, $\varphi$, is calibrated to 59.11, which is consistent with a Calvo (1983) price-setting specification where prices change on average once every four quarters.

The likelihood that the nominal interest rate falls to and remains at zero depends critically on both the parameters of the discount factor and technology processes. Richter and Throckmorton (2014) show that a clear tradeoff exists between the persistence and the standard deviation of the stochastic shock processes. As the persistence of a process increases, the standard deviation of that shock must decline, otherwise our numerical algorithm will not converge to an minimum state variable (MSV) solution. The failure to converge occurs because the economy either remains at the ZLB too long when the shocks are very persistent or falls to the ZLB too frequently when the processes are highly volatile. We chose the parameters in (1) and (4) so that (1) they are constant across all models; (2) they generate ZLB events when simulating the model; and (3) they match the data as closely as possible. Specifically, we set the persistence of the discount factor, $\rho_{\beta}$, equal to 0.8 and the standard deviation of the shock, $\sigma_z$, equal to 0.0025. Those values follow Fernández-Villaverde et al. (2012) who assume that a discount factor shock has a half life of about 3 quarters. Steady-state technology, $\bar{z}$, is normalized to 1, the persistence of the technology shock, $\rho_z$, is 0.9, and the standard deviation of the shock, $\sigma_z$, equals 0.0025. In the data, deviations of log real GDP from trend are 1.85% per quarter and deviations of the log difference in the PCE price index are 0.29% from 1983 to 2007. The equivalent values in our models are smaller than is observed, but they do not include many real world shocks and sources of persistence needed to match the data.

In the policy sector, the steady-state gross inflation rate, $\bar{\pi}$, is set to 1.006, which implies an annual inflation rate target of 2.4%. That value equals the average growth rate of the U.S. PCE chain-type price index from 1983-2007. In our baseline calibration, we set the coefficients on inflation and output in the policy rule to 1.5 and 0.1, but we also consider other values.

The model is solved using the policy function iteration algorithm described in Richter et al. (2013), which is based on the theoretical work on monotone operators in Coleman (1991).

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$^7$The parameter values we chose for the stochastic processes do not generate average ZLB events that are consistent with observed ZLB events. However, it is possible for longer ZLB events to still occur. For example, in Model 1 ($y_i = \bar{y}$) the average ZLB event is only 2 quarters, but the maximum ZLB event in a 500,000 quarter simulation is 19 quarters, which is closer to ZLB events observed in the data. Furthermore, Gust et al. (2013) note that most financial market participants expected the federal funds rate to remain below 25 basis points for only 3-4 quarters.
solution method discretizes the state space and uses time iteration to solve for the updated decision rules until the tolerance criterion is met. We use piecewise linear interpolation to approximate future variables that show up in expectations, since this approach more accurately captures the kink in the decision rules than continuous functions, and Gauss–Hermite quadrature to numerically integrate. These techniques capture the expectational effects of going to and returning to the ZLB. For a more formal description of the numerical algorithm and convergence see appendix A.

Benhabib et al.’s (2001) finding that constrained New Keynesian models contain two perfect foresight steady-state equilibria has generated considerable discussion in the literature on whether or not a unique MSV solution exists in stochastic models with a ZLB constraint. Richter and Throckmorton (2014) provide strong evidence that the numerical algorithm used in this paper converges to a determinate MSV solution. They show the region of the parameter space where our numerical algorithm converges to an MSV solution is identical to the determinacy region that Davig and Leeper (2007) analytically derive in a Fisherian economy with a Markov-switching monetary policy rule. That result means our numerical algorithm is non-convergent whenever the policy parameters are outside the analytical determinacy region and convergent whenever the long-run Taylor Principle is satisfied. This exercise does not constitute a formal proof of uniqueness in models with a ZLB constraint, but it does provide a good benchmark for our numerical algorithm, because a model with a ZLB constraint is similar to a Markov-switching model where the central bank pegs the nominal interest rate in one state and obeys the Taylor Principle in the other state.

The models are simulated using draws from the distributions for the discount factor and technology shocks. The state space is discretized to minimize extrapolation of the decision rules during the simulation. As an example, figure 2 shows the simulated distributions for Model 1 when the central bank targets the steady-state output gap. We simulate the model for 500,000 quarters to obtain a large sample of ZLB events. Although not reported above, we find that the likelihood of a discount factor shock causing the nominal interest rate to fall to zero is lower when the central bank targets the potential output gap rather than the steady-state output gap.

Figure 2a shows the unconditional distributions of technology, the discount factor, and the nominal interest rate. The state space for technology lies within ±2.5% of its steady state, which is normalized to unity. The state space for the discount factor lies between ±1.9% of its steady state, which equals 0.995. Over these states, the quarterly net nominal interest rate is distributed over a range of 0% to 3.6%, with a large mass (5% of quarters) between 0 and 20 basis points.

Figure 2b shows the distribution of the discount factor and technology conditional on the ZLB binding. When technology is substantially above its steady state, the nominal interest rate hits its ZLB. That result is consistent with the finding in Fernández-Villaverde et al. (2012) that high levels of technology are associated with low interest rates. Kiley (2003) uses U.S. data to show that periods of high labor productivity growth have been associated with relatively low inflation and argues this result could be caused by the Fed’s policy rule as our models suggest.

4 Model 1: States of the Economy, Economic Dynamics, and the ZLB

The New Keynesian model without capital, outlined in section 2.1, contains two state variables, the discount factor and technology. This section presents the complete solution to that model, key cross

\[ \hat{x}_t = 100(x_t - \bar{x})/\bar{x} \] and a tilde denotes a net rate (i.e., for some gross rate \( x \), \( \tilde{x}_t = 100(x_t - 1) \)).
sections of that solution, impulse responses to technology shocks, and simulation statistics. We compare these results across alternative monetary policy rules. Each variable is shown in percent deviations from its steady state, except inflation and the interest rates, which are net percentages.

Figure 3 shows three-dimensional contour plots of the nominal interest rate and adjusted output over the entire state space. The contour plots provide a complete picture of the model solution for both variables when the central bank targets the steady-state output gap \( y^*_t = \bar{y} \). The shaded areas represent the states of the economy where the net nominal interest rate, \( \tilde{r} \), equals zero. This shaded area reveals the nominal interest rate only hits the ZLB when either technology or the discount factor are unusually high. A higher discount factor makes the household more patient, which reduces their demand and causes both inflation and the real interest rate to fall until the nominal interest rate hits its ZLB. When the central bank aggressively targets the steady-state output gap, a higher level of technology lowers inflation and the real interest rate when the ZLB does not bind. When the ZLB binds, higher technology pushes up the real interest rate which reduces demand. Looking at the highest discount factor in figure 3, output exhibits the same unconventional response, even when technology is below its steady state. Indeed, many ZLB studies assume an elevated discount factor is the source of the ZLB event being examined.

The contours in figure 3 are useful because they provide the solution for every possible com-
Figure 3: Model 1 \((y^*_t = \bar{y})\) decision rules as a function of the technology \((\hat{z}_{-1})\) and the discount factor \((\hat{\beta}_{-1})\) states. Each variable is in percent deviations from its deterministic steady state, except the nominal interest rate, which is a net percentage. The shaded region indicates where the ZLB binds.

Combination of both shocks, but they also can be difficult to read. Thus, we focus on specific cross sections of the state space. The solid line in figure 3 shows the cross section where the technology state is held constant at its steady state \((\hat{z}_{-1} = 0)\). Two-dimensional representations of that cross section are shown in figure 4. The shaded region indicates where the ZLB binds, which begins where the discount factor is 0.9% above its steady state. A high discount factor indicates that households have a strong desire to save. In that case, household demand is depressed, which reduces output, inflation, and the nominal interest rate. At the ZLB expected inflation falls, which increases the real interest rate. That higher real interest rate then makes current consumption more costly and causes even lower household demand in discount factor states where the ZLB binds.

The dashed line in figure 3 shows the cross section where the discount factor is held constant at 0.9% above its steady-state value \((\hat{\beta}_{-1} = 0.9)\), which is the minimum value where the ZLB binds when technology is at its steady state. Two-dimensional representations of that cross section are shown in figure 5, which also includes results with alternative weights on the steady-state output gap \((\phi_y)\). The entire shaded region indicates where the ZLB binds when \(\hat{z}_{-1} = 0\) and \(\phi_y = 0\). Larger values of \(\phi_y\) cause the ZLB to first bind in slightly higher technology states, as the darker shaded regions show. The unconventional response of the economy to a positive technology shock is smaller as the value of \(\phi_y\) declines. With \(\phi_y = 0.05\) \((\phi_y = 0)\), the response of output is positive in technology states up to 0.75 (1.4) above its steady state. Moreover, in the high technology states where the economy does contract, output and inflation are more stable with a lower \(\phi_y\). For example, when \(\phi_y = 0\), output never falls below its initial ZLB level \((\bar{y}^{adj} = -1.25)\), even in the highest technology state. In contrast, when \(\phi_y = 0.1\), output falls from −1% to −3% when technology increases from \(\hat{z}_{-1} = 0\) to \(\hat{z}_{-1} = 2.5\). These results confirm the finding in Braun and Körber (2011) that a shorter expected duration at the ZLB can reverse the unconventional dynamics, since the expected duration of the ZLB increases in higher technology states. Their paper, however, sets \(\phi_y = 0\) and does not consider alternative monetary policy specifications.
To better understand these results, we begin by examining the region of the state space where the ZLB does not bind. In low technology states, workers are less productive and firms’ per unit marginal cost of production is higher. Firms choose higher prices and have a low demand for labor. With less output available for consumption, the household works more to moderate the decline in consumption. The higher labor supply dominates the drop in labor demand so that the equilibrium level of labor is higher and the real wage is lower. The household also believes technology will slowly return to its steady state and, as a result, expects its future consumption to increase. Higher expected future consumption is reflected in an elevated real interest rate. A larger value of $\phi_y$ in technology states where the ZLB does not bind keeps output, labor, and the real wage rate closer to their steady states, but that additional stability comes at the expense of more inflation and a higher nominal interest rate. The real interest rate in that case is mostly unaffected.

The last area to consider are the technology states where the ZLB binds. In those states, higher technology continues to push down per unit production costs and firms react by lowering their prices. The additional decline in expected inflation combined with a zero nominal interest rate forces the real interest rate to rise. The household reduces its consumption to capitalize on the higher returns, which results in the paradox of thrift. Aggregate demand falls because everyone wants to save more at the higher real interest rate, but in the aggregate, it is not possible. Thus, the household reduces its consumption, which lowers output until actual and desired savings are equal. With less consumption, the household increases its labor supply. Firms respond to the reduction in demand by further lowering their prices and decreasing their labor demand. The drop in labor demand dominates the increase in labor supply, so that both total hours and the real wage decline. This is an example of the paradox of toil [Eggertsson (2010)]. At the ZLB, everyone wants to work
more, but the higher real interest rate lowers demand, which causes firms to reduce employment.

With less weight on the steady-state output gap, inflation is more stable in all technology states. Thus, the real interest rate rises less at the ZLB, which helps maintain household demand in high technology states. Higher labor demand raises equilibrium hours, which mitigates the decline in the real wage. In short, a tension exists at the ZLB between the supply-side effects of technology and the demand-side effects of the real interest rate. If the central bank responds less aggressively to the steady-state output gap when the ZLB does not bind (i.e., a lower $\phi_y$), then the demand-side effects at the ZLB are weaker and both real and nominal variables are less volatile.

Figure 6 shows generalized impulse response functions (GIRFs) to a 1% positive technology
shock when the central bank targets the steady-state output gap under two cases: (1) a non-ZLB case (solid line) where the discount factor remains at its steady state so that the nominal interest rate is above its ZLB; and (2) a ZLB case (dashed line) where the discount factor is set to its mean value over a 500,000 quarter simulation under the condition that the ZLB binds and technology is at its steady state.\(^\text{10}\) Basically, this figure compares the GIRFs to a positive technology shock at two different states—one away from and the other at the ZLB. GIRFs have a couple of advantages over decision rules. One, they provide a clearer quantitative comparison between economic dynamics at and away from the ZLB. Two, GIRFs are based on an average of Monte Carlo simulations of the model where the realization of shocks is consistent with household expectations.

\[\text{Adjusted Output (}\hat{y}^{adj}\text{)}\]

\[\text{Real Interest Rate (}\hat{r}/E[\hat{\pi}]\text{)}\]

\[\text{Inflation Rate (}\hat{\pi}\text{)}\]

\[\text{Nominal Interest Rate (}\hat{r}\text{)}\]

\[\text{Labor Hours (}\hat{n}\text{)}\]

\[\text{Real Wage Rate (}\hat{w}\text{)}\]

Figure 6: Model 1 \((y^*_t = \bar{y})\) generalized impulse response functions to a 1% positive technology shock. The steady-state case (solid line) is initialized at the model’s steady state. The ZLB case (dashed line) is initialized at the average state vector conditional on the ZLB binding in a 500,000 quarter simulation.

The impulse responses in the non-ZLB case are standard and follow the intuition from the decision rules. On average, a 1% positive technology shock increases adjusted output, lowers

\(^{10}\)The general procedure for computing GIRFs is outlined in Koop et al. (1996). To compute the GIRFs, we calculate the mean of 10,000 Monte Carlo simulations of the model conditional on a 1% technology shock and a random shock in the first quarter. We then take the percentage change (or difference for the interest rates and inflation) between the two means. In the ZLB case, the responses of the variables converge to zero since the discount factor is reverting to its steady state independently from the initial shock to technology. See Appendix B for further details.
firms’ per unit marginal cost of production, and causes inflation and the nominal interest rate to fall. According to the Taylor rule, the nominal interest rate falls more than the inflation rate, so the real interest rate declines, which increases consumption. The positive technology shock also raises productivity, which decreases the equilibrium level of labor and increases the real wage rate.

In the ZLB case, a 1% positive technology shock initially increases adjusted output by only 0.05% on average. This sluggish response occurs because the ZLB binds in 87% of the simulations after the positive technology shock, which means the nominal interest rate cannot fall by as much as it does in the non-ZLB case. The positive technology shock also lowers per unit production costs which helps to push down prices. With prices falling and the nominal interest rate stuck at zero, an increase in technology sharply raises the real interest rate. That sharp response then limits the increase in output and causes labor to fall further than in the non-ZLB case. Our results in figure 5, however, indicate that the responses of output and the real wage depend critically on the value of $\phi_y$. If $\phi_y = 0$, a positive technology shock increases adjusted output more on impact than when $\phi_y > 0$ because the absence of a policy response to output limits the upward pressure on the real interest rate at the ZLB. From period 2 onward, adjusted output increases as the economy exits the ZLB due to the mean reversion in both technology and the discount factor. The nominal interest rate rises far enough above zero by period 8 that the ZLB case effectively mirrors the non-ZLB case. In both cases, technology returns to its steady state about 20 quarters after the initial shock.

Figure 7 plots the same cross section of the state space that is shown in the dashed line in figure 3 across three specifications of monetary policy: (1) the central bank does not respond to output ($\phi_y = 0$, solid line); (2) the central bank responds to the steady-state output gap ($y_t^* = \bar{y}$, $\phi_y = 0.1$, dashed line); and (3) the central bank responds to the potential output gap ($y_t^* = y_t^p$, $\phi_y = 0.1$, circle markers). The shaded region indicates where the ZLB binds, but the specific level of technology where the ZLB initially binds depends on the monetary policy rule. When $y_t^* = \bar{y}$ ($y_t^* = y_t^p$), the ZLB first binds when the technology state is 0.1% (0.25%) above its steady state. The most interesting difference between the policy rules with $y_t^* = \bar{y}$ and $y_t^* = y_t^p$ is that higher technology states at the ZLB generate further increases in output and the real wage rate with a potential output gap as opposed to a decline with the steady-state output gap. Moreover, adjusted output falls in 49.4% of the Monte Carlo simulations used to compute a GIRF initialized at the ZLB with a steady-state output gap but only 1.8% of the simulations with a potential output gap.

Unlike steady-state output, which is constant, potential output positively co-moves with technology. In technology states below steady state, potential output is lower and the gap is less negative than the steady-state output gap. Since low technology leads to inflation, inflation is positively related to the the potential output gap, but negatively related to the steady-state output gap. Thus, inflation is more stable when the central bank targets the potential output gap. On the other hand, technology above steady state lowers inflation and the ZLB constraint forces the real interest rate to rise. That higher real rate encourages the household to save more, which puts downward pressure on demand. Lower demand dampens the upward force on output from the lower production costs. Which effect dominates depends on if the real interest rate rises enough to offset the positive effects of higher technology. Since the real interest rate is inversely related to the expected inflation rate at the ZLB, the real interest rate will rise less when the central bank targets the potential output gap since inflation is more stable. Therefore, output (the output gap) will be higher (lower) when the central bank targets the potential output gap as opposed to the steady-state output gap. While it is known that the expected duration of the ZLB (determined by the technology state) can reverse unconventional dynamics at the ZLB, the specification of monetary policy also plays a role.
In terms of the monetary policy rule, the Federal Reserve has favored a version of the Taylor rule that puts weight on both inflation and the output gap. We noted above that the specification of the output gap qualitatively affects how the economy behaves in high technology states. Next, we examine how it affects the likelihood of hitting the ZLB using 500,000 quarter simulations of the models. Our main result is that policymakers who more aggressively target the steady-state (potential) output gap will increase (decrease) the likelihood of hitting the ZLB.

Table 2 shows the effect of reducing the weight on output ($\phi_y$) while holding the weight on inflation at $\phi_\pi = 1.5$. We begin with the original Taylor (1993) specification, $\phi_y = 0.125$, and
reduce it by increments of 0.025. With the steady-state output gap \((y^*_t = \bar{y})\), the ZLB binds in 2.73\% of the simulated quarters when \(\phi_y = 0.125\). This value monotonically decreases with \(\phi_y\) and equals 2.33\% when \(\phi_y = 0\). Decreasing the weight on the steady-state output gap raises the volatility of output but has no meaningful effect on the volatility of inflation. Overall, there is not much of a tradeoff between the volatility of output and inflation. The results are reversed when the central bank reacts to the potential output gap \((y^*_t = y^*_{in})\). Placing more weight on the potential output gap reduces the likelihood of hitting the ZLB and the standard deviations of output and inflation fall. These results are consistent with the main finding of Adam and Billi (2006) that the central bank’s optimal policy is to aggressively reduce the nominal interest rate after an adverse shock when it targets potential output. This policy reduces visits to the ZLB and welfare losses.

<table>
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<tr>
<th>(\phi_y)</th>
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Table 2: Volatility implications of alternative weights on the output gap. Model 1: No capital, technology and discount factor shocks. \(\phi_{\pi} = 1.50, \rho_z = 0.90, \sigma_z = 0.0025, \rho_\beta = 0.80, \) and \(\sigma_\beta = 0.0025.\)

Table 3 reports the results when we fix \(\phi_y = 0.125\) and change the weight on the inflation gap \((\phi_{\pi})\). The frequency of ZLB events is more sensitive to changing \(\phi_{\pi}\) than changing \(\phi_y\). With \(y^*_t = \bar{y}\), the probability of hitting the ZLB falls from 2.73\% of the simulated quarters in the baseline case to 0.43\% when \(\phi_{\pi} = 3\). Moreover, the standard deviations of output and inflation fall as \(\phi_{\pi}\) is increased. Increasing \(\phi_{\pi}\) also reduces the likelihood of hitting the ZLB when \(y^*_t = y^*_{in}\). In the baseline case, the ZLB binds in 1.56\% of the quarters and 0.43\% when \(\phi_{\pi} = 3\).

<table>
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<th>Std. Dev. (% of mean)</th>
<th>ZLB Binds % of quarters</th>
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Table 3: Volatility implications of alternative weights on the inflation gap. Model 1: No capital, technology and discount factor shocks. \(\phi_y = 0.125, \rho_z = 0.90, \sigma_z = 0.0025, \rho_\beta = 0.80, \) and \(\sigma_\beta = 0.0025.\)
5 Model 2: States of the Economy and the ZLB

This section shows how our findings change when capital is incorporated into a New Keynesian model. In Model 1, the household can only smooth consumption by varying its labor supply. The presence of capital in Model 2 gives the household another margin to smooth consumption. The addition of another state variable complicates the presentation of the complete solution to the model. We initially fix technology at its steady state so the complete solution can be presented with contour plots. This allows us to focus on the dynamics created by the discount factor process, which is commonly used to generate ZLB events in the literature. Thus, this model initially contains two state variables—the discount factor and the endogenous capital stock. We then reintroduce the technology process to compare the dynamics between Models 1 and 2 in response to a technology shock. Also in this section, we focus on the dynamics for the steady-state output gap specification of the Taylor rule since we believe it better reflects the behavior of actual monetary policy.\footnote{Potential output is defined as the level of output under flexible prices. However, this poses a problem for a model with a ZLB, since sticky prices are necessary for a solution to exist. One alternative is to solve the flexible price model without the ZLB, but this creates an implausibly large output gap given that monetary policy is constrained.}

Figure 8 shows the three-dimensional contour plots of the nominal interest rate, output, consumption, and investment over the entire state space. Presenting a complete picture of the model solution is informative in models with an endogenous state variable like capital since it shows the interaction between the two states. The curvature of the ZLB (shaded) region is due to the quadratic capital adjustment costs. When capital is at its steady state ($\hat{k}_{-1} = 0$), the ZLB binds when $\hat{\beta}_{-1} = 1.22$. As capital rises, the nominal interest rate initially hits zero at lower values of the discount factor. In general, the qualitative behavior of consumption, inflation, and the nominal interest rate are similar to the model without capital. The household’s ability to invest in capital, however, causes consumption to be less volatile and generates stronger expectational effects.

We focus our analysis on two cross sections of the contour map in figure 8. The endogeneity of capital makes selecting particular cross sections in Model 2 more difficult than in Model 1. In Model 1, the discount factor and technology states are independent and, therefore, any one realization of the discount factor is just as likely regardless of the technology state. In Model 2, the capital and discount factor states are not independent so that the level of capital is likely below (above) its steady state when the discount factor is also below (above) its steady state.

Figure 9 shows two cross sections from the contour map in figure 8. The solid line is the cross section where the capital is fixed at its steady state ($\hat{k}_{-1} = 0$). The dashed line represents the cross section where capital increases with the discount factor along the diagonal of the state space ($\hat{k}_{-1} = \hat{k}_{diag}$). The darker (entire) shaded region indicates the area of the state space where the ZLB binds in the steady-state (diagonal) cross section. We begin by examining the behavior of the economy when the ZLB does not bind. Regardless of the capital state, a higher discount factor makes the household more patient, which increases their desire to invest in capital and postpone consumption. The higher discount factor also encourages the household to supply more labor. As for the firms, the additional investment in capital pushes up the marginal product of labor, which encourages firms to raise their output and labor demand. The increase in output then puts downward pressure on firm prices so inflation falls. In equilibrium, labor increases and the real wage rate falls as the discount factor increases. Lastly, a higher capital stock pushes down the marginal product of capital which causes its real rental rate to fall.

In the diagonal cross section where capital increases with the discount factor ($\hat{k}_{-1} = \hat{k}_{diag}$), the
marginal product of capital falls as the discount factor rises which leads to a more rapid decline in the real rental rate. From the household’s perspective, a lower rental rate makes investment less attractive as a consumption smoothing channel. The household responds by moderating both their decline in consumption and their increase in labor supply than in the case where capital is fixed at its steady state ($\hat{k}_{-1} = 0$). Those more modest responses in the real rental rate, investment, consumption, and labor are illustrated by their flatter decision rules when the ZLB does not bind.

In the diagonal (steady-state) cross section, the ZLB binds when the discount factor is more than 0.8% (1.2%) above its steady state. The qualitative properties of the decision rules when the ZLB binds are similar across both cross sections. The mechanism that distorts the economy in Model 2 is similar to Model 1. As the discount factor rises, both inflation and the real interest rate continue to fall. When the nominal rate hits zero, the real interest rate rises as inflation continues to
Figure 9: Model 2 ($y^*_t = \bar{y}$) decision rules as a function of the discount factor state ($\hat{\beta}_{-1}$). The solid line is the cross section of the state space where the capital state is fixed at its steady-state value ($\hat{k}_{-1} = 0$) and the dashed line is the diagonal cross section where the capital state changes with the discount factor state ($\hat{k}_{-1} = \hat{k}_{\text{diag}}$). Each variable is in percent deviations from its deterministic steady state, except inflation and the nominal interest rate, which are net percentages. The dark (entire) shaded region indicates where the ZLB binds when $\hat{k}_{-1} = 0$ ($\hat{k}_{-1} = \hat{k}_{\text{diag}}$).
fall. That higher real rate further encourages households to postpone consumption and motivates them to supply more labor. Firms respond to the lower demand by further reducing their prices and sharply cutting their labor demand. That decline in labor demand dominates the increase in labor supply so that both the equilibrium level of labor and the real wage rate fall. Lower consumption then pushes down output which causes the household to further reduce investment in order to further smooth its consumption. Thus, the paradoxes of toil and thrift both occur—despite the household wanting to work more to smooth consumption and save more to benefit from higher real interest rates, hours and investment both fall. Those findings demonstrate that our model with capital faces the same unconventional dynamics as the model without capital.

The Importance of Nonlinearities We apply the policy function iteration algorithm to log-linearized versions of Model 1 and Model 2, where the only nonlinearity is the ZLB constraint. This solution method is similar to the procedure employed in Nakov (2008), where linear splines are used to approximate the kink in the decision rules. We then compare the resulting linear decision rules to their nonlinear counterparts to demonstrate the importance of using the fully nonlinear model. The benefit of solving the nonlinear and linear models in this manner is that differences in the decision rules are entirely due to whether or not the model is linearized.

Figure 10: Model 1 ($\hat{y}_t = \bar{y}$) decision rules as a function of the technology state (left panel) and the discount factor state (right panel). The solid line (dashed line) corresponds to the decision rules based on the constrained nonlinear (linear) model. Each variable is in percent deviations from its deterministic steady state. The dark (entire) shaded region indicates where the ZLB binds in the fully nonlinear (linear) model.

Figure 10 compares cross sections of the linear and nonlinear decision rules for output in Model 1 when the central bank targets the steady-state output gap. The left (right) panel shows the decision rule as a function of the technology state (discount factor state). The linear decision rules are a fairly accurate approximation of the nonlinear decision rules for both the technology and the discount factor shocks so long as the ZLB does not bind. The linear and nonlinear decision rules for output then diverge as the economy moves deeper into the ZLB region for both shocks. That separation is initiated by the inability of monetary policy to compensate for growing price

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12 The rental rate of capital falls at the ZLB, but the household expects that the future rental rate will increase since they believe the discount factor will return to its steady state. That result is consistent with a rising real interest rate.
adjustment costs, which are different due to the linearization of the quadratic price adjustment cost function. Furthermore, the location of the ZLB kink is nearly identical for both the linear and nonlinear decision rules. Those results, in contrast to Fernández-Villaverde et al. (2012), indicate that the linear model provides a fairly good approximation of the nonlinear model without capital in most states. We, however, will show that decision rules from the linear model with capital are far less accurate because of the expectational effects caused by the ZLB constraint.

Figure 11 compares the linear and nonlinear decision rules in Model 2 as a function of the discount factor state \((\hat{\beta}_{-1})\). The ZLB first binds in the linear (nonlinear) model when the discount factor is 1.4\% (0.8\%) percent above its steady state. That difference has two important implications. One, simulations of the linear model indicate that the economy hits the ZLB far less frequently than in the nonlinear model. In our baseline calibration, the ZLB never binds in the linear model, while it binds in 1.15\% of the time in the nonlinear model. That result is one reason why ZLB studies that linearize the model must specify much larger shocks to generate ZLB events.

Two, the decision rules differ between the linear and nonlinear model when the ZLB does not bind, because the expectational effects of visiting the ZLB are weaker in the linear model. Thus, the linear model cannot accurately quantify the effects of discount factor shocks even when the ZLB does not bind. The reason the linear and nonlinear models generate such different results in the model with capital is because Model 2 has two assets—capital and bonds, while Model 1
only has bonds. Arbitrage implies that the expected rates of return on investment and bonds are identical, which means the expected future rental rate of capital equals the current real interest rate.

Another critical difference between the linear and nonlinear model is that the real interest rate falls in the nonlinear model when the discount factor is high, but not high enough for the ZLB to bind. In that situation, the household in the nonlinear model places a higher probability on the ZLB binding and the unconventional dynamics. Those beliefs factor into the household’s expectations and, when the probability is high enough, it causes the household to expect the rental rate of capital will decline and consumption growth will slow. Those effects then cause the real interest rate to fall before the ZLB is hit. The linear model, however, is unable to generate these important expectational effects. Overall, linearization drastically impacts the predictions of Model 2.

6 Model 1 and Model 2 Comparisons

This section shows capital qualitatively and quantitatively affects dynamics at the ZLB. We compare the impulse responses of the model without capital (Model 1) to those responses from the model with capital (Model 2). We evaluate impulse responses because a cross section requires assumptions about how the capital state in Model 2 co-moves with the exogenous state variables. To conduct such an experiment, technology is stochastic with the same parameter values in both models. We also assume that the central bank targets the steady-state output gap \( y^* = \bar{y} \) and set \( \phi_y = 0.025 \) in both models. A small weight on \( \phi_y \) is necessary to make a direct comparison because our numerical algorithm does not converge for higher values of \( \phi_y \) in Model 2.

![Comparison of the generalized impulse response functions to a 1% positive technology shock in Model 1 (solid line) and Model 2 (dashed line). The initial state vector is equal to the average state vector in a 500,000 quarter simulation conditional on the ZLB binding. The central bank targets the steady-state output gap \( y^* = \bar{y} \).](image)

Figure 12 plots the GIRFs to a 1% positive technology shock in Model 1 (solid line) and Model 2 (dashed line). In this experiment, the discount factor is initially set to its mean value that causes...
the ZLB to bind in a 500,000 quarter simulation of the model where technology shocks are set to zero (Model 1: $\hat{\beta}_{-1} = 1$, Model 2: $\hat{\beta}_{-1} = 1.4$). The unconventional dynamics are not present in Model 1 when $\phi_y$ is small, because the positive supply-side effects of higher technology dominate the negative demand-side effects of a higher real interest rate. As $\phi_y$ increases, the adverse demand-side effects overcome the beneficial supply-side effects so that output and labor hours both decline in response to a positive technology shock at the ZLB. Output, in contrast, declines on impact in Model 2 even when $\phi_y$ is small. The responses of the real interest rate, inflation, and labor are qualitatively the same in both models but quantitatively larger in Model 2. From period 2 onward, technology and the discount factor mean revert and the GIRFs from both models converge.

Next, we show the impact of capital on the volatility of output and inflation for a range of values for $\phi_y$. Model 1 and Model 2 are simulated for 500,000 quarters under the assumption that the central bank targets the steady-state output gap. We fix technology at its steady state to examine a broader range of values for $\phi_y$. Table 4 shows the effect of reducing the weight on the output gap ($\phi_y$) while holding the weight on inflation ($\phi_{\pi}$) at 1.5. The value of $\phi_y$ is initially set slightly below the original Taylor (1993) specification, $\phi_y = 0.1$, and is reduced by increments of 0.025.

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</tbody>
</table>

Table 4: Volatility implications of alternative weights on the output gap. Comparison between Model 1 and Model 2. The only stochastic component in both models is discount factor shocks. $\phi_{\pi} = 1.50$, $\rho_{\beta} = 0.80$, and $\sigma_{\beta} = 0.0025$.

Our results show capital decreases the frequency of ZLB events for every value of $\phi_y$, which is consistent with Christiano (2004). A larger value of $\phi_y$ in Model 1 decreases the likelihood of ZLB events. At first glance, that result appears to contradict our Model 1 findings with the steady-state output gap target reported in table 2, column 2. Since technology is fixed at its steady state, the steady-state output gap and the potential output gap are equivalent. Thus, the qualitative result is consistent with the findings from Model 1 with the potential output gap that is presented in table 2, columns 5. Model 2, however, does generate two opposite results from Model 1. One, a higher $\phi_y$ increases the likelihood ZLB events in Model 2. Two, both output and inflation volatility decline as $\phi_y$ increases in Model 1, whereas a tradeoff exists between lower output volatility and higher inflation volatility in Model 2. Those differences are noteworthy since many central banks have emphasized output stabilization since the end of the Great Recession.

7 Conclusion

This paper examines monetary policy when it is constrained by the ZLB using models with and without capital. Our analysis focuses on discount factor shocks since they are viewed as the likely cause of many ZLB events. Nonetheless, we also examine technology shocks because they are an
important source of aggregate fluctuations in many dynamic models. We use these models to analyze why technology shocks at the ZLB may have unconventional effects, what factors influence the likelihood of hitting the ZLB, and the tradeoffs faced by the central bank under a dual mandate.

Three main findings emerge: (1) A positive technology shock can generate lower consumption, labor, and output when the ZLB binds and the central bank targets the steady-state output gap. Those unexpected results usually disappear when the central bank reacts to the potential output gap; (2) When the central bank targets the steady-state output gap in the model with capital, a positive technology shock at the ZLB generates more contractionary dynamics than in the model without capital; (3) The constrained linear model provides a good approximation of the nonlinear model without capital, but differences exist between the solutions in the model with capital.

Despite the large amount of research on the ZLB, many important questions remain. For example, do the medium- to long-run benefits of returning to normal policy outweigh the short-run costs of a higher nominal interest rate? What are the benefits of forward guidance and quantitative easing in a dynamic model that accounts for expectational effects, and how does the impact of discount factor and technology shocks change as a result of these policies? Answering these questions and others requires careful treatment of expectations and is the subject of ongoing research.
REFERENCES


A Numerical Algorithm

A formal description of the numerical algorithm begins by writing the model compactly as

$$ E[f(v_{t+1}, w_{t+1}, v_t, w_t)] \Omega_t = 0, $$

where $f$ is a vector-valued function that contains the equilibrium system, $v$ is a vector of exogenous variables, $w$ is a vector of endogenous variables, and $\Omega_t = \{M, P, z_t\}$ is the household’s information set in period $t$, which contains the structural model, $M$, its parameters, $P$, and the state vector, $z$. In model 1, $v = z = (\beta, z)$ and $w = (c, \pi, y, n, w, r)$. In model 2 with both stochastic processes, $v = (\beta, z)$, $z = (k, \beta, z)$, and $w = (c, \pi, y, n, w, r, k, i, r^k, q)$.

Policy function iteration approximates the vector of decision rules, $\Phi$, as a function of the state vector, $z$. The time-invariant decision rules for the exogenous model are

$$ \Phi(z_t) \approx \hat{\Phi}(z_t). $$

We iterate on $\Phi = (n, \pi)$ for Model 1 and $\Phi = (n, \pi, i)$ for Model 2 so that we can easily solve for future variables that enter the household’s expectations using $f$. Each continuous state variable in $z$ is discretized into $N^d$ points, where $d \in \{1, \ldots, D\}$ and $D$ is the dimension of the state space. The discretized state space is represented by a set of unique $D$-dimensional coordinates (nodes). In general, we set the bounds of continuous stochastic state variables to encompass 99.999% of the probability mass of the distribution. We specify 251 grid points for each continuous state variable and use 31 Gauss-Hermite weights for each continuous shock. These techniques minimize extrapolation and ensure that the location of the kink in the decision rules is accurate.

The following outline summarizes the policy function algorithm we employ for our models. Let $i \in \{0, \ldots, I\}$ index the iterations of the algorithm and $n \in \{1, \ldots, \Pi^{D-1}_{d=1}N^d\}$ index the nodes.

1. Obtain initial conjectures for the approximating functions on each node from the log-linear model without the ZLB imposed. The approximating functions are $\hat{c}_0$ and $\hat{\pi}_0$ for Model 1 and $\hat{n}_0$, $\hat{\pi}_0$, and $\hat{i}_0$ for Model 2. We use gensys.m to obtain these conjectures.

2. For $i \in \{1, \ldots, I\}$, implement the following steps:

   (a) On each node, solve for $\{y_t, r_t\}$, given $\hat{c}_{i-1}(z^n_t)$ and $\hat{\pi}_{i-1}(z^n_t)$ in Model 1 and given $\hat{n}_{i-1}(z^n_t)$, $\hat{\pi}_{i-1}(z^n_t)$, and $\hat{i}_{i-1}(z^n_t)$ in Model 2 with the ZLB imposed.

   (b) Linearly interpolate $\{c_{i+1}, \pi_{i+1}\}$ in Model 1 and $\{n_{i+1}, \pi_{i+1}, i_{i+1}\}$ in Model 2 given $\{c^n_{i+1}, \pi^n_{i+1}, i^n_{i+1}\}$. Each of the $M$ values $c^n_{i+1}$ are Gauss-Hermite quadrature nodes. We use Gauss-Hermite quadrature to numerically integrate, since it is accurate for normally distributed shocks. We use piecewise linear interpolation to approximate future variables, since this approach more accurately captures the kink in the decision rules than continuous functions such as cubic splines or Chebyshev polynomials.\footnote{Aruoba and Schorfheide (2013) use a linear combination of two Chebyshev polynomials—one that captures the dynamics when the ZLB binds and one that captures the dynamics when the Taylor principle holds. While this approach is more accurate than using one Chebyshev polynomial, there is no guarantee that it will accurately locate the kink. Moreover, Chebyshev polynomials can lead to large approximation errors due to extrapolation. With linear interpolation, a dense state space will lead to more predictable extrapolation and more accurately locate the kink.}

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(c) We use the nonlinear solver, csolve.m, to minimize the Euler equation errors. On each node, numerically integrate to approximate the expectation operators,

\[
\mathbb{E} \left[ f \left( x_{t+1}^m, x_t^n \right) | \Omega_t \right] \approx \frac{1}{\pi} \sum_{m=1}^{M} f(\hat{x}_{t+1}, \hat{x}_t^m) \phi(\varepsilon_{t+1}^m),
\]

where \( x \equiv (v, w) \) and \( \phi \) are the respective Gauss-Hermite weights. The superscripts on \( x \) indicate which realizations of the state variables are used to compute expectations. The nonlinear solver searches for \( \hat{c}_i(z^n_t) \) and \( \hat{\pi}_i(z^n_t) \) in Model 1 and \( \hat{n}_i(z^n_t), \hat{\pi}_i(z^n_t), \) and \( \hat{i}_i(z^n_t) \) in Model 2 so that the Euler equation errors are less than \( 1^{-4} \) on each node.

3. Define \( \text{maxdist}_i \equiv \max \{|\hat{c}_i - \hat{c}_{i-1}|, |\hat{\pi}_i - \hat{\pi}_{i-1}|\} \) in Model 1 and \( \text{maxdist}_j \equiv \max \{|\hat{n}_i - \hat{n}_{i-1}|, |\hat{\pi}_i - \hat{\pi}_{i-1}|, |\hat{i}_i - \hat{i}_{i-1}|\} \) in Model 2. Repeat the steps in item 2 until one of the following conditions is satisfied.

- If for all \( n \), \( \text{maxdist}_i < 10^{-13} \) for 10 consecutive iterations, then the algorithm converged to a MSV solution. In Model 1, since the state is composed of only exogenous variables, the solution is bounded so long as the decisions rules are positive and finite. In Model 2, simulations of the model must not be explosive.
- Otherwise, we say the algorithm is non-convergent for one of the following reasons:
  - \( i = I = 500,000 \) (Algorithm times out)
  - For all \( n \) and any \( i, \hat{\pi}_i < 0.5 \), or for any \( n \), \( \hat{c}_i < 0 \) in Model 1 or \( \hat{n}_i < 0 \) in Model 2 (Approximating functions drift)
  - Define \( \text{dir}_i = \text{maxdist}_i - \text{maxdist}_{i-1} \). For all \( n \), \( \text{dir}_i \geq 0 \) and \( \text{dir}_i - \text{dir}_{i-1} \geq 0 \) for 50 consecutive iterations (Algorithm diverges)

**B Generalized Impulse Response Functions**

The generalized impulse responses functions (GIRFs) are based on the average of 10,000 Monte Carlo simulations of the model. The advantage of this approach is that the realization of shocks are consistent with the household’s expectation that the stochastic processes will mean revert when the GIRF is initialized away from the model’s stochastic steady state. The general procedure for calculating GIRFs is laid out in Koop et al. (1996). We apply the following steps to our models:

1. Find the state vector at which to initialize each case:
   
   (a) **Non-ZLB Case**: Simulate the model without shocks until it converges to its stochastic steady state, \( z_0^{ss} \).
   
   (b) **ZLB Case**: Simulate the model for 500,000 quarters using random draws of discount factor shocks. The initial state vector is the average state vector conditional on the ZLB binding, \( z_0^{ZLB} \). The average discount factor when the ZLB binds is 1% above its steady state.

2. Draw random shocks to technology and the discount factor, \( \{\varepsilon_{z,t}, \varepsilon_{\beta,t}\}_{t=0}^{N} \), from their independent normal distributions. Simulate each case for \( R \) different draws of the sequence of
shocks beginning at the alternative initial state vectors, $z_{0}^{ss}$ and $z_{0}^{zlb}$. This yields $R$ equilibrium paths for each case, $\{x_{t}^{j}(z_{0}^{ss})\}_{t=0}^{N}$ and $\{x_{t}^{j}(z_{0}^{zlb})\}_{t=0}^{N}$, where $j \in \{1, 2, \ldots, R\}$. We set $N = 20$ and $R = 10,000$.

3. Using the same $R$ draws of shocks from step 2, replace the technology shock in period one with a 1% shock (i.e., set $\varepsilon_{z,1} = 0.01$ for all $j \in \{1, 2, \ldots, R\}$). Simulate each case with these alternate sequences of shocks. This yields $R$ equilibrium paths for each case, $\{x_{t}^{j}(z_{0}^{ss}, \varepsilon_{z,1})\}_{t=0}^{N}$ and $\{x_{t}^{j}(z_{0}^{zlb}, \varepsilon_{z,1})\}_{t=0}^{N}$.

4. Average across the $R$ simulations from step 2 and step 3 to obtain average paths, given by,

$$\bar{x}_{t}(z_{0}^{ss}) = \frac{1}{R} \sum_{j=1}^{R} x_{t}^{j}(z_{0}^{ss}), \quad \bar{x}_{t}(z_{0}^{ss}, \varepsilon_{z,1}) = \frac{1}{R} \sum_{j=1}^{R} x_{t}^{j}(z_{0}^{ss}, \varepsilon_{z,1}),$$

$$\bar{x}_{t}(z_{0}^{zlb}) = \frac{1}{R} \sum_{j=1}^{R} x_{t}^{j}(z_{0}^{zlb}), \quad \bar{x}_{t}(z_{0}^{zlb}, \varepsilon_{z,1}) = \frac{1}{R} \sum_{j=1}^{R} x_{t}^{j}(z_{0}^{zlb}, \varepsilon_{z,1}).$$

5. The difference between the two average paths for each case is a GIRF. In our figures, a variable in either case with a hat is calculated as $100(\bar{x}_{t}(z_{0}^{s}, \varepsilon_{z,1}) - \bar{x}_{t}(z_{0}^{s}))$, and with a tilde is calculated as $100(\tilde{x}_{t}(z_{0}^{s}, \varepsilon_{z,1}) - \tilde{x}_{t}(z_{0}^{s}))$, where $s \in \{ss, zlb\}$. 

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