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Abstract

Private information may limit insurance possibilities when a few agents form a partnership to pool idiosyncratic risk. We show that these insurance possibilities can improve if the partnership's income depends on capital accumulation and production, because cheating distorts investment. As agents' weights in the partnership increase, they are more affected by the investment distortion, and their incentives to misreport under the full information allocation are reduced. In the long run, either one of the partners is driven to immiseration, or both partners' lifetime utilities are approximately equal. The second case is only possible with capital accumulation. The theory's testable implications are in line with empirical evidence on the organization of small-business partnerships.

JEL Classifications: D82, D86, D92, G32

Keywords: Partnerships, Small businesses, Private information, Ownership structure

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1 Introduction

Private information limits insurance possibilities when a few agents form a partnership to pool idiosyncratic risk. We show that if the partnership's income depends on capital accumulation and production, insurance possibilities improve because cheating distorts investment. This occurs because cheating implies that the misrepresenting partner will receive higher utility today at the expense of reducing the other partner's current consumption and reducing investment. The latter reduces future value of the partnership, and therefore provides incentives to prevent cheating. This mechanism is more significant when the agent's weight in the partnership is larger, as he cares more about the future of the partnership. In the long run, either one of the partners is driven to immiseration, or both partners' lifetime utilities are approximately equal. We show that the second case is only possible in economies with capital accumulation. The theory's testable implications are in line with empirical evidence on the organization of small-business partnerships.[1] The result transcends this model and also holds in other applications with private information, a small number of agents, and capital accumulation (e.g., economic unions among countries or states).

We study the optimal arrangement between two risk-averse agents who are partners in the ownership of a production technology and capital. Partners face preference shocks that vary from period to period.[2] In this environment, the optimal contract maximizes the weighted sum of the partners' lifetime utility and the solution determines how to split output between investment and consumption of each agent. Ideally, each partner's consumption should depend on his preference shock. However, if shocks are private information, the optimal arrangement must also deal with incentives to misreport. Thus, private information limits the extent of the partnership's insurance capabilities.

To understand the role of capital accumulation for incentives under private information, consider first the simpler case of an endowment economy. In that case, the incentives to cheat exist because reporting a high preference shock increases consumption of the reporting agent at the expense of reducing consumption of the other agent. Thus, unless promises about future consumption are modified as a function of the report, agents always want to report the highest value of the preference shock. In contrast, with capital accumulation, if a partner chooses to cheat, she will consume more this period at the cost (at least partially) of disinvestment, which will reduce her future consumption. We show that this force is more important for providing incentives for agents with larger shares in the partnership (represented by their Pareto

[1] See Vereshchagina (2015), Encinosa III, Gaynor, and Rebitzer (2007), Fehr, Kremhelmer, and Schmidt (2008), Espino, Kowloski, and Sanchez (forthcoming), and Hellmann and Wasserman (forthcoming), among others.

[2] As in Diamond and Dybvig (1983) and Atkeson and Lucas (1992).

weights), as they more strongly internalize the future of the partnership. Thus, in addition to incorporating capital accumulation, it is crucial to consider a small number of agents. To the best of our knowledge, this result was not previously demonstrated or understood, since past work looked at only a continuum of agents or an endowment economy.[3]

We made two additional assumptions that help simplify the analysis: partners are ex-ante identical and the preference shock can be either high or low. Under these conditions we derive three key results as summarized by Propositions 1, 2, and 3. First, Proposition 1 shows that an agent has no incentives to cheat under the full information allocation when his weight is above a threshold smaller than one. To understand this result, note that if an agent's weight is equal to one, all the extra funds she receives after cheating are only financed by disinvesting in the partnership, since the consumption of the other agent is already zero. She also fully internalizes the effect of the investment distortion, because given that her weight is one, she will receive all future output. Thus, she would be strictly worse off misrepresenting her preference shock when her weight is exactly one. Now, as a small change in the weight changes consumption and investment only slightly (by continuity of the full information allocation), cheating will not be desirable even if the weight were slightly smaller than one. Although this result demonstrates that at some point the incentives to cheat vanish completely, the main takeaways are: (i) there is a force created by the behavior of investment in the full information allocation that helps provide incentives for truthful revelation, and (ii) this force is increasing in the value of the agent's weight in the partnership.

Proposition 2 shows that, under certain conditions, both agents have no incentives to cheat under the full information allocation for a partnership with equal weights. In particular, the additional assumption is that the spread between the low and the high value of the preference shock is sufficiently large. To understand the result, consider the extreme case in which consumption is not valued if the low preference shock is realized. In this case, agents with the low value of the preference shock would be strictly worse off misrepresenting their shock; they would obtain extra consumption today, when it is not valued, at the expense of lower consumption tomorrow (because investment decreases), when it is valued. Similarly, if their valuation of consumption in the low state is close to zero and each agent weight is sufficiently high, the same forces are present and there are no incentives to cheat; i.e., the full information plan is incentive compatible for both agents.

Proposition 3 shows that in the long run private information becomes irrelevant and the agents' weights remain unchanged. This may happen because either one of the partners has all

[3]A notable exception is Marcet and Marimon (1992). The key difference is that in their work one of the agents has linear preferences and deep pockets. As we show in Section 5.1 this assumption eliminates the disinvestment mechanism studied here.

the weight in the partnership, or the weights are approximately equal. This proposition hinges on the existence of a region of weights such that both incentive compatibility constraints are not binding. This result is important because it implies that the immiseration result, which has been widely studied in private information problems, does not hold. More generally, when the condition in Proposition 2 is not satisfied, for instance, because the preference shock is less coarse, the main takeaway is that due to the incentives provided by capital accumulation, the economy spends more time in the surroundings of equal weights than in an endowment economy.

The framework presented in this paper has several features that resemble the organization of small-business partnerships.[4] Small businesses are recognized to be an important driver of the economy (see Hurst and Pugsley (2011)). These businesses are concentrated among skilled craftsmen, lawyers, real estate agents, doctors, small shopkeepers, and restaurateurs, among others. There are five characteristics of small-business partnerships that are well captured by our framework: small number of owners, shared ownership, production possibilities, liquidity shocks, and internal financing. We argue that our theory can account for two common findings in the study of partnerships: (i) increasing the number of partners worsens incentives problems, and (ii) partners having equal shares of ownership facilitates the provision of incentives. Moreover, our theory has other testable predictions about the dynamics of ownership shares that are, a priori, harder to rationalize with other existing theories. In particular, under private information, our theory implies that: (i) ownership shares change more frequently when firm ownership is unequally distributed, and (ii) the distribution of ownership shares moves over time towards either equal distribution of ownership or sole-proprietorship. We argue these two findings are in line with empirical evidence we present in Espino, Kowloski, and Sanchez (forthcoming).

Related literature. This paper is related to long standing theoretical literature studying private information problems. Previous work had made at least one of three assumptions: (i) resources available are not affected by the agents' decisions —"endowment economy" (Thomas and Worral, 1990); (ii) there is a continuum of agents (Atkeson and Lucas, 1992); (iii) one of the agents is risk neutral with deep pockets (Marcet and Marimon, 1992; Clementi and Hopenhayn, 2006).[5] We find that these assumptions are not innocuous. When we relax these assumptions and consider a production economy with a finite number of risk-averse agents, we find that

[4]Notice that we refer to these firms as small-business partnerships to differentiate them from large public companies. We do not mean partnership in the legal status sense. The legal status and tax treatment differences are ignored in this analysis.

[5]See also Green (1987), Spear and Srivastava (1987), Quadrini (2004), Clementi, Cooley, and Giannatale (2010), and Cole, Greenwood, and Sanchez (2012).

investment provides incentives to prevent cheating, in particular for partners with large weight in the partnership.

Our work is also related to theoretical and empirical literature studying the organization of partnerships. The theoretical literature in this case is more scarce, mainly because the standard assumptions made in previous private information setups do not apply to partnerships. Although there are a few notable exemptions, they all consider a static environment and therefore ignore investment and the evolution of ownership shares, which are the main ingredients for our analysis. Bhattacharyya and Lafontaine (1995) study how to split revenue in franchises in a static environment in which the unobservable effort of both partners affects output. More recent theoretical work focuses on the choices of businesses to organize as partnerships or corporations (Kaya and Vereshchagina, 2014; Levin and Tadelis, 2005, among others).

Empirical work on partnerships presents evidence in line with our findings. Drawing on previous research, Wasserman (2012) documents the prevalence of equal equity splits in startups.[6] Other works study medical practices (Gaynor and Gertler, 1995) and law partnerships (Kevin Lang, 1995) in which the partners get together to share risk, as in our framework. Their main finding is that increasing the number of partners (diminishing the share each partner has) reduces the risk-sharing ability of the partnership. This result is in line with our finding that having all partners with large ownership shares facilitates the provision of incentives.

The paper is organized as follows. Section 2 describes the model and shows how the problem of organizing a partnership can be represented recursively using Pareto weights and capital as state variables. Section 3 characterizes the optimal allocation under full and private information. Section 4 argues that allowing for capital accumulation is crucial for our results. The robustness of the result to other assumptions is discussed in Section 5. Section 6 asserts that our theory can be used to understand the organization of small-business partnerships. Finally, Section 7 provides conclusions. The recursive formulation of the partnership problem is provided in Appendix A while the rest of the proofs are provided in Appendices B and C.

2 Model

Time is discrete and the time horizon is infinite. At date 0, two agents, indexed by $i = 1, 2$, start operating a decreasing return to scale technology that delivers a profits function $f(k)$ with $f'(k) >$ and $f''(k) < 0$. They start with capital $k_0 = k_{0,1} + k_{0,2}$, contributed by agents 1 and 2, respectively.[7] Capital depreciates at the rate $\delta \in (0, 1)$. Given technological assumptions,

[6] Since most of the start-ups are small, our results for small-business partnerships can also be applied to startups. More on this in Section 6.

[7] This paper studies neither the formation nor the break-up of the contract. The formation could be determined by Nash bargaining between agents who would split the benefits of the partnership (raising capital

there exists some $\overline{k}$ such that $X = [0, \overline{k}]$ denotes the sustainable levels of capital.[8]

Agents have preferences à la Diamond and Dybvig (1983) as they face liquidity shocks.[9] At the beginning of date t, agent i privately observes his shock $s_{i,t} \in S_{i,t} = \{s_L, s_H\}$, where $s_H > s_L$ and define $S_i^t = \times_{h=0}^{t} S_{i,h}$, $S_t = S_{1,t} \times S_{2,t}$ and $S^t = \times_{j=0}^{t} S_t$.

These shocks are assumed to be i.i.d. across time and agents, where $\pi(s_{i,t}) > 0$ is the probability of $s_{i,t}$. Let $s_t = (s_{1,t}, s_{2,t}) \in S_t$ be the joint shock at date t with probability $\pi(s_t) = \pi(s_{1,t}) \times \pi(s_{2,t})$; s_{-i} denotes a liquidity shock that excludes agent i's element (e.g., $s_{-1} = s_2$) and $s^t = (s_0, ..., s_t) \in S^t$ denotes a partial history of events from date 0 to date t. The probability at date 0 of a partial history s^t is $\pi(s^t) = \pi(s_0)...\pi(s_t)$.

Given a consumption path $\{c_{i,t}\}_{t=0}^{\infty}$ such that $c_{i,t} : S^t \to \mathbb{R}_+$, agent i's preferences are represented by

$$E\left\{\sum_{t=0}^{\infty} \beta^t \ s_{i,t} \ u(c_{i,t})\right\} = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) \ s_{i,t} \ u(c_i(s^t)),$$

where $u : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, strictly concave, and twice differentiable; $\lim_{c_t \to 0} u'(c_t) = +\infty$; and $\beta \in (0, 1)$. A higher value of the agent's liquidity needs implies that he is willing to take more resources to consume more today compared with the future.

Let $K' = \{K_{t+1}\}_{t=0}^{\infty}$ be an *investment plan* that every period allocates next-period capital, given a history of joint reports (i.e. $K_{t+1} : S^t \to \mathbb{R}_+$ for all t), given K_0. Similarly, let $C = \{(C_{1,t}, C_{2,t})\}_{t=0}^{\infty}$ be a *distribution plan*, where $C_{i,t} : S^t \to \mathbb{R}_+$ for all t, $i = 1, 2$.

Any (C, K') satisfying these properties is called a *sequential plan* and it is feasible if

$$K(s^t) + \sum_{i=1}^{2} C_i(s^t) \leq f(K(s^{t-1})) + (1 - \delta)K(s^{t-1}), \tag{1}$$

for all s^t and all t.

Intuitively, feasibility at the firm's level means that part of the output is reinvested in the partnership, $K(s^t) - (1 - \delta)K(s^{t-1})$, and part is paid out to the partners, $C_1(s^t) + C_2(s^t)$. Importantly, note that there is no external finance available to the partnership.[10] In what follows, we refer to K as the stock of capital as well as the size of the partnership indistinctly.

and providing insurance). To allow for break-ups, the model could be extended by adding some type of lack of commitment, along the lines of Wang (2011) or Amador, Werning, Hopenhayn, and Aguiar (2015).

[8] All the analysis regarding the solution method also applies to the general case in which there is an arbitrary number of partners I.

[9] This class of preferences is standard in the literature; see Tirole (2005), Chapter 12. Moreover, liquidity shocks are multiplicative as in Atkeson and Lucas (1992).

[10] In Section 5 we study the case in which partners can obtain unlimited resources at a given interest rate r.

Given a sequential plan (C, K'), agent i's utility at date t given the partial history s^t is

$$U_{i,t}(C, K' \| s^{t-1}) = \sum_{j=0}^{\infty} \beta^j \sum_{(s_t,...,s_{t+j})} \pi(s_t, ..., s_{t+j}) \; s_{i,t+j} \; u(C_i(s^{t-1}, s_t, ..., s_{t+j})).$$

As liquidity shock realizations are privately observed, any mechanism for allocating investment and consumption must be incentive compatible.[11] A sequential plan (C, K') is *incentive compatible* if no agent has incentives to misreport his liquidity shocks and so truth-telling is the best response; i.e. for each i,

$$\begin{aligned} &\sum_{s_{-i}} \pi(s_{-i}) \left(s_i \; u(C_i(s^{t-1}, s_i, s_{-i})) + \beta \; U_{i,t+1}(C, K' \| (s^{t-1}, s_i, s_{-i}))\right) \\ &\geq \sum_{s_{-i}} \pi(s_{-i}) \left(s_i \; u(C_i(s^{t-1}, \widetilde{s}_i, s_{-i})) + \beta \; U_{i,t+1}(C, K' \| (s^{t-1}, \widetilde{s}_i, s_{-i}))\right) \end{aligned} \tag{2}$$

for all $t \geq 0$, all s^{t-1} and all $(s_i, \widetilde{s}_i)$.[12]

To solve for the constrained efficient arrangement, we consider a fictitious planner who chooses among plans that are incentive compatible and feasible. Let $\Psi^*(K)$ be the utility possibility correspondence—that is, the levels of utility of the partners that can be attained by a corresponding plan that is incentive compatible and feasible at K,

$$\begin{aligned} \Psi^*(K) \equiv \big\{ w \in \mathbb{R}^2_+ : \exists \; (C, K') \text{ satisfying } (1) - (2) \\ \text{and } w_i \leq U_i(C, K', z^*) \; \forall \; i, \;\; K_0 = K \big\}. \end{aligned}$$

As Ψ^* is a continuous, compact-valued, and convex correspondence (see Espino (2005)), the set of constrained efficient plans can be parameterized by $(\theta_1, \theta_2) \in \mathbb{R}^2_+$. We say that (C^*, K'^*) is constrained efficient if is the corresponding plan sustaining the levels of lifetime utility that solves

$$h^*(K, \theta) = \max_{w \, \in \, \Psi^*(K)} (\theta_1 w_1 + \theta_2 w_2), \tag{3}$$

for some $(\theta_1, \theta_2) \in \mathbb{R}^2_+$. That is, $h^*(K, \theta)$ captures the Pareto frontier of the utility possibility correspondence Ψ^* by solving for the highest level of weighted lifetime utility given weights (θ_1, θ_2).[13] Hereafter we restrict the welfare weights to add up to 1; that is, $\Delta \equiv \{\theta \in \mathbb{R}^2_+ :$

[11] This restriction is without loss of generality since the Revelation Principle holds and it allows us to restrict attention to mechanisms that rely on truthful reports of these shocks.

[12] We restrict to temporary incentive compatibility. A more general concept of Bayesian implementation can shown to be equivalent in this particular dynamic environment. See Espino (2005).

[13] Our approach to define the objective of the partnership as maximizing the weighted sum of the partners

$\theta_1 + \theta_2 = 1\}$.[14]

It is more convenient to write this problem recursively. Our recursive representation of the problem adapts the method developed by Spear and Srivastava (1987) and Abreu, Pearce, and Stacchetti (1990). We characterize the constrained efficient frontier by giving a Pareto weight to each agent.[15] These weights, together with capital, become endogenous state variables that summarize the history. In a nutshell, our approach can be interpreted as a combination of Abreu, Pearce, and Stacchetti (1990) and Marcet and Marimon (1994)'s Lagrangean method.[16] In Appendix A we provide an algorithm capable of finding the value function h^* (and its corresponding policy functions) and argue that h^* satisfies

$$h^*(k, \theta) = \max_{(c,w',k')} \sum_{i=1}^{2} \theta_i \left\{ \sum_{s} \pi(s) \left[s_i \ u(c_i(s)) + \beta \ w_i'(s) \right] \right\}, \tag{4}$$

subject to

$$k'(s) + \sum_{i=1}^{2} c_i(s) = f(k) + (1-\delta)k \tag{5}$$

$$\begin{aligned} &\sum_{s_{-i}} \pi(s_{-i}) \left(s_i u(c_i(s_i, s_{-i})) + \beta \ w_i'(s_i, s_{-i}) \right) \\ &\geq \sum_{s_{-i}} \pi(s_{-i}) \left(s_i u(c_i(\widetilde{s}_i, s_{-i})) + \beta \ w_i'(\widetilde{s}_i, s_{-i}) \right) \end{aligned} \tag{6}$$

for all $(s_i, \widetilde{s}_i)$ and

$$c_i(s) \geq 0, \qquad w_i'(s) \geq 0 \qquad \text{for all } s \text{ and all } i, \tag{7}$$

$$\min_{\theta' \in \Delta} \left[h^*(k'(s), \theta') - \sum_{i=1}^{2} \theta_i'(s) \ w_i'(s) \right] \geq 0 \text{ for all } s. \tag{8}$$

Note that the optimization problem takes as given (k, θ) to distribute output between current consumption to the agents and investment, and assigns continuation utility levels. The optimization problem defined in condition (8) characterizes the set of continuation utility levels attainable at (k', θ').[17] The values of θ' that attain the minimum in (8) at (k, θ) for state s,

utility is in the spirit of Magill and Quinzii (1996), Chapter 6, Section 31.

[14] This restriction is innocuous because solutions are homogeneous of degree 0 with respect to (θ_1, θ_2).

[15] The idea of substituting utility levels with Pareto weights is borrowed from Lucas and Stokey (1984).

[16] As Marcet and Marimon (1994), we sidestep the requirement that future utilities must lie in the utility correspondence next period by mapping utility levels into Pareto weights. See also Mele (2014), Messner, Pavoni, and Sleet (2012), and Beker and Espino (2013).

[17] The same condition was used for the same purpose by Lucas and Stokey (1984), equation (5.7), and more recently by Beker and Espino (2011), equation (15).

denoted by $\theta'(k, \theta; s)$, are the next-period weights that are consistent with the entitlement of continuation utilities.[18]

In what follows, we say that a sequential plan (C, K') is generated by the set of policy functions $(\widehat{c}_i(k, \theta; s), \widehat{k}'(k, \theta; s), \widehat{\theta}'(k, \theta; s))$ solving (4)-(8) as

$$\begin{aligned} C_i(s^t) &= \widehat{c}_i(K(s^{t-1}), \theta(s^{t-1}); s_t), \\ \theta(s^{t-1}, s_t) &= \widehat{\theta}'(K(s^{t-1}), \theta(s^{t-1}); s_t), \\ K(s^{t-1}, s_t) &= \widehat{k}'(K(s^{t-1}), \theta(s^{t-1}); s_t), \end{aligned} \tag{9}$$

for all t and all $s^t \in S^t$, given θ_0 and K_0.

3 Characterization

In this section we characterize the solution of the model. First, we briefly describe the optimal allocation in an economy with full information (i.e., preference shocks are observable). Next, we characterize the constrained efficient allocation under private information.

3.1 Insurance under Full Information

Consider the case of the economy with full information (i.e. liquidity shocks are perfectly observable). Since $\theta_1 = 1 - \theta_2$ and the model is symmetric between both agents, hereafter we refer to agent 1's weight directly as θ. The proofs of the results in this section are provided in Appendix B.

There are two sources to finance the extra consumption that an agent receives after reporting high liquidity needs—namely, *redistribution* and *disinvestment*. Consider the increment of agent 1's consumption as he reports high liquidity needs compared with the case in which he reports low needs. His consumption is still conditional on agent 2's report, s_2. Define the share of this increment that is financed by means of disinvestment as

$$\text{DInv}(k, \theta, s_2) = \frac{k'(k, \theta; s_L, s_2) - k'(k, \theta; s_H, s_2)}{c_1(k, \theta; s_H, s_2) - c_1(k, \theta; s_L, s_2)}; \tag{10}$$

that is, the fraction of the higher consumption financed by investing less.

Figure 1a describes the disinvestment share and shows that it is increasing in the agent's

[18] It is well-known that there is an issue regarding renegotiation-proofness in dynamic contracts, even in simpler settings (see Wang, 2000). However, in our setting with investment, capital can be manipulated to make sure that constrained efficient plans are always renegotiation-proof. Details are available upon request. We thank an anonymous referee about the need to mention this property.

weight.[19] As partner 1's weight increases, the marginal cost of financing higher needs of consumption through redistribution increases because partner 2's consumption decreases. Hence, the share financed with disinvestment increases. Moreover, the lower the shock reported by partner 2, the higher the disinvestment. The intuition is that when partner 2 reports s_L, his current consumption is less desirable than future consumption. Hence, investment (and so next period capital) is higher and therefore it is efficient to increase partner 1's consumption with more disinvestment.

Lemma 1 characterizes the main features of partners' consumption and investment. First, efficiency dictates that welfare weights are kept constant. This result is especially interesting because it will contrast with the behavior of welfare weights under private information. To grasp the intuition of this result, think about a fictitious planner who wants to distribute utility across agents optimally. The valuations of delivering one more unit of utility to agents 1 and 2 are θ_1 and θ_2, respectively. On the other hand, the valuations of delivering one more unit of continuation utility to agents 1 and 2 at s are $\beta\pi(s)\theta_1$ and $\beta\pi(s)\theta_2$, respectively. Consequently, the relative valuation remains unchanged at θ_1/θ_2 and this implies that the normalized weights must satisfy $\theta'(s) = \theta$. This reasoning makes evident the difference with the case under private information. There, continuation utilities are additionally manipulated to provide incentives and so their valuations can differ.

Lemma 1 also describes how consumption depends on welfare weights and liquidity needs shocks. Part of the increase in a agent's payout after reporting high liquidity needs is financed by means of disinvestment; i.e., next period capital is smaller when an agent reports high liquidity needs than when he reports low liquidity needs.

Lemma 1 (Full Information)**.** *Under full information:*

1. *Welfare weights do not change; i.e.* $\theta'(k, \theta; s) = \theta$ *for all* s*, all* k *and all* θ*.*

2. *If* $u(c) = c^{1-\sigma}/(1-\sigma)$*, the optimal investment and distribution policy of consumption under full information satisfy:*

    (a) *The fraction of total consumption that is paid to agent* i *is increasing in his liquidity needs, decreasing in the other agent's liquidity needs, and increasing in his weight.*

[19] For all figures we use numerical results derived with the following assumptions. The utility function is $\frac{c^{(1-\sigma)}}{1-\sigma}$, the profit function is $f(K) = K^\alpha$ with $0 < \alpha < 1$, and the shocks are $s(L) = 1-\epsilon$ and $s(H) = 1+\epsilon$. The parameter $\sigma = 0.5$, as in other studies of private information such as those by Hopenhayn and Nicolini (1997) and Pavoni (2007). The parameter $\alpha = 0.7$ is in the range of estimations of Cooper and Ejarque (2003). We set $\delta = 0.07$ and $\beta = 0.97$ as is standard in the literature. We consider $\epsilon = 0.6$ just for illustrative proposes.

(b) The level of investment is decreasing in the agents' needs of liquidity; i.e.

$$k'(k,\theta;s_L,s_2) > k'(k,\theta;s_H,s_2),$$
$$k'(k,\theta;s_1,s_L) > k'(k,\theta;s_1,s_H).$$

3.2 Incentives to Cheat and the Size of the Pareto weight

The remainder of this section studies insurance with privately observed liquidity socks. First, we describe the key difference with previous results. We show that when welfare weights are relatively large the corresponding incentive compatibility constraints become slack. Since incentive to cheat disappears when the weights are big enough, we call this result "too big to cheat." Next, we discuss the intuition for the driving mechanism. Appendix C contains all the proofs in this section.

Proposition 1 (Too big to cheat)**.** *Given k, there exists some value of the agent 1's welfare weight $\overline{\theta}(k) \in (0,1)$ such that the agent 1's incentive compatibility constraint does not bind for all (k,θ) with $\theta \in \left[\overline{\theta}(k), 1\right]$.*
Similarly, the agent 2's incentive compatibility constraint does not bind for all (k,θ) with $\theta \in \left[0, 1-\overline{\theta}(k)\right]$.

The underlying intuition for this result can be grasped as follows. Cheating implies that the agent misrepresenting high liquidity needs will receive higher consumption. The resources for that extra consumption are obtained from two sources: (i) decreasing consumption of the other agent and (ii) reducing investment. Note that this higher consumption is not necessarily beneficial for this agent. It would be beneficial if it is financed with a reduction in the other agent consumption, but it may not be beneficial if it is financed with a reduction in investment, because it implies that future output will be lower. The magnitude of the second force depends (and is increasing) in the value of the Pareto weight which represents how much of future output will be consumed by this agent.

If one agent's weight is equal to one, all the extra funds he receives after cheating are only financed by disinvestment because the consumption of the other agent is already zero. He also fully internalizes the effect of the investment distortion, because given that the Pareto weight is one, he will own all future output. Therefore, when the agent's Pareto weight is equal to one, he would be strictly worse off misrepresenting his liquidity needs. Now, since small changes in the Pareto weight change consumption and investment only slightly (continuity), cheating will not be desirable even if the Pareto weight were slightly smaller than 1. Of course, as the Pareto weight decreases further, at some point the gains of cheating are larger than the costs and the full information allocation is no longer incentive compatible.

Although this result demonstrates that at some point the incentives to cheat vanish completely (the multiplier of the incentive compatibility constraint is zero), the main takeaway should be different. It should be that (i) as long as the agent's weight is greater than zero, the distortion in investment creates a force that helps in providing incentives for truthful revelation and (ii) that force is increasing in the value of the agent's Pareto weight.

When the Pareto weight is smaller than the threshold described above, the constrained efficient allocation changes with respect to the full information allocation to provide incentives for truthful revelation. In particular, it changes in three dimensions: consumption, continuation weights, and investment. The distortions in consumption are standard in the literature; i.e., the constraint efficient allocation provides less consumption insurance in order to provide incentives. We now study how continuation weights and investment are distorted in the constrained efficient allocation.

Continuation weights are manipulated to provide incentives for truthful revelation of the shock. Lemma 2 characterizes the evolution of Pareto weights under private information. In general, to provide incentives to report low preference shocks, the constraint efficient allocation punishes the report of a high preference shock by decreasing his future weight. When agent 1's incentive compatibility constraint does not bind, there is no need to provide incentives for agent 1, and as a consequence future weights are independent of his report. But weights could still depend (and, in general, will) on the other partner's report. By symmetry, if agent 2 reports a high shock, his weight decreases, and the weight of agent 1 increases. Finally, note that when both incentive compatibility constraints are slack and there is no need to provide incentives, future weights are equal to current weights and independent of reports about preference shocks.[20]

Lemma 2 (Dynamics of weights)**.** *Partner 1 weight evolves as follows:*

1. $\theta_1'(k, \theta; s_L, s_2) \geq \theta_1'(k, \theta; s_H, s_2)$ *for* $s_2 \in \{s_L, s_H\}$.

2. *Moreover, if the incentive compatibility constraint of partner 1 does not bind, then* $\theta_1'(k, \theta; s_L, s_2) = \theta_1'(k, \theta; s_H, s_2)$ *for* $s_2 \in \{s_L, s_H\}$.

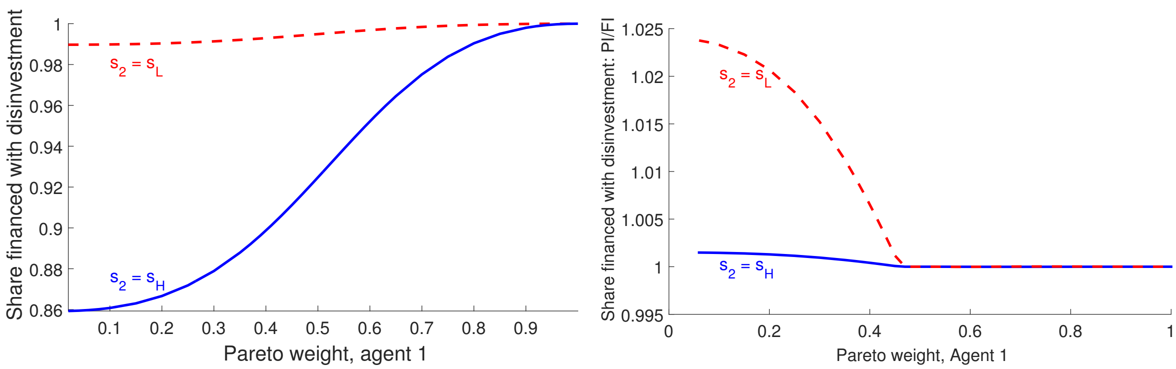
3. *Moreover, if no incentive compatibility constraint bind, then* $\theta_1'(k, \theta; s_1, s_2) = \theta$ *for* $(s_1, s_2) \in \{s_L, s_H\} \times \{s_L, s_H\}$.

Finally, capital accumulation is also distorted and disinvestment is stronger under private information than under full information. This is shown in Figure 1b, which displays the dis-

[20]There is a subtle difference here between future utility and future Pareto weight. Imagine the case in which both incentives constraints are slack. As we mentioned, in that case the future Pareto weights will be independent of the report. However, promised utilities will not be independent of the report, as capital accumulation does depend on the report even in the full information allocation.

investment share for private information relative to that of full information (i.e., the ratio of equation 10 in private information to full information). Note that this ratio is above one for low values of θ, when partner 1 incentive constraint is binding. To understand this, recall that the reduction in investment following a report of a high preference shock is the cost of cheating. Thus, to prevent cheating, it is natural that the optimal contract prescribes more disinvestment. The fact that investment is distorted under private information is the key difference with the work of Marcet and Marimon (1992). As we discuss below, this occurs because agents are risk averse in our setup.

Figure 1: **Disinvestment**



(a) Full information

(b) Distortions under private information

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3.3 Existence of the Too-Big-To-Cheat

In the general setting in which both partners face liquidity shocks, we show a sufficient condition such that private information does not matter in the long-run. This condition is that the full information plan is a *strictly incentive compatible plan* (i.e., a plan that satisfies both incentive compatibility constraints with strict inequality) at $\theta = 1/2$ for all k. This feature of the plan relies on the characteristics of the partnership and so we next check that this set is non-empty.

The following proposition analyzes the role of one of the parameters that describes the characteristics of the model. In particular, it characterizes the size of the liquidity needs shocks that make it possible to attain the full information plan with an equally shared ownership structure. In order to do that, we parameterize liquidity shocks so that $s_L = 1 - \epsilon$ and $s_H = 1 + \epsilon$ for both partners.

Proposition 2 (Existence of "too big to cheat" region)**.** *For any partnership, there exists some $\epsilon^* \in (0,1)$ such that for all $\epsilon \in (\epsilon^*, 1)$ the full information plan is strictly incentive compatible for both agents at $\theta = 1/2$ for all k.*

The proposition above states both partners have no incentive to cheat with $\theta = 1/2$ if the low value of the preference shock is sufficiently low, relatively to the high value of the shock. To understand the result, consider the case of $\epsilon = 1$. This implies that consumption is not valued if the low preference shock is realized; i.e. $s_L = 0$. In this case, agents with the low value of the preference shock would be strictly worse off misrepresenting their shock; they would obtain extra consumption today, when it is not valued, at the expense of lower consumption tomorrow (because investment decreases), when it is valued. If ϵ is close to 1 and each agent weight is sufficiently high, the same forces are present and there are no incentives to cheat. Thus, for $\theta = 1/2$, there exists an $\epsilon^* < 1$ such that for all $\epsilon \in (\epsilon^*, 1)$ the full information plan is incentive compatible for both agents.

While this result is shown with the difference between the high and the low value of the shock, similar forces are at work when other parameters are changed. For instance, while increasing ϵ helps to provide incentives because the agent assigns a very low value for consumption today, increasing β would be similar because it increases the weight that the agents assigns to the distortion on investment.

In the remainder of this section we study the case in which the full information plan is strictly incentive compatible at $\theta = 1/2$ for all k.

3.4 Long-run Convergence

The following result establishes that private information does not matter in the long run.

Proposition 3 (In the long run, sole proprietorship or equally owned partnership)**.** *Suppose that the full information plan is strictly incentive compatible at $\theta = 1/2$ for all k. Then, there exists $\theta^* \in (0, 1/2)$ such that*

1. *If $(k_t, \theta_t) \in [k_{\min}(\theta_t), k_{\max}(\theta_t)] \times [\theta^*, 1-\theta^*]$ at some t, then the partnership's size and ownership structure, (k_t, θ_t), stay in a region in which the constrained efficient plan and the full information plan coincide and $\theta_{t+n} = \theta_t$ for all $n \geq 0$.*

2. *If $(k_t, \theta_t) \in [k_{\min}(\theta_t), k_{\max}(\theta_t)] \times [0, \theta^*]$ at some t, the partnership's size and ownership structure, (k_t, θ_t), reach a region in which the constrained efficient plan and the full information plan coincide with probability 1; i.e. $\theta_t \to \{0, [\theta^*, 1-\theta^*]\}$ a.s.*

3. *If* $(k_t, \theta_t) \in [k_{\min}(\theta_t), k_{\max}(\theta_t)] \times [1-\theta^*, 1]$ *at some* t*, the partnership's size and ownership structure,* (k_t, θ_t)*, reach a region in which the constrained efficient plan and the full information plan coincide with probability 1; i.e.* $\theta_t \to \{[\theta^*, 1-\theta^*], 1\}$ *a.s.*

As long as the full information plan is strictly incentive compatible at $\theta = 1/2$ for all k, Proposition 3 states that in the long run private information becomes irrelevant and the Pareto weights remains unchanged. This may happen because (i) one of the agents' Pareto weight is equal to one, or (ii) both Pareto weights are approximately equal to 1/2. Notice that the too-big-to-cheat region defined in Proposition 2 can be reached either immediately (as the initial weights and the initial capital stock starts there) or in the long run (as the weights and the capital stock converge as time and uncertainty unfold).[21]

The fact that in the long run private information does not matter resembles previous results.[22] What is new in our setup is that this may happen in a region in which both agents have positive weights. As we mentioned before, this is possible due to the inclusion of capital accumulation.

Proposition 3 hinges on the existence of a region in the space of capital and Pareto weights such that both incentive compatibility constraints are not binding (Proposition 2 shows this case exists for some configuration of the preference shock). This result is important because it implies that the immiseration result, which has been widely studied in private information problems, does not hold. More generally, when the condition in Proposition 2 is not satisfied, for instance because the preference shock is less coarse, the takeaway should be that due to the incentives provided by capital accumulation, the economy will spend more time in the surroundings of equal weights than in an endowment economy.

4 The Role of Investment

What role do investment and capital accumulation play in our results? In this section we argue that capital accumulation is crucial for our results.

We study the same problem as above but assume that the resources available cannot be affected by investment and thus are given every period. Thus, this framework resembles that of Atkeson and Lucas (1992), but with only two agents.[23]

[21]Note that Proposition 3 assumes that k_t starts in the ergodic set for capital under full information, $[k_{min}(\theta_t), k_{max}(\theta_t)]$. We performed numerical exercises with very small or very large k_t and found that under private information capital converges to this interval in about 65 periods.

[22]As in Thomas and Worral (1990), the simplified argument is the following: imagine Pareto weights converged and private information still matters. Then future Pareto weights must be spread to provide incentives, contradicting the initial statement that Pareto weights converged.

[23]To facilitate the comparison in the examples below, the level of resources in the economy without capital

In the endowment economy the full information plan violates the incentive compatibility constraints for both agents for all $\theta \in (0, 1)$. To understand this result, consider the incentive compatibility constraint under the full information allocation in the endowment economy. Continuation utilities are independent of the reports about preference shocks, and therefore the incentive compatibility constraint only depends on consumption. Consumption is strictly increasing in the preference shock for all $\theta \in (0, 1)$. As a result, the incentive compatibility constraint is always violated. At $\theta = 0$ and $\theta = 1$ the agent is indifferent between cheating or not because future utility and consumption are independent of the report. In contrast, in the production economy the full information allocation dictates that investment, and as a consequence, continuation utilities depend on the reports. Hence, at $\theta = 0$ and $\theta = 1$, the agent would be strictly worse off by cheating.[24]

Figure 2 illustrates this point further. The dashed line represents the welfare gains of moving from private to full information in an endowment economy. There are two main results: (i) the maximum welfare gains are about 0.1 percent in terms of consumption equivalent units[25] and (ii) the maximum value of welfare gains occurs when the partners have equal weights. The solid blue line displays the results of performing the same exercise but for our benchmark economy with capital accumulation. In sharp contrast, in the production economy we find that (i) welfare gains are one order of magnitude smaller; and (ii) these gains are actually zero when both partners have the same weight. In general, the last result means that equal weights minimize the cost of private information in a production economy.

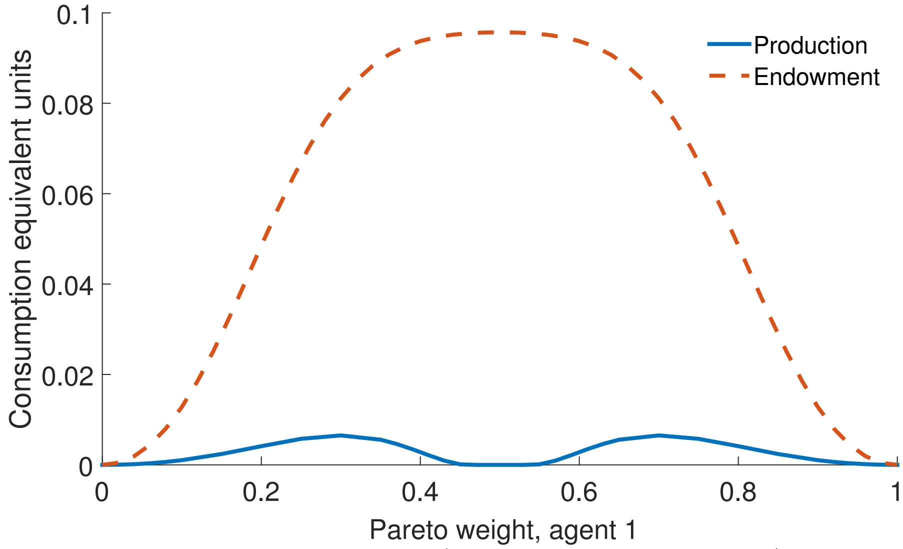
Up to this point, we consider the case in which the conditions of Proposition 2 are satisfied and the full information plan is strictly incentive compatible at $\theta = 1/2$ for all k. However, capital accumulation still mitigates incentives to cheat even if this condition does not hold. To show this, we simulate both the production and the endowment economy 1,000 times for 100 periods starting at $\theta = 0.5$ for the case in which $\epsilon < \epsilon^*$ and the full information plan is not incentive compatible at $\theta = 1/2$. Recall that under full information weights do not change, so θ would be constant at 0.5 forever. Figure 3 shows the resulting distribution of θ over those 100 periods for both the endowment and the production economy under private information. The production economy (solid blue line) spends more time close to $\theta = 0.5$ than the endowment economy (red dashed line). This result confirms that the main mechanism highlighted in this paper—that capital accumulation mitigates the role of private information—is at work even

accumulation are set at the output produced with the mean of the steady-state level of capital in the economy with capital accumulation.

[24]Technically, this explains why a continuity argument can be used in the production economy to show Proposition 1 but not in the endowment economy.

[25]Note that this number is slightly more than 10 times higher than the gains from eliminating business cycles estimated by Lucas (1987) and similar to those found by Krusell and Smith (1999) in an economy with large heterogeneity.

Figure 2: **Welfare gains of moving from private to full information**



Note: Consumption equivalent units is defined as $100\left((H^{*}(k,\theta)/H^{**}(k,\theta))^{\frac{1}{1-\sigma}}-1\right)$*. For the production economy, we show the cost for the value of capital such that* $F(k)$ *is equal to the endowment of the economy without capital accumulation.*

when the technical condition imposed in Proposition 2 is not satisfied.

5 Discussion of Key Assumptions

In this section we discuss the role played by internal financing, the coarseness of state space for the liquidity need shocks, and present some numerical simulations to study the long run when the condition needed for Proposition 2 does not hold.

5.1 External Financing

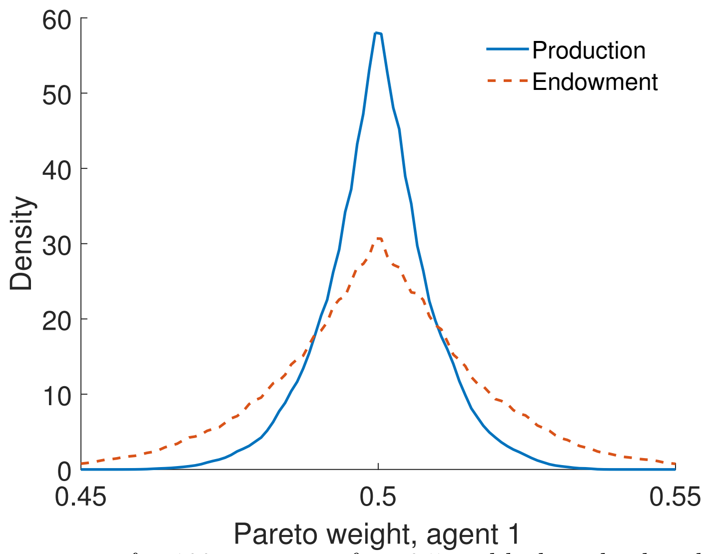
To study the role of external financing, we consider an extreme case in which the partners have perfect access to capital market. To operationalize this, we contemplate an environment similar to Marcet and Marimon (1992), in which agent 2 is risk neutral, faces no shocks, and has deep pockets.[26]

In this setup, for all $\theta \in (0,1)$, the first-order condition that characterizes optimal investment of the private information plan implies

$$1 = \beta\left(f'(k^{*}) + (1-\delta)\right).$$

[26]The assumption that agent 2 does not face shocks is made only for simplicity.

Figure 3: **Evolution of Pareto weights**



Note: We simulate 1000 economies for 100 starting at $\theta_0 = 0.5$ and look at the distribution of θ under both the production and the endowment economy.

This finding is analogous to the result in Marcet and Marimon (1992), who find that private information does not distort optimal investment. This result is a direct consequence of evaluating the investment decision with the intertemporal marginal rate of substitution of the risk-neutral agent.

Note that the capital stock in this setup will jump directly from k_0 to k^* and remain constant forever (in particular, the reports of the agent with private information do not change investment). Therefore, under full information, future utility is independent of the reports, and as we explained in the case of the endowment economy, this implies that the full information plan violates the incentive compatibility constraints for all $\theta \in (0, 1)$.

5.2 Continuous State Space for the Liquidity Need Shocks

We have assumed that the liquidity shock can take only two values but it is useful to briefly discuss the case with continuous state space for the liquidity need shocks. We argue that there is less incentive to misrepresent the shock once we allow for capital accumulation even if the shock is continuous. For this purpose, we assume the agent is offered the full information plan and compare the reported liquidity need shock both in the endowment economy and in the economy with capital accumulation.

First, with no investment possibilities, if the partners are offered the full information plan, any partner would always have incentive to report the highest liquidity shock as consumption is increasing in the report, independently of his share of ownership in the partnership. At the individual level, there are no incentives to report something smaller since the increment in payout would always come from a reduction in consumption of the other agents. Importantly, this result holds, with both discrete and continuous state space for the liquidity needs shocks.

In contrast, in the economy with investment, the size of the optimal misrepresentation (reported θ - actual θ) under the full information plan depends on the Pareto weight. As the partner's Pareto weight increases, his desired report approaches his realized shock because the part of his increment in consumption that is financed with disinvestment is increasing in his size. In the limit, for $\theta = 1$ the full information plan is actually incentive compatible. In particular, the further the reported shock from the truth, the larger the loss of cheating (because investment is more distorted). Now, for a Pareto weight close to 1, by continuity of the allocation on the Pareto weight, the only misreport that can be desirable is a marginal one, which is available only if the shock is continuous (if the shock is discrete, this means cheating is undesirable). This is the key difference with the model without capital accumulation, in which, as we argued above, the desired misreport would be the largest irrespective of the coarseness of the shock.

5.3 Long-run Convergence when the Too-Big-To-Cheat Region Does Not Exist

Consider the alternative case in which the full information plan is not strictly incentive compatible at $\theta = 1/2$ for all k.[27] As in this case the incentive compatibility constraint of at least one partner is always binding, it follows by the same arguments as in Proposition 3 that partnerships do not exist in the long run since, if it converges, only one of the partners will own 100 percent of the partnership.

We provide some numerical examples to highlight that both the initial Pareto weights and the assumption of strict incentive compatibility are key to determining the long run outcome. Table 1 shows the results of simulating 1,000 partnerships until the ownership structure converged. We start partnerships with different initial Pareto weights, $\theta_0 \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$.

Under full information the ownership structure in the long run coincides with the initial distribution. In contrast, there are two possibilities under private information, depending upon whether the full information plan is strictly incentive compatible at $\theta = 1/2$ for all k or not. The first case under private information is illustrated for $\epsilon > \epsilon^*$ while columns TBTC, IM1 and IM2 refer to convergence to θ^*, $\theta = 0$ and $\theta = 1$, respectively. In line with the results of Proposition 3, as long as $\theta_0 < \theta^*$ the partnership converges to either $\theta = 0$ (i.e. immiseration

[27]Suppose, for instance, that $\epsilon < \epsilon^*$.

Table 1: **Long-run convergence**

θ_0		$\epsilon > \epsilon^\star$ **TBTC**	**IM1**	**IM2**	$\epsilon < \epsilon^\star$ **TBTC**	**IM1**	**IM2**
0.1	$E(\theta)$	0.47	0.00			0.00	1.00
	% convergence	12	88	0	0	95	5
0.2	$E(\theta)$	0.47	0.00			0.00	1.00
	% convergence	30	70	0	0	86	14
0.3	$E(\theta)$	0.47	0.00			0.00	1.00
	% convergence	50	50	0	0	75	25
0.4	$E(\theta)$	0.47	0.00			0.00	1.00
	% convergence	79	21	0	0	62	38
0.5	$E(\theta)$	0.50				0.00	1.00
	% convergence	100	0	0	0	48	52

Note: We simulate N=1000 economies of length T = 1,000,000, and keep the last t=1,000 periods. TBTC is defined as $E[\theta] \in [0.45, 0.55]$*, IM1 is defined as* $E[\theta] \leq 0.1$ *and IM2 is defined as* $E[\theta] \geq 0.9$*.*

of partner 1) or θ^*. The closer is θ_0 to θ^*, the more frequently the partnership converges to the TBTC region with more equally distributed ownership structure. If $\theta_0 \in [\theta^*, 1/2]$, then $\theta_t = \theta_0$ for all t because the full information plan can be attained immediately. The second case under private information is illustrated for $\epsilon < \epsilon^*$. As noted above, only sole proprietorship can be reached with positive probability in the long run. The closer is θ_0 to 0, the more frequently the partnership converges to $\theta = 0$ (i.e. immiseration of partner 1) and vice versa.

5.4 Symmetry

Although throughout the paper we assume that the partnership is symmetric (the agents are ex-ante identical), our theory is certainly more general. This assumption is important for the result that the too-big-to-cheat region is around 50-50. If partners were asymmetric we would find two thresholds, θ_1 and θ_2, such that this region would be $[\theta_1, \theta_2]$. Note that if the asymmetry is sufficiently large this region might not include $\theta = 1/2$. For instance, if only agent 1 faces preference shocks, then the too-big-to-cheat region is $[\theta_1, 1]$. As we will show below, studying the symmetric case is appealing for the application we consider.[28]

[28]As the model's prediction is that the ownership structure would spend more time (or in the cross section, is more likely) around the too-big-to-cheat region, we can potentially test the validity of this assumption. For our application, we argue this assumption has empirical support.

6 Application: Small-business Partnerships

The framework presented above has several features that resembles the organization of small-business partnerships. Small-business are recognized to be an important driver of the economy. For example, firms with fewer than 20 employees in the US represent about 87 percent of all firms, and 19 percent of total employment in 2005 (See Hurst and Pugsley (2011)). They also show that most small businesses intend to provide an existing service to an existing market, have little desire to grow big, and non-pecuniary benefits (being one's own boss, having flexibility of hours, etc.) play a first-order role in the business formation decision. These businesses are concentrated among skilled craftsmen, lawyers, real estate agents, doctors, small shopkeepers, and restaurateurs, among others. Notice that we refer to these firms as small-business partnerships to differentiate them from large public companies.[29] We do not mean partnership as in the legal status.[30]

This paper contributes to the study of the internal organization of small-business partnerships and shows how private information can distort the investment policies of those firms. We first argue that the model captures the main features of small-business partnerships. Then, we discuss empirical evidence in line with some predictions of our theory.

6.1 Why This Is a Good Model of Small-business Partnerships?

There are five characteristics of small-business partnerships that are well captured by our framework.

1. **Small number of owners:** According to the Kauffman Firm Survey, most of the starting businesses have very few partners: 60 percent have only one owner, almost 30 percent have two owners, and about 6 percent have three owners. This fact is robust across different data sources and countries. In the Survey of Small Business Finances, about 50 percent of the firms have one owner, slightly more than 30 percent have two owners, and 7.5 percent have three owners. In the CAF Survey, which covers different countries in Latin America, 60 percent of the small businesses have one owner, 27 percent have two owners, and 7 percent have three owners.[31] Hence, our assumption of two agents seems adequate to understand the behavior of small business partnerships.

2. **Production technology:** We model the resources of the partnerships as derived from production, instead of endowment, and allow for investment to accumulate capital. It

[29]According to Magill and Quinzii (1996) (1996, p. 331), partnerships (in contrast to corporations) are characterized by not having access to the stock market and limited access to loans.

[30]The legal status and tax treatment differences are ignored in this analysis.

[31]See Espino, Kowloski, and Sanchez (forthcoming) for more details.

is important to take into account investment because in reality the share of investment to output is about 20 percent across all firms, and this share is even larger for smaller businesses.[32] The modeling of production follows, for instance, Lucas (1978). As we show in Section 4, allowing for capital accumulation is not only a reasonable assumption for a firm's production but also key for our results.

3. **Liquidity shocks:** A preference shock is introduced in our theory to capture, as in Diamond and Dybvig (1983) and in all the literature that followed, the existence of a random need for liquidity on the part of individuals. As in Diamond and Dybvig (1983), the realization of the shock is unobservable. In our model, and in reality—as highlighted in Wasserman (2012)—a partner's personal life may affect his commitment and contributions to the business. For example, extreme and unexpected health problems can catch all parties by surprise.[33] Given the uncertainty about liquidity needs, the partnership may be useful as an insurance mechanism. Although direct evidence on the relevance of this mechanism is hard to obtain, Hurst and Pugsley (2011) provide evidence that some non pecuniary benefits play a first-order role in the business formation decision. One of such a benefit could certainly be the insurance provided by the ownership of the business.[34] Moreover, Berk and DeMarzo (2014) argue that the small business owner is forced to hold a large fraction of his or her wealth in a single asset—the company—and therefore is likely to be undiversified and value the insurance provided from this asset (the company).

4. **Internal Financing:** Classical references and recent papers have found that access to external financing is limited and small businesses rely heavily on internal resources. Butters and Lintner (1945) provide some of the earliest research to support this theory. They examine the early histories of several industries and conclude that many small companies, even companies with promising growth opportunities, find it extremely difficult or impossible to raise outside capital on reasonably favorable terms, and the majority of small firms finance their growth almost exclusively through retained earnings. More recently, Hubbard (1998) and Carpenter and Petersen (2002) show that growth of most small firms

[32] See, for example, Evans (1987a) and Evans (1987b).

[33] Interesting, anecdotal evidence of this type of shock is available. While Microsoft was still a private company, co-founder Paul Allen was diagnosed with Hodgkin's lymphoma, which caused him to quit the company, leaving Bill Gates as the sole active founder during the crucial three years before it became a public company. We thank an anonymous referee for the references. See https://en.wikipedia.org/wiki/Paul_Allen.

[34] The predictions of the model are the same if the shock is reinterpreted as a shock to the cost of effort, under some assumptions. For example, assume that preferences depend on consumption and leisure and each partner must do work in a fixed amount, L, measured in efficiency units. Every period, the productivity varies, and the individuals must adjust the effort to compensate for changes in productivity. If leisure and consumption are substitutes, when the agent reports low productivity (e.g., back pain), she is reporting that effort was high, so that is equivalent to reporting a high marginal utility of consumption (a high preference shock) as in our model.

is constrained by the availability of internal finance. Similarly, Berk and DeMarzo (2014) argue that one limitation of small businesses is that they do not have access to outside equity capital, so they have relatively little capacity for growth. Finally, note that outside the US, where in general financial systems are less developed, this argument is even stronger.

Although we assume that all the resources for investment must be raised internally, that assumption is made only to simplify the analysis and sharpen the results. As far as outside financing is limited, and firms must adjust investment to respond to liquidity needs, the new mechanism, which we want to highlight, will be in place. In contrast, if we assume perfect access to capital markets, investment is not distorted and the problem is equivalent to a model without investment (endowment economy).[35]

In addition to no access to capital markets, we assume that the partners do not set resources aside (precautionary savings) to use in case of liquidity needs. This assumption, which greatly simplifies the analysis, is also supported by empirical evidence. Gentry and Hubbard (2000) show that the portfolios of entrepreneurial households, even wealthy ones, are very undiversified, with the bulk of assets held within active businesses. Similarly, Kaplan, Violante, and Weidner (2014) document that many households are *wealthy hand-to-mouth*: they hold little or no liquid wealth despite owning sizable quantities of illiquid assets. Hence, many of these entrepreneurial households choose to tap into business earnings after a negative shock.

5. **Shared ownership:** We interpret Pareto weights, θ_i, as ownership shares of the business. These firms are privately-owned and these shares should not be interpreted as equity shares that can be traded in the capital market. We propose that partners directly solve problem (4)-(8) and how much they own of the business is mapped into how much their utility is weighted in the problem. Even though we do not study the initial determination of shares (we take them as given), those weights could be determined by Nash bargaining between partners that make different contributions to the formation of the firm (the initial contribution could be the business idea or financial resources). These initial weights would be increasing in the initial contribution of each individual to the partnerships.

6.2 Testable Implications

Some features of our theory are common to existing models of partnerships while others are particular to this paper. In terms of the more general implications, there are two characteristics

[35] See Section 5.1.

that are highlighted in several studies: (i) small businesses usually have ownership shares equally distributed among partners, and (ii) most small businesses have 2 or 3 owners. These facts are well described using representative data for the US by Vereshchagina (2015) and Espino, Kowloski, and Sanchez (forthcoming) and using information on medical practices alone by Encinosa III, Gaynor, and Rebitzer (2007).

First, our theory can account for the fact that most of partnerships have equal shares of ownership because we have argued that such structure facilitates the provision of incentives under private information. Second, although we considered a setup with only two partners, our theory can be used to think about the effect of adding more partners. If the number of partners is $n > 2$, each member would have a smaller share even if the ownership structure is equally distributed (just because $1/n$ may be too small). This implies that private information problems would be exacerbated. Thus, our theory can also be used to understand why most small businesses have 2 or 3 owners.

Since the previous facts can also be accounted for other theories (e.g. Fehr, Kremhelmer, and Schmidt, 2008), we now focus on other, more distinct, predictions of our theory. In particular, our theory has testable predictions about the dynamics of ownership shares that are, a priori, harder to rationalize with other existing theories. Under private information, our theory implies that: (i) ownership shares change more frequently when firm ownership is unequally distributed, and (ii) the distribution of ownership shares moves over time towards either equal distribution of ownership or sole-proprietorship.

With regard to the first testable implication, Espino, Kowloski, and Sanchez (forthcoming) find that partnerships with equally distributed shares, which according to the theory would be less affected by private information, changed their ownership structure less frequently than those with unequally distributed shares. Although one possibility is that there are observable differences between equal and unequal partnerships that may account for the dissimilarity in the dynamics of the ownership structure, Espino, Kowloski, and Sanchez (forthcoming) show that this finding is robust to adding controls for age, number of owners, legal status, profitability, and number of employees.

To evaluate the second testable implication, Espino, Kowloski, and Sanchez (forthcoming) compare family firms and private firms. This comparison is useful because in family firms private information problems are less severe than in private firms. The results show that, conditional on other observable features of the firms, private firms are more likely to converge over time to an ownership structure with equal distribution of ownership. Related results are obtained comparing small firms in different countries, with different values in an index of corporate transparency.

Finally, Ivashina and Lerner (2016) analyze the evolution of ownership shares in private equity funds which are other types of partnerships. They find evidence consistent with the last two predictions of the theory. Specifically, when ownership shares are unequally distributed, they tend to move towards more equally distributed. Moreover, partners with lower shares are significantly more likely to leave the partnership.

Overall, the findings suggest that private information may be important to shape the organization of small business and partnerships.

7 Conclusion

Previous studies showed that private information is important for determining the extent to which agents can pool idiosyncratic risk in partnerships. The analysis in this paper suggests that insurance capabilities of partnerships improve as a consequence of the introduction capital accumulation. Under certain conditions, we show that (i) when a partner's Pareto weight is sufficiently large, her incentives to cheat under the full information allocation disappear, (ii) the full information allocation may be incentive compatible when both agents have equal Pareto weights, and (iii) in the long run, either one of the partners is driven to immiseration, or both partners' lifetime utilities are approximately equal.

Our theory has several features that resemble small-business partnerships: small number of owners, shared ownership, production possibilities, liquidity shocks, and internal financing. Therefore, we use our theory to derive predictions for the organization of these businesses: (i) increasing the number of partners worsens incentives problems, (ii) equal shares of ownership facilitates the provision of incentives, (iii) ownership shares change more frequently when firm ownership is unequally distributed, and (iv) the distribution of ownership shares moves over time towards either equal distribution of ownership or sole-proprietorship. We argue these findings can be found in the data.

Although our application focused on small-business partnerships, our theory can also be applied to different settings. For instance, a partnership could be reinterpreted as an economic union among several countries. Then, the size of the countries (in terms of how much wealth they have relative to the union) would be important for determining the extent to which misreporting must be prevented by the union's structure and regulations. In an economic union between a large and a small country, our results suggest that the small country would have incentives to misreport if the union regulations are not carefully designed. Moreover, our theory predicts that adding more countries to the union—and thereby reducing each member's share—would exacerbate information problems.

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Appendices

A Recursive formulation

In this Appendix we show how to write the partnership problem recursively. Our analysis here generalizes to I partners with privately observed shocks to liquidity needs since our alternative recursive approach does not depend on our 2-partner assumption. Abusing our notation, we denote $s \in \{s_L, s_H\}^I$ and $(s_i, s_{-i}) \in \{s_L, s_H\} \times \{s_L, s_H\}^{I-1}$.

Let $\Delta^I \equiv \{\theta \in \mathbb{R}^I_+ : \sum_{i=1}^I \theta_i = 1\}$ and $\|h\| = \sup_{(k,\theta)} \left\{| \ h(k,\theta) \ |: \theta \in \Delta^I\right\}$ and define

$$F \equiv \{h : X \times \mathbb{R}^I_+ \to \mathbb{R}_+ : \ h \text{ is continuous and } \ \|h\| < \infty\},$$

as the set of continuous and bounded functions mapping $X \times \mathbb{R}^I_+$ into $\mathbb{R}_+$, and denote

$$\overline{F} \equiv \{h \in F : h \text{ is HOD 1 and concave in } k\}.$$

as the subset of functions that are homogeneous of degree 1 and concave.

Given the metric induced by $\|.\|$, observe that $\left(\overline{F}, \|.\|\right)$ is a closed subset of the Banach space $(F, \|.\|)$ and thus a Banach space itself. Define the operator T defined on $\overline{F}$ as follows

$$(Th)(k,\theta) = \sup_{(c,w',k')} \sum_{i=1}^I \theta_i \left\{ \sum_s \pi(s) \left[s_i \ u(c_i(s)) + \beta \ w'_i(s) \right] \right\}, \tag{11}$$

subject to

$$k'(s) + \sum_{i=1}^I c_i(s) = f(k) + (1-\delta)k \tag{12}$$

$$\begin{aligned} &\sum_{s_{-i}} \pi(s_{-i}) \left(s_i u(c_i(s_i, s_{-i})) + \beta \ w'_i(s_i, s_{-i}) \right) \\ &\geq \sum_{s_{-i}} \pi(s_{-i}) \left(s_i u(c_i(\widetilde{s}_i, s_{-i})) + \beta \ w'_i(\widetilde{s}_i, s_{-i}) \right) \end{aligned} \tag{13}$$

for all $(s_i, \widetilde{s}_i)$ and

$$c_i(s) \geq 0, \qquad w'_i(s) \geq 0 \qquad \text{for all } s \text{ and all } i, \tag{14}$$

$$h(k'(s), \theta') \geq \sum_{i=1}^I \theta'_i(s) \ w'_i(s) \qquad \text{for all } \theta' \text{and } s. \tag{15}$$

Now define

$$\Psi(k)(h) \equiv \big\{ w \in \mathbf{R}^I_+ : \exists (c, k', w') \text{ such that (12)-(15) are satisfied}$$

$$\text{and } w_i = \sum_s \pi(s)[s_i u(c_i(s)) + \beta w_i'(s)]\}.$$

Given $h \in \overline{F}$, let $\mathcal{W}(k)(h)$ denote the constraint correspondence defined by (12)-(15) at $k \in X$. Any $(c, w', k') \in \mathcal{W}(k)(h)$ will be referred to as a *feasible, incentive-compatible recursive plan with respect to* h. We say that $h \in \overline{F}$ is *preserved under* T if $h(k,\theta) \leq (Th)(k,\theta)$ for all (k,θ). Importantly, notice that it is straightforward to check that

$$(Th)(k,\theta) = \sup_{w \in \Psi(k)(h)} \sum_{i=1}^{I} \theta_i w_i$$

The following result establishes that the correspondence $\Psi(.)(h)$ is well behaved.[36]

Lemma 3. *$\Psi(.)(h)$ is a continuous compact-valued correspondence for all $h \in \overline{F}$.*

It follows that the sup in the operator T is attained. In the next lemma, we establish the convexity of $\Psi(k)(h)$, a property that is key to our approach.

Lemma 4. *$\Psi(k)(h)$ is convex for all $k \in X$ and all $h \in \overline{F}$.*

Proof. Let w and $\widetilde{w} \in \Psi(k)(h)$ as $(c, w', k'), (\widetilde{c}, \widetilde{w}', \widetilde{k}') \in \mathcal{W}(k)(h)$ are the corresponding feasible, incentive-compatible recursive plans with respect to h.

We need to show that $w^\lambda = \lambda w + (1-\lambda)\widetilde{w} \in \Psi(K)(h)$ for any $\lambda \in [0,1]$. In order to do that, define for each i and all s

$$u(c_i^\lambda(s)) = \lambda u(c_i(s)) + (1-\lambda)u(\widetilde{c}_i(s)),$$

$$k'^\lambda(s) = \lambda k'(s) + (1-\lambda)\widetilde{k}'(s))$$

$$w'^\lambda(s) = \lambda w'(s) + (1-\lambda)\widetilde{w}'(s))$$

Notice that the strict concavity of u implies that $c_i^\lambda(s) \leq \lambda c_i(s) + (1-\lambda)\widetilde{c}_i(s)$ for all i, all s.

Step 1. Notice that by construction, it follows that

$$w_i^\lambda = \lambda w + (1-\lambda)\widetilde{w} = \sum_s \pi(s)[s_i u(c_i^\lambda(s)) + \beta\ w_i'^\lambda(s)]$$

for all i.

Step 2. Feasibility. As (c, w', k') and $(\widetilde{c}, \widetilde{w}', \widetilde{k}')$ are both feasible and $c_i^\lambda(s) \leq \lambda c_i(s) + (1-\lambda)\widetilde{c}_i(s)$ for all i, all s as mentioned, it follows immediately that

$$k'^\lambda(s) + \sum_{i=1}^{I} c_i^\lambda(s) \leq f(k) + (1-\delta)k$$

for all s and so $(c^\lambda, w'^\lambda, k'^\lambda)$ is also feasible.

[36]The proof is omitted as it follows by standard arguments. Details are available upon request.

Step 3. Incentive Compatibility. As the liquidity shocks are multiplicative, it follows by the linear construction that $(c^{\lambda}(s), w'^{\lambda}(s), k'^{\lambda}(s))$ satisfy (13).

Step 4. Take any $\theta' \in \Delta^I$ and notice that since $h \in \overline{F}$ is concave in k, it follows that

$$h(k'^{\lambda}, \theta') \geq \lambda h(k', \theta') + (1-\lambda) h(\widetilde{k}', \theta')$$
$$\geq \lambda \sum_{i=1}^{I} \theta_i'(s)\ w_i'(s) + (1-\lambda) \sum_{i=1}^{I} \theta_i'(s)\ \widetilde{w}_i'(s) = \sum_{i=1}^{I} \theta_i'(s)\ w_i'^{\lambda}(s).$$

Since $\theta' \in \Delta^I$ is arbitrary, condition (15) is satisfied. Therefore, we can conclude that $(c^{\lambda}, w'^{\lambda}, k'^{\lambda})$ is a feasible incentive-compatible recursive plan with respect to h. □

The next result is useful to characterize convex set and used to make our alternative approach computationally simpler (see also Lucas and Stokey (1984)).

Lemma 5. *For any $h \in \overline{F}$, $w \in \Psi(k)(h)$ if and only if $w \geq 0$*

$$Th(k, \theta) \geq \sum_{i=1}^{I} \theta_i\ w_i \qquad \textit{for all } \theta \in \Delta^I. \tag{16}$$

Proof. See Rockafellar (1970), Theorem 13.1. □

Remark. For computational purposes, it is convenient to recall that Condition (16) holds if and only if

$$\min_{\widetilde{\theta} \in \Delta^I} \left[Th\left(K, \widetilde{\theta}\right) - \sum_{i=1}^{I} \widetilde{\theta}_i w_i \right] \geq 0.$$

Our method complements the traditional APS approach as it identifies attainable levels of next-period utility by iterating directly on the utility possibility frontier with no need to know the utility possibility correspondence that describe the utility possibility set a priori. Our alternative approach can be summarized as follows. Following the methods developed by Abreu, Pearce, and Stacchetti (1990), one would need to construct an operator defined on correspondences and iterate on that space. Taking advantage of the convexity of our problem, we instead iterate on the (convex) frontier similar in spirit to Marcet and Marimon (1994)'s Lagrangean method.

The next result below is similar in spirit to APS's celebrated self-generation (Abreu, Pearce, and Stacchetti (1990).).

Lemma 6 (Self-generating)**.** *If $h \in \overline{F}$ is preserved under T, then*

$$(Th)(k, \theta) \leq h^*(k, \theta)$$

for all (k, θ).

Proof. Take any arbitrary $\left(\widehat{c}_0(s_0), \widehat{w}'_0(s_0), \widehat{k}'_1(s_0)\right)_{s_0 \in S} \in \mathcal{W}(k_0)(h)$ and notice that this implies, in particular, that

$$\sum_{i=1}^{I} \theta'_i \; w'_{i,0}(s_0) \leq h(\widehat{k}'_1(s_0), \theta') \tag{17}$$

for all $\theta' \in \Delta^I$. On the other hand, as h is preserved under T, this last condition implies that given $\widehat{k}'_1(s_0)$

$$h(\widehat{k}'_1(s_0), \theta') \leq (Th)(\widehat{k}'_1(s_0), \theta') \tag{18}$$

for all $\theta' \in \Delta^I$. Hence, as we couple conditions (17) and (18), we conclude that

$$\theta' \widehat{w}'_0(s_0) \leq (Th)(\widehat{k}'_1(s_0), \theta')$$

for all $\theta' \in \Delta^I$ and therefore $\widehat{w}'_0(s_0) \in \Psi(k'(s_0))(h)$ as a direct implication of Lemma 5. Importantly, this implies that there exists some $\left(\widehat{c}_1(s_0, s_1), \widehat{w}'_1(s_0, s_1), \widehat{k}'_2(s_0, s_1)\right)_{s_1 \in S} \in \mathcal{W}(\widehat{k}'_1(s_0))(h)$ such that

$$\widehat{w}'_0(s_0) = \sum_{s_1} \pi(s_1) \left[s_i \; u(\widehat{c}_{i,1}(s_0, s_1)) + \beta \; \widehat{w}'_{i,1}(s_0, s_1)\right]$$

for each $s_0 \in S$.

As we repeat this strategy T times, we can conclude that for any arbitrary $\theta_0 \in \Delta^I$

$$\begin{aligned}
&\sum_{i=1}^{I} \theta_{i,0} \; \left\{\sum_{s_0} \pi(s_0) \left[s_i \; u(\widehat{c}_{i,0}(s_0)) + \beta \; \widehat{w}'_{i,0}(s_0)\right]\right\} \\
&= \sum_{i=1}^{I} \theta_{i,0} \; \left\{\sum_{s_0} \pi(s_0) s_i \; u(\widehat{c}_{i,0}(s_0)) \right. \\
&\left. + \beta \sum_{s_0} \pi(s_0) \; \sum_{s_1} \pi(s_1) \left[s_i \; u(\widehat{c}_{i,1}(s_0, s_1)) + \beta \; \widehat{w}'_{i,1}(s_0, s_1)\right]\right\} \\
&= \sum_{i=1}^{I} \theta_{i,0} \; E\left(\sum_{t=0}^{T} \beta^t \; s_{i,t} u(\widehat{c}_{i,t})\right) + \beta^{T+1} \sum_{i=1}^{I} \theta_{i,0} \; E\left(\widehat{w}'_{i,T+1}\right)
\end{aligned}$$

Condition (15) implies that

$$\sup \sum_{i=1}^{I} \theta_{i,0} \; E\left(\widehat{w}'_{i,T+1}\right) \leq \|h\|,$$

and so taking limits on both sides as $T \to \infty$, it follows from the Dominated Convergence Theorem that

$$\sum_{i=1}^{I} \theta_{i,0} \; \left\{\sum_{s_0} \pi(s_0) \left[s_i \; u(\widehat{c}_{i,0}(s_0)) + \beta \; \widehat{w}'_{i,0}(s_0)\right]\right\}$$

$$\leq \sum_{i=1}^{I} \theta_{i,0} \; E\left(\sum_{t=0}^{\infty} \beta^t \; s_{i,t} u(\widehat{c}_{i,t})\right) \tag{19}$$

as $\beta \in (0,1)$.

Consider the sequential plan $(\widehat{c}, \widehat{k}')$ stemming from above. It is immediate that this plan is sequentially feasible by construction. Now we argue that it is incentive compatible as well. To see this, denote recursively $W_{i,t}(s^t) = \widehat{w}'_{i,t}(s_0, ..., s_t)$ and observe that by construction

$$\begin{aligned}
&\left|U_{i,t}(\widehat{c}, \widehat{k}'; s^t) - W_{i,t}(s^t)\right| \\
&= \beta \left|\sum_{s_{t+1}} \pi(s_{t+1}) \left(U_{i,t}(\widehat{c}, \widehat{k}'; s^t, s_{t+1}) - W_{i,t+1}(s^t, s_{t+1})\right)\right| \\
&\leq \beta \sup_{s_{t+1}} \left|U_{i,t}(\widehat{c}, \widehat{k}'; s^t, s_{t+1}) - W_{i,t+1}(s^t, s_{t+1})\right| \\
&\leq \beta^k \sup_{(s_{t+1}, ... s_{t+k})} \left|U_{i,t}(\widehat{c}, \widehat{k}'; s^t, s_{t+1}, ..., s_{t+k}) - W_{i,t+k}(s^t, s_{t+1}, ..., s_{t+k})\right|.
\end{aligned}$$

Observe that $0 \leq W_{i,t}(s^t) \leq \|h\| < \infty$ for all i and all s^t while $\widehat{c}$ is uniformly bounded by construction. Taking the lim sup as $k \to \infty$ for this last expression, we can conclude that $U_{i,t}(\widehat{c}, \widehat{k}')(s^t) = W_{i,t}(s^t)$ for all i and all s^t and so sequential incentive compatibility follows immediately.

Since both $\left(\widehat{c}_0(s_0), \widehat{w}'_0(s_0), \widehat{k}'_1(s_0)\right)_{s_0} \in \mathcal{W}(k_0)(h)$ and the corresponding sequential plan $(\widehat{c}, \widehat{k})$ are arbitrary, we take the sup on both sides of (19) to conclude that

$$\begin{aligned}
Th(k, \theta) &= \sup_{(\widehat{c}, \widehat{w}', \widehat{k}') \in \mathcal{W}(k_0)(h)} \sum_{i=1}^{I} \theta_{i,0} \; \left\{\sum_{s_0} \pi(s_0) \left[s_i \; u(\widehat{c}_i) + \beta \; \widehat{w}'_i\right]\right\} \\
&\leq \sup \sum_{i=1}^{I} \theta_i \; E\left(\sum_{t=0}^{\infty} \beta^t \; s_{i,t} u(\widehat{c}_{i,t})\right) \\
&= h^*(k, \theta).
\end{aligned}$$

and this completes the proof. □

Now we are prepared to prove our two main results of this section.

Proposition 4. *h^* is a fixed point of T.*

Proof. Given (K, θ), take any $w \in \Psi^*(K)$ for which (C, K') denotes the corresponding feasible incentive-compatible plan. Observe that

$$\sum_{i=1}^{I} \theta_i \; U_i(C, K') = \sum_{i=1}^{I} \theta_i \sum_{s_0 \in S_0} \pi(s_0) \; \left[s_{i,0} \; u(C_i(s_0)) + \beta \; U_{i,1}(C, K' \| (s_0))\right]$$

Notice that $(U_{i,1}(C,K'\|(s_0)))_{i=1}^{I} \in \Psi^*(K(s_0))$ for all s_0. It follows by definition of h^* (see (3)) that

$$h^*(K'(s_0),\theta') \geq \sum_{i=1}^{I} \theta'_i \; U_{i,1}(C,K'\|(s_0))$$

for all $\theta' \in \Delta^I$ and all s_0. Therefore, $(C_i, U_{i,1}(C,K'), K')_{i=1}^{I} \in \mathcal{W}(K_0)(h^*)$ and then

$$\sum_{i=1}^{I} \theta_i \; U_i(C,K') \leq (Th^*)(K,\theta)$$

Since weak inequalities are preserved in the limit, we can conclude that

$$h^*(K,\theta) = \sup_{(C,K')} \sum_{i=1}^{I} \theta_i U_i(C,K') \leq (Th^*)(K,\theta),$$

for all (K,θ) (i.e. h^* is preserved under T). Thus, Lemma 6 implies that $h^*(K,\theta) = (Th^*)(K,\theta)$ for all (K,θ). □

Importantly, it can be shown that the following version of the Principle of Optimality holds.[37]

Remark 1. *A plan (C^*, K'^*) is constrained efficient at K_0 if and only if it is generated by the set of policy functions*

$$\begin{aligned}
\widehat{C}_i(s^t) &= \widehat{c}_i(\widehat{K}(s^{t-1}), \theta(s^{t-1}); s_t), \\
\theta(s^{t-1}, s_t) &= \widehat{\theta}'(\widehat{K}(s^{t-1}), \theta(s^{t-1}); s_t), \\
\widehat{K}(s^{t-1}, s_t) &= \widehat{k}'(\widehat{K}(s^{t-1}), \theta(s^{t-1}); s_t),
\end{aligned} \tag{20}$$

Thus, the value of any plan that can be attained with an incentive-compatible, feasible sequential plan (C,K') can also be attained by splitting output between total current payouts and investment and then delivering current payouts and contingent future ownership shares to each partner.

Now we provide an algorithm capable of finding the value function h^* and its corresponding policy functions.

Let $\widehat{T}$ be the operator solving the recursive problem for the full information case, discussed in Section 3.1 (i.e. the incentive compatibility constraints (13) are ignored), and h^{**} is the corresponding value function such that $h^{**} = \widehat{T}h^{**}$. Evidently, $Tf \leq \widehat{T}f$ for all $f \in \overline{F}$ and $h^*(k,\theta) \leq h^{**}(k,\theta)$ for all (k,θ).

Proposition 5. *Let $h_0 = h^{**}$ and denote $h_n = T^n(h^{**})$. Then, $\{h_n\}$ is a monotone decreasing sequence of continuous functions and $\lim_{n\to\infty} h_n = h^*$ uniformly.*

Proof. It is a routine exercise to show that T is a monotone operator (i.e. if $f \geq g$, then $Tf \geq Tg$). This property and Proposition 4 imply that $h^* = Th^* \leq Th^{**} \leq \widehat{T}h^{**} = h^{**}$.

[37] Proof available upon request.

Step 1. Since $h_n = T^n h^{**}$, then monotonicity implies that $h_n \geq h_{n+1} \geq h^*$ for all n.

Step 2. T preserves concavity with respect to k. In addition, Proposition 3 makes possible to apply the Theorem of the Maximum to conclude that $T : \overline{F} \to \overline{F}$; i.e. h_n is continuous for all n.

Step 3. As $\{h_n\}$ is a monotone decreasing sequence of uniformly bounded continuous functions on a compact set (S, X), Dini's theorem implies that there exists a continuous function $h_\infty \geq h^*$ such that $h_n \to h_\infty$ uniformly.[38]

Step 4. h_∞ is preserved under T.

Given (k, θ), $h_\infty(k, \theta) \leq h_n(k, \theta)$ implies that there exists $(c^n, w'^n, k'^n) \in \mathcal{W}(k)(h_n)$ such that for all n

$$h_\infty(k, \theta) \leq h_{n+1}(k, \theta) = \sum_{i=1}^{I} \theta_i \sum_s \pi(s) \left[u(c_i^n(s)) + \beta \; w_i'^n(s)\right] \tag{21}$$

and

$$h_n(k'^n(s), \theta') \geq \sum_{i=1}^{I} \theta_i' \; w_i'^n(s), \quad \text{for all } s \text{ and all } \theta'. \tag{22}$$

Since (c^n, w'^n, k'^n) lies in a compact set and, thus, it has a convergent subsequence with limit point $\left(\widehat{c}, \widehat{w}', \widehat{k}'\right)$. Suppose for notational simplicity that the convergent subsequence is the sequence itself and notice that both feasibility and incentive compatibility are preserved in the limit. In addition, as h_∞ is (uniformly) continuous, condition (22) implies that in the limit

$$h_\infty(\widehat{k}'(s), \theta') \geq \sum_{i=1}^{I} \theta_i' \; \widehat{w}_i'(s), \quad \text{for all } s \text{ and all } \theta'.$$

Therefore, $\left(\widehat{c}, \widehat{w}', \widehat{k}'\right) \in \mathcal{W}(k)(h_\infty)$ and condition (21) implies that in the limit

$$h_\infty(k, \theta) \leq \sum_{i=1}^{I} \theta_i \sum_s \pi(s) \left[u(\widehat{c}_i(s)) + \beta \; \widehat{w}_i'(s)\right]$$

Finally, since the recursive allocation plan $\left(\widehat{c}, \widehat{w}', \widehat{k}'\right) \in \mathcal{W}(k)(h_\infty)$ is arbitrary, we can conclude that for all (k, θ)

$$\begin{aligned}(Th_\infty)(k, \theta) &\geq \sum_{i=1}^{I} \theta_i \sum_s \pi(s) \left[u(\widehat{c}_i(s)) + \beta \; \widehat{w}_i'(s)\right] \\ &\geq h_\infty(k, \theta),\end{aligned}$$

and thus h_∞ is preserved under T by definition.

Step 5. This implies that $h_\infty \leq h^*$ due to Lemma 6) and therefore $h_\infty = h^*$.

□

[38]Rudin, Walter R. (1976) Principles of Mathematical Analysis, Third Edition, McGraw-Hill. See Theorem 7.13 on page 150 for the monotone decreasing case.

Our method complements the traditional APS approach in terms of tractability as it greatly simplifies the computational burden. Of course, the APS approach outperforms ours if the utility possibility correspondence is not convex valued.

B Full Information - Proofs

This Appendix provides the proof of Lemma 1.

Proof of Lemma 1. 1. To show this result, we use the first order condition with respect to w'. To save space, this conditions can be found in equations (36)-(39) below by setting the multipliers of the incentive compatibility constraint, ϕ, equal to zero. As shown there, since $\theta_2 = 1-\theta_1$, we can derive equations 41-44, which, given that all ϕs are equal to zero, imply that $\theta'(s) = \theta$.

2. In addition:

   (a) The necessary and sufficient first order conditions of the full information problem imply

$$c_1(s) = \frac{(\theta s_1)^{1/\sigma}}{(\theta s_1)^{1/\sigma} + ((1-\theta)\, s_2)^{1/\sigma}} \left(f(k) + (1-\delta)k - k'(s)\right), \tag{23}$$

$$c_2(s) = \frac{((1-\theta)\, s_2)^{1/\sigma}}{(\theta s_1)^{1/\sigma} + ((1-\theta)\, s_2)^{1/\sigma}} \left(f(k) + (1-\delta)k - k'(s)\right). \tag{24}$$

   Hence the payout of partner i is increasing in his liquidity needs, decreasing in the other partner's liquidity needs, and increasing in his ownership shares.

   (b) Under Full information the problem reduces to choosing total payouts and investment in a sole proprietorship with an "aggregate" investor with preferences

$$S(\theta, s)\frac{C^{1-\sigma}}{1-\sigma},$$

   where $C = c_1 + c_2$ denotes total payouts and

$$S(\theta, s) = \left((\theta s_1)^{1/\sigma} + ((1-\theta)\, s_2)^{1/\sigma}\right)^{\sigma}$$

   is an "aggregate" liquidity shock. The problem reduces to

$$h^{**}(k,\theta) = \max_{c,k'} \left\{ \sum_s \pi(s) \left[S(\theta, s)\, u(C(s)) + \beta h^{**}\left(k'(s), \theta\right)\right] \right\}$$

   subject to

$$k'(s) + C(s) = f(k) + (1-\delta)\, k\ .$$

The Euler equation is

$$S(\theta, s)\, C(s)^{-\sigma} = \beta \left(f'\left(k'(s)\right) + 1 - \delta\right) \sum_{s'} S\left(\theta, s'\right) C\left(s'\right)^{-\sigma}$$

We can write the Euler equation as

$$S(\theta, s) = \beta \frac{\left(f'\left(k'(s)\right) + 1 - \delta\right)}{\left(f(k) + (1-\delta) k - k'(s)\right)^{-\sigma}} \sum_{s'} S\left(\theta, s'\right) C\left(k'(s)\, s'\right)^{-\sigma} \tag{25}$$

and note that the right hand side is decreasing in k'. Hence, as $S(\theta, s)$ increases, $k'(s)$ decreases. Moreover, note that the ratio of aggregate shocks

$$\frac{S(\theta, s_H, s_2)}{S(\theta, s_L, s_2)} = \frac{\left(\left(\theta s_H\right)^{1/\sigma} + \left((1-\theta) s_2\right)^{1/\sigma}\right)^{\sigma}}{\left(\left(\theta s_L\right)^{1/\sigma} + \left((1-\theta) s_2\right)^{1/\sigma}\right)^{\sigma}} \tag{26}$$

is greater than one and increasing in θ which concludes the proof.

□

C Private Information - Proofs

In this Appendix we prove Propositions 1, 2, 3, and Lemma 2. In order to do that, we write down the problem and the system of equations that characterizes the solution. The problem can be written as follows

$$h^*(k, \theta) = \max_{c, w', k'} \left\{ \sum_{s^1 \in \{L,H\}} \sum_{s^2 \in \{L,H\}} \sum_{i=1}^{2} \theta_i \pi(s_1)\, \pi(s_2) \left[s_i u\left(c_i(s_1, s_2)\right) + \beta w_i'(s_1, s_2) \right] \right\} \tag{27}$$

subject to

$$k'(s_1, s_2) + \sum_{i=1}^{2} c_i(s_1, s_2) = f(k) + (1-\delta) k \tag{28}$$

$$\begin{aligned} & \pi(s_L)\left(s_L u\left(c_1(s_L, s_L)\right) + \beta w_1'(s_L, s_L)\right) + \pi(s_H)\left(s_L u\left(c_1(s_L, s_H)\right) + \beta w_1'(s_L, s_H)\right) \\ \geq\ & \pi(s_L)\left(s_L u\left(c_1(s_H, s_L)\right) + \beta w_1'(s_H, s_L)\right) + \pi(s_H)\left(s_L u\left(c_1(s_H, s_H)\right) + \beta w_1'(s_H, s_H)\right) \end{aligned} \tag{29}$$

$$\begin{aligned} & \pi(s_L)\left(s_L u\left(c_2(s_L, s_L)\right) + \beta w_2'(s_L, s_L)\right) + \pi(s_H)\left(s_L u\left(c_2(s_H, s_L)\right) + \beta w_2'(s_H, s_L)\right) \\ \geq\ & \pi(s_L)\left(s_L u\left(c_2(s_L, s_H)\right) + \beta w_2'(s_L, s_H)\right) + \pi(s_H)\left(s_L u\left(c_2(s_H, s_H)\right) + \beta w_2'(s_H, s_H)\right) \end{aligned} \tag{30}$$

$$\min_{\theta' \in \Delta} \left[h\left(k'(s_1, s_2), \theta'(s_1, s_2)\right) - \sum_{i=1}^{2} \theta_i'(s_1, s_2)\, w_i'(s_1, s_2) \right] \geq 0 \tag{31}$$

Let $\lambda(s_1, s_2)$, ϕ_1, ϕ_2 and $\mu(s_1, s_2)$ be the Lagrange multipliers of (28),(29),(30) and (31) respectively. The necessary and sufficient first order conditions are as follows.

First, the first order conditions with respect to consumption for agent 1 and 2, respectively, imply that for all s_1 and s_2

$$(\theta\pi(s_L)s_L + \phi_1 s_L)\pi(s_2)u'(c_1(s_L, s_2)) = \lambda(s_L, s_2) \tag{32}$$

$$(\theta\pi(s_H)s_H - \phi_1 s_L)\pi(s_2)u'(c_1(s_H, s_2)) = \lambda(s_H, s_2) \tag{33}$$

$$((1-\theta)\pi(s_L)s_L + \phi_2 s_L)\pi(s_1)u'(c_2(s_1, s_L)) = \lambda(s_1, s_L) \tag{34}$$

$$((1-\theta)\pi(s_H)s_H - \phi_2 s_L)\pi(s_1)u'(c_2(s_1, s_H)) = \lambda(s_1, s_H) \tag{35}$$

Second, the first order conditions with respect to continuation utilities for agent 1 and 2, respectively, imply that for all s_1 and s_2

$$(\theta\pi(s_L) + \phi_1)\pi(s_2)\beta = \mu(s_L, s_2)\theta_1'(s_L, s_2) \tag{36}$$

$$(\theta\pi(s_H) - \phi_1)\pi(s_2)\beta = \mu(s_H, s_2)\theta_1'(s_H, s_2) \tag{37}$$

$$((1-\theta)\pi(s_L) + \phi_2)\pi(s_1)\beta = \mu(s_1, s_L)\theta_2'(s_1, s_L) \tag{38}$$

$$((1-\theta)\pi(s_H) - \phi_2)\pi(s_1)\beta = \mu(s_1, s_H)\theta_2'(s_1, s_H) \tag{39}$$

Finally, the first order conditions with respect to capital for an interior solution and making use of the corresponding envelope condition deliver the Euler equation

$$\begin{aligned}\lambda(k,\theta)(s_1, s_2) &= \mu(k,\theta)(s_1, s_2)\left(f'\left(k'(k,\theta)(s_1, s_2)\right) + (1-\delta)\right) \\ &\quad \times \sum_{(s_1', s_2')} \lambda\left(k'(k,\theta)(s_1, s_2), \theta'(k,\theta)(s_1, s_2)\right)(s_1', s_2').\end{aligned} \tag{40}$$

As $\theta_2 = 1 - \theta_1$, we get from (36)-(39) that

$$\theta_1'(s_L, s_L) = \frac{\left(\theta + \frac{\phi_1}{\pi(s_L)}\right)}{\left(1 + \frac{\phi_1}{\pi(s_L)} + \frac{\phi_2}{\pi(s_L)}\right)} \tag{41}$$

$$\theta_1'(s_L, s_H) = \frac{\left(\theta + \frac{\phi_1}{\pi(s_L)}\right)}{\left(1 + \frac{\phi_1}{\pi(s_L)} - \frac{\phi_2}{\pi(s_H)}\right)} \tag{42}$$

$$\theta_1'(s_H, s_L) = \frac{\left(\theta - \frac{\phi_1}{\pi(s_H)}\right)}{\left(1 - \frac{\phi_1}{\pi(s_H)} + \frac{\phi_2}{\pi(s_L)}\right)} \tag{43}$$

$$\theta_1'(s_H, s_H) = \frac{\left(\theta - \frac{\phi_1}{\pi(s_H)}\right)}{\left(1 - \frac{\phi_1}{\pi(s_H)} - \frac{\phi_2}{\pi(s_H)}\right)} \tag{44}$$

C.1 Proof of Proposition 1

Proof of Proposition 1. First, we show that the full information plan satisfies the partner 1's incentive compatibility constraints at $\theta = 1$.

Consider the recursive problem (27)-(31) for the case in which the incentive compatibility constraints are absent and let $(c(k,\theta;s), c_2(k,\theta;s), k'(k,\theta;s), \theta'(k,\theta;s), w_1'(k,\theta;s), w_2'(k,\theta;s))$ be the set of continuous policy functions such that $\phi_i = 0$ for $i = 1,2$.

Notice that in this case Lemma 1 implies that $\theta'(k,\theta;s) = \theta$ for all s and all (k,θ).

Since $\theta'(k,\theta;s) = \theta = 1$, then $h(k'(s),1) = w_1'(s)$ for all (s,θ,k) and the value function reduces to

$$h(k,1) = \pi(s_L)\left[s_L u(c_1(k,1;s_L,s_2)) + \beta w_1'(k,1;s_{L,}s_2)\right] + \pi(s_H)\left[s_H u(c_1(k,1;s_H,s_2)) + \beta w_1'(k,1;s_H,s_2)\right]$$

for all s_2. Notice that as $\theta'(k,\theta;s) = \theta = 1$, then s_2 plays no allocative role. Therefore, consumption, future promised utilities, and capital accumulation are independent of s_2.

Suppose that the corresponding full information plan is not incentive compatible; i.e.

$$s_L\ u(c_1(k,1;s_L,s_2)) + \beta\ h(k'(k,1;s_L,s_2),1) < s_L\ u(c_1(k,1);s_H,s_2) + \beta\ h(k'(k,1;s_H,s_2),1)$$

while

$$c_1(k,1;s_L,s_2) + k'(k,1;s_L,s_2) = f(k) + (1-\delta)k$$

$$c_1(k,1;s_H,s_2) + k'(k,1;s_H,s_2) = f(k) + (1-\delta)k$$

This implies that $(c_1(k,1;s_H,s_2), k'(k,1;s_H,s_2))$ is feasible at $s_1 = s_L$ and

$$w_1'(k,1;s_H,s_2) = h(k'(k,1;s_H,s_2)).$$

This contradicts that $(c_1(k,1;s_L,s_2), k'(k,1;s_L,s_2), w_1'(k,1;s_L,s_2))$ is the unique solution at $(k,1)$.

To complete the proof we need to argue that, conditional upon k, we can find some $\overline{\theta}(k) < 1$ such that the private information plan satisfies the partner 1's incentive compatibility constraint. As the private information plan coincides with the full information plan at $\theta = 1$, the partner 1's incentive compatibility constraint must hold with strict inequality when $\theta = 1$. Since the policy functions in the private information case are continuous, it must be the case that there exists some $\overline{\theta}(k) < 1$ such that the partner 1's incentive compatibility constraint does not bind for $\theta \in [\overline{\theta}(k), 1]$.

It follows by symmetry that the partner 2's incentive compatibility constraint does not bind for all (θ,k) with $(1-\theta) \in \left[0, 1-\overline{\theta}(k)\right]$. □

C.2 Proof of Proposition 2

Proof of Proposition 2. Consider the solution to the full information problem evaluated at $\theta = 1/2$ and let $s_L = 1-\epsilon$ and $s_H = 1+\epsilon$. Given ϵ, consider the incentive compatibility constraint of partner

1 that needs to be satisfied to complete the proof

$$\begin{aligned}&\textstyle\sum_{s_2} \pi(s_2) \left((1-\epsilon)u(c_1(k,1/2)(1-\epsilon,s_2)) + \beta\ w_1'(k,1/2)(1-\epsilon,s_2)\right) \\ \geq &\textstyle\sum_{s_2} \pi(s_2) \left((1-\epsilon)u(c_1(k,1/2)(1+\epsilon,s_2)) + \beta\ w_1'(k,1/2)(1+\epsilon,s_2)\right).\end{aligned} \tag{45}$$

As we consider the full information plan, Lemma 1 implies that $\theta'(k,1/2;s_1,s_2) = 1/2$ for all (s_1,s_2) and all k. This implies that $w_1'(k,1/2;s_1,s_2) = w_2'(k,1/2;s_1,s_2)$ for all (s_1,s_2) and all k. Hence

$$h(k'(k,1/2;s_1,s_2),1/2) = w_1'(k,1/2;s_1,s_2). \tag{46}$$

As h is strictly increasing in k and investment is decreasing in the liquidity shocks (Lemma 1), it follows by (46) that $w_1'(1-\epsilon,s_2) > w_1'(1+\epsilon,s_2)$ for all s_2. By the Theorem of the Maximum, policy functions can be parameterized continuously with respect to ϵ. For each k, since (45) holds with strict inequality as ϵ goes to 1, we can conclude that there exists some $\epsilon^*(k) \in (0,1)$ such that the full information plan is strictly incentive compatible for partner 1 for all $\epsilon \in (\epsilon^*(k),1)$ at k.

Finally, standard arguments prove that $\epsilon^*(k)$ varies continuously

$$\epsilon^* \equiv \max\{\epsilon^*(k) : k \in [k_{\min}(1/2), k_{\max}(1/2)]\} \in (0,1)$$

and ϵ^* is well-defined. Therefore, the full information plan is strictly incentive compatible for partner 1 for all $\epsilon \in (\epsilon^*,1)$ *for all* k.

It follows by symmetry it is also strictly incentive compatible for partner 2.

□

C.3 Proof of Lemma 2

Proof of Lemma 2. Note that:

1. Equations (41)-(44) imply that if $\phi_1 = \phi_2 = 0$, then $\theta'(k,\theta,s_1,s_2) = \theta$ for $(s_1,s_2) \in \{s_L,s_H\} \times \{s_L,s_H\}$.

2. Equations (41)-(44) imply that

$$\theta_1'(s_L,s_L) - \theta_1'(s_H,s_L) = \left(\frac{\phi_1}{\pi(s_H)\pi(s_L)}\right) \frac{(1-\theta) + \frac{\phi_2}{\pi(s_L)}}{\left(1+\frac{\phi_1}{\pi(s_L)}+\frac{\phi_2}{\pi(s_L)}\right)\left(1-\frac{\phi_1}{\pi(s_H)}+\frac{\phi_2}{\pi(s_L)}\right)} \tag{47}$$

$$\theta_1'(s_L,s_H) - \theta_1'(s_H,s_H) = \frac{\phi_1}{\pi(s_H)} \frac{\frac{(1-\theta)}{\pi(s_L)} - 2\frac{\phi_2}{\pi(s_H)}}{\left(1+\frac{\phi_1}{\pi(s_L)}-\frac{\phi_2}{\pi(s_H)}\right)\left(1-\frac{\phi_1}{\pi(s_H)}-\frac{\phi_2}{\pi(s_H)}\right)} \tag{48}$$

   Hence, if $\phi_1 = 0$, then $\theta'(k,\theta,s_L,s_2) = \theta'(k,\theta,s_H,s_2)$ for all $s_2 \in \{s_L,s_H\}$.

3. Note that if $s_2 = s_L$, then (47) implies that $\theta'(k,\theta,s_L,s_L) \geq \theta'(k,\theta,s_H,s_L)$.

4. If $s_2 = s_H$, then (42) and (44) implies that $\frac{\theta'(k,\theta,s_L,s_H)}{\theta'(k,\theta,s_H,s_H)} \geq 1$ if $\phi_2 \leq (1-\theta)\pi(s_H)$ and this condition is satisfied due to (39).

□

C.4 Proof of Proposition 3

To prove Proposition 3, the following result is key. Let $\{\theta_t\}_{t=0}^{\infty}$ be the stochastic process for ownership shares generated by the set of policy functions as in (20). That is, $\theta_t : S^\infty \to [0,1]$, where $\theta_t(s^\infty)$ denotes a particular realization at date t.

Lemma 7. *Suppose that the full information plan is strictly incentive compatible at $\theta = 1/2$ for all k. Suppose that $\theta_0 \in [0, 1/2]$. The ratio of ownership shares satisfies the following properties:*

1. *It is a nonnegative martingale; i.e., for all t and all s^t,*

$$\mathbf{E}\left[\frac{\theta_{t+1}}{(1-\theta_{t+1})} \parallel s^t\right] = \frac{\theta_t(s^\infty)}{(1-\theta_t(s^\infty))} \in [0,1] \qquad s^\infty - a.s.$$

2. *There exists a random variable $\widehat{\theta}$ on $(S^\infty, \mathcal{B}(S^\infty))$. such that*

$$\frac{\theta_t(s^\infty)}{(1-\theta_t(s^\infty))} \to \frac{\widehat{\theta}(s^\infty)}{\left(1-\widehat{\theta}(s^\infty)\right)} \qquad s^\infty - a.s.$$

Proof of Lemma 7. Consider first the case in which $\theta \in [0, 1/2]$. As the full information plan is strictly incentive compatible at $\theta = 1/2$ for all k, conditions (36)-(39) for both partners and the fact that $\phi_2 = 0$ for $\theta \leq 1/2$ imply that[39]

$$\frac{\theta'(k,\theta,1-\theta;s_L,s_2)}{(1-\theta'(k,\theta,1-\theta;s_L,s_2))} = \frac{\theta + \phi_1(k,\theta,1-\theta)/\pi(s_L)}{(1-\theta)} = \frac{\theta}{(1-\theta)} + \frac{\phi_1(k,\theta,1-\theta)}{(1-\theta)\ \pi(L)},$$

$$\frac{\theta'(k,\theta,1-\theta;s_H,s_2)}{(1-\theta'(k,\theta,1-\theta;s_H,s_2))} = \frac{\theta - \phi_1(k,\theta,1-\theta)/\pi(s_H)}{(1-\theta)} = \frac{\theta}{(1-\theta)} - \frac{\phi_1(k,\theta,1-\theta)}{(1-\theta)\ \pi(H)},$$

for all s_2. This implies that

$$\mathbf{E}\left[\frac{\theta'(k,\theta,1-\theta;s_1,s_2)}{(1-\theta'(k,\theta,1-\theta;s_1,s_2))}\right] = \frac{\theta}{(1-\theta)}. \tag{49}$$

We argue now that this expectation (49) is bounded by 1.

[39] In the paper, the state variables were denoted by (k,θ). Here we abuse notation and make the state (k,θ_1,θ_2).

Note first that for all $\theta \in [\theta^*, 1/2]$ no incentive compatibility constraint binds and so

$$\frac{\theta'(k,\theta,1-\theta;s_1,s_2)}{(1-\theta'(k,\theta,1-\theta;s_1,s_2))} = \frac{\theta}{(1-\theta)} \leq 1 \tag{50}$$

for all k, all (s_1, s_2).

Note that $\frac{\theta'(k,\theta,1-\theta;s_1,s_2)}{(1-\theta'(k,\theta,1-\theta;s_1,s_2))}$ is homogeneous of degree 0 with respect to $(\theta, 1-\theta)$. In addition, it is an standard exercise to show that $\frac{\theta'\left(k,\frac{\theta}{(1-\theta)},1;s_1,s_2\right)}{\left(1-\theta'\left(k,\frac{\theta}{(1-\theta)},1;s_1,s_2\right)\right)}$ is increasing in $\frac{\theta}{(1-\theta)}$. This implies that for all $\theta \leq \theta^*$

$$0 \leq \frac{\theta'\left(k,\theta,1-\theta;s\right)}{\left(1-\theta'\left(k,\theta,1-\theta;s\right)\right)} \leq \frac{\theta'\left(k,\frac{\theta}{(1-\theta)},1;s\right)}{\left(1-\theta'\left(k,\frac{\theta}{(1-\theta)},1;s\right)\right)} \leq \frac{\theta'\left(k,\frac{\theta^*}{(1-\theta^*)},1;s\right)}{\left(1-\theta'\left(k,\frac{\theta^*}{(1-\theta^*)},1;s\right)\right)} = \frac{\theta^*}{(1-\theta^*)} \leq 1. \tag{51}$$

for all k and all s.

Conditions (50) and (51) imply that $\frac{\theta_t(s^\infty)}{(1-\theta_t(s^\infty))} \in [0,1]$ as $\theta_0 \in [0, 1/2]$ and (49) reads

$$\mathbf{E}\left[\frac{\theta_{t+1}}{(1-\theta_{t+1})} \parallel s^t\right] = \frac{\theta_t(s^\infty)}{(1-\theta_t(s^\infty))} \qquad s^\infty - a.s.$$

Hence, $\{\frac{\theta_t(s^\infty)}{(1-\theta_t(s^\infty))}\}_{t=0}^\infty$ follows a bounded martingale and so it follows by the martingale convergence theorem that

$$\frac{\theta_t(s^\infty)}{(1-\theta_t(s^\infty))} \rightarrow \frac{\widehat{\theta}(s^\infty)}{\left(1-\widehat{\theta}(s^\infty)\right)} \qquad s^\infty - a.s.$$

for some random variable $\widehat{\theta}$ on $(S^\infty, \mathcal{B}(S^\infty))$. □

Proof of Proposition 3. Suppose that the full information plan is strictly incentive compatible at $\theta = 1/2$ for all k. Using the same arguments developed in the proof of Proposition 1, it follows by continuity of the full information policy functions that there exists $\theta^* \in (0, 1/2)$ such that if $(\theta, k) \in [\theta^*, 1-\theta^*] \times [k_{\min}(\theta), k_{\max}(\theta)]$, both partners' incentive compatibility constraints do not bind.

1. Suppose that $(\theta_t, k_t) \in [\theta^*, 1-\theta^*] \times [k_{\min}(\theta_t), k_{\max}(\theta_t)]$ at some t. It follows by definition of θ^* that no partner incentive compatibility constraint binds in this case. Consequently, the private information plan and the full information plan coincide and that implies by Lemma 1 that $\theta'(s)(\theta, k) = \theta$ for all (s, θ, k) and $k' \in [k_{\min}(\theta), k_{\max}(\theta)] = [k_{\min}(\theta'), k_{\max}(\theta')]$. and so $\theta_{t+n} = \theta_t$ for all $n \geq 0$.

2. Suppose that $(\theta_t, k_t) \in [0, \theta^*) \times [k_{\min}(\theta_t), k_{\max}(\theta_t)]$ at some t. Notice that, by definition of θ^*, the partner 2's incentive compatibility constraint does not bind. It follows by Lemma 7 that the ratio of ownership shares is a non-negative martingale.

Using the notation of Lemma 7, take any arbitrary $s^\infty \in \Omega = \{s^\infty \in S^\infty : \theta_t(s^\infty) \rightarrow \widehat{\theta}(s^\infty) \in [0, 1/2]\}$.

If $\widehat{\theta}(s^\infty) = 0$, then the limiting plan reaches a full information plan as it is a sole-proprietorship.

If $\widehat{\theta}(s^\infty) \in [\theta^*, 1/2]$, it follows by part 1. above that the limiting plan coincides with a full information plan as no incentive compatibility constraint binds.

We need to show that $\widehat{\theta}(s^\infty) \notin (0, \theta^*)$; i.e., the limiting plan can converge to a plan where some ICC is binding only for zero-probability sequences.

Step 2.1. $\widehat{\theta}(s^\infty) < \overline{\theta}(k) \leq \theta^*$ for all k.

It follows by definition that the partner 1's incentive compatibility constraint binds for all k. Suppose that the state (s_H, s_L) occurs infinitely often and consider that infinite subsequence $\{(s_{1,t_n}, s_{2,t_n})\}_{n=0}^\infty$ in which $(s_{1,t_n}, s_{2,t_n}) = (s_H, s_L)$ for all n. Since $\{k_{t_n}\}_{n=0}^\infty$ is a sequence in a compact set, it must have a convergent subsequence with limit $\widehat{k}(s^\infty) \in \times \left[k_{\min}(\widehat{\theta}(s^\infty)), k_{\max}(\widehat{\theta}(s^\infty))\right]$. To simplify notation, suppose that it is the sequence $\{k_{t_n}\}_{n=0}^\infty$ itself. Since $\theta_{t_n+1} = \theta'(\theta_{t_n}, k_{t_n}; s_H, s_L)$, it follows by continuity that taking the limit

$$\widehat{\theta}(s^\infty) = \theta'(\widehat{\theta}(s^\infty), \widehat{k}(s^\infty); s_H, s_L);$$

i.e. the the partner 1's incentive compatibility constraint does not bind. But this contradicts that $\widehat{\theta}(s^\infty) \notin (0, \theta^*)$ and consequently, as in Thomas and Worral (1990), $\{\theta_t\}_{t=0}^\infty$ can converge to some number in the interval $(0, \theta^*)$ only for sequences where (s_H, s_L) occurs only finitely often. Those events occur with zero probability.

Step 2.2: $\widehat{\theta}(s^\infty) < \overline{\theta}(k) \leq \theta^*$ for some $k \in (k_{min}(\widehat{\theta}(s^\infty)), k_{max}(\widehat{\theta}(s^\infty)))$.

Let $\widehat{k}(s^\infty)$ be defined such that $\widehat{\theta}(s^\infty) = \overline{\theta}(\widehat{k}(s^\infty))$ and notice that $\widehat{k}(s^\infty) \in (k_{min}(\widehat{\theta}(s^\infty)), k_{max}(\widehat{\theta}(s^\infty)))$.

As long as $\widehat{\theta}(s^\infty) \geq \overline{\theta}(k_t(s^\infty))$, then it follows that $\theta'\left(\widehat{\theta}(s^\infty), k_t(s^\infty)\right)(s_{1,t}, s_{2,t}) = \widehat{\theta}(s^\infty)$ for all $(s_{1,t}, s_{2,t})$ (i.e. no incentive compatibility constraint binds). Since s^∞ belongs to a set with positive probability and under the assumption that $\widehat{\theta}(s^\infty) < \overline{\theta}(k)$ for some $k \in (k_{min}(\widehat{\theta}(s^\infty)), k_{max}(\widehat{\theta}(s^\infty)))$, there exists some finite T such that $\widehat{\theta}(s^\infty) < \overline{\theta}(k_T(s^\infty))$. But then the full information plan does not satisfy the partner 1's incentive compatibility constraint at $(\widehat{\theta}(s^\infty), k_T(s^\infty))$ and so the argument follows as in Step 2.1.

3. Notice that symmetry implies that

$$c_1(k, \theta, 1-\theta; s) = c_2(k, 1-\theta, \theta; s),$$
$$k'(k, \theta, 1-\theta)(s) = k'(k, 1-\theta, \theta; s)$$

for all s, for all k and for all $\theta \in [0, 1]$.

Therefore, the analysis for the case in which If $(\theta_t, k_t) \in [1-\theta^*, 1] \times [k_{\min}(\theta_t), k_{\max}(\theta_t)]$ at some t, is analogous to 2. above.

□