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Endogenous Credit Limits with Small Default Costs*

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Abstract

We analyze an exchange economy of unsecured credit where borrowers have the option to declare bankruptcy in which case they are temporarily excluded from financial markets. Endogenous credit limits are imposed that are just tight enough to prevent default. Economies with temporary exclusion differ from their permanent exclusion counterparts in two important properties. If households are extremely patient, then the first–best allocation is an equilibrium in the latter economies but not necessarily in the former. In addition, temporary exclusion permits multiple stationary equilibria, with both complete and with incomplete consumption smoothing.

JEL classification: D51; D91; G33

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1 Introduction

The role of limited contract enforcement in dynamic general equilibrium has been explored extensively in key papers by Eaton and Gersovitz [11], Kehoe and Levine [14], Kocherlakota [17], and Kiyotaki and Moore [16], all of which seek to explain why individual consumption, aggregate output and asset prices fluctuate more than aggregate consumption, productivity or dividends . Limited commitment has been used to investigate anomalies in asset pricing (Alvarez and Jermann [1,2], Azariadis and Kaas [6]), international business cycles (Kehoe and Perri [13]), economic growth (Marcet and Marimon [20]), consumption patterns and social security issues (Krueger and Perri [18,19], Andolfatto and Gervais [3]). All these models describe environments in which a shortage of collateral rules out complete risk sharing or consumption smoothing. One institution that improves the distribution of consumption over households is unsecured credit backed by limiting defaulters' subsequent trading in asset markets. The literature typically assumes that an omnipotent credit authority or auctioneer excludes defaulting agents for the rest of their lives from any asset trade. Such a penalty is clearly the strongest possible punishment in the absence of collateral.

This paper explores the consequences of weaker punishment arrangements. For example, Bulow and Rogoff [8] and Hellwig and Lorenzoni [12] impose *one–sided* exclusion which permits defaulters to accumulate assets but banishes them permanently from all borrowing. This paper works out the consequences of temporary exclusion from *both sides* of asset markets. To this end, we consider a stochastic pure–exchange economy in which defaulters are readmitted to asset trading with positive probability. When the punishment period is over, bankrupt households regain full access to all markets. We maintain the common assumption in the literature of a complete market of state–contingent claims supported by default–deterring credit limits. We believe that temporary exclusion is an important feature since real–world bankruptcy procedures never come close to perpetual market exclusion.

Under permanent market exclusion, Alvarez and Jermann [1] prove two results: one, autarky is the unique (and constrained efficient) equilibrium if the autarkic interest rate exceeds the economy's growth rate. Two, when the autarky equilibrium is inefficient, there is a better equilibrium with improved risk sharing. Moreover, Kehoe and Levine [14, Proposition 2] establish a kind of "folk theorem": this constrained efficient equilibrium is first best provided that the common discount factor is sufficiently large. This paper considers a particular equilibrium refinement, namely, the robustness to small explicit default costs. For our class of pure-exchange economies we find that permanent market exclusion gives rise to a unique robust equilibrium. In particular, autarky is not robust to this refinement, unless it is the unique equilibrium.

We show that economies with temporary exclusion differ from their permanent exclusion counterparts in two important ways. First, a higher discount factor does not necessarily help to

implement the first best allocation. With temporary exclusion of defaulters and no matter how low the (positive) readmission probability is, the first–best allocation is not self-enforcing, even when agents are extremely patient, unless they are also sufficiently risk averse. Intuitively, very patient households care less about temporary exclusion penalties but are more interested in the expected value of consumption after they are readmitted to asset trade. Default raises both consumption risk during the punishment phase but also the expected value of consumption in the long run (i.e. after readmission to financial markets).

Second, temporary exclusion can give rise to the existence of multiple robust equilibria. In particular, we prove that autarky is always a robust equilibrium if the readmission probability is sufficiently high (i.e. punishment is sufficiently weak). Besides the robust autarky equilibrium, additional stationary equilibria can emerge; the first–best allocation may be one of them. To understand why multiple robust equilibria occur under temporary exclusion although they are impossible under permanent exclusion, we note that temporary exclusion introduces a *dynamic complementarity* between future and current asset prices. Consider the example of a defaulter who loses the ability to save in the default period but is readmitted to trade in all subsequent periods. Then an increase in *future* interest rates reduces the continuation value from default (because the agent forgoes asset trade in the default period) which relaxes credit constraints today. In turn, a higher volume of borrowing necessitates a higher *current* interest rate to clear the credit market today. This dynamic complementarity can trigger equilibrium multiplicity provided that the impact of future market prices on the value of default is sufficiently strong. Note that such a link is absent in economies with infinite market exclusion where default leads to permanent autarky whose payoff is independent of future prices. We present a more elaborate discussion of this argument in Section 4.

We are aware of only a few contributions dealing with temporary asset market exclusion of defaulting borrowers. Athreya [5] and Chatterjee et al. [9] develop quantitative equilibrium models with incomplete asset markets, characterizing optimal default behavior and equilibrium loan price schedules. In the sovereign default literature, temporary market exclusion of defaulting international borrowers is also a common assumption (Arellano [4], Cuadra, Sanchez and Sapriza [10]). However, these contributions neither discuss equilibrium multiplicity nor the role of discounting which are the focus of this paper.[1] Azariadis and Lambertini [7] consider a deterministic overlapping–generations economy with three–period lived individuals, also demonstrating the existence of multiple equilibria. In their paper endogenous debt constraints are based on one–period exclusion by construction since individuals die in the period after default. Our paper shows that similar results can be obtained in stochastic economies with infinitely–lived agents.

The paper is organized as follows. After introducing the economic environment and equilibrium

[1] We note that the duration of market exclusion is an exogenously given parameter in this literature, as it is in our work. Endogenizing this parameter is beyond the scope of this paper.

concepts in Section 2, we discuss first–best allocations and the role of discounting in Section 3. Section 4 considers equilibria with binding credit constraints and establishes our main result on equilibrium multiplicity and robustness.

2 The environment

States and agents

We consider a pure exchange economy in discrete time $t = 0, 1, \ldots$ with a unit mass of consumers $i \in [0, 1]$ who face idiosyncratic income risk. There is a single non–durable consumption good in each period. Every consumer's endowment of this good follows a two–state Markov process between a high level $y(H) = \lambda > 1$ and a low level $y(L) < 1$. We define transition probabilities $\pi(H|H) = \pi_H$, $\pi(L|H) = 1 - \pi_H$, $\pi(L|L) = \pi_L$, $\pi(H|L) = 1 - \pi_L$, and we normalize mean income to one, which implies that $y(L) = 1 - (\lambda - 1)\frac{1-\pi_L}{1-\pi_H}$. We write $s_t^i \in \{H, L\}$ for consumer i's income state in period t and $s^{t,i} = (s_0^i, \ldots, s_t^i)$ for the consumer's income state history in period t. We let $\pi(s^{t,i})$ denote the unconditional probability of history $s^{t,i}$. To simplify notation, we drop index i from all these expressions whenever possible.

Expected utility of consumer i from a consumption plan $(c^i(s^t))_{t\geq 0}$ is

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) u(c^i(s^t)) \ . \tag{1}$$

The period utility function u is differentiable, strictly increasing and strictly concave. $\beta < 1$ is the discount factor.

Financial markets and contract enforcement

Each period consumers trade a complete set of state–contingent claims on next period's consumption good. We write $q(s^t, s)$ for the date–t price of a claim on the consumption good in history $s^{t+1} = (s^t, s)$, and $a^i(s^t, s)$ for consumer i's trade of this security. Consumer i's budget constraint in period t is then

$$c^i(s^t) + q(s^t, H)a^i(s^t, H) + q(s^t, L)a^i(s^t, L) \leq y(s_t) + a^i(s^t) \ , \tag{2}$$

for any history $s^t \in \{H, L\}^{t+1}$. In period $t = 0$, consumers hold initial claims on the consumption good $(a^i(L), a^i(H))$.

As is standard in the literature on limited commitment, we assume that no part of an agent's endowment can be collateralized. In the absence of any contract enforcement, consumers could not trade securities and would live in autarky. We allow *limited enforcement* of financial contracts through the following mechanism. At the beginning of every period, anyone unwilling to redeem debt may declare bankruptcy. When a consumer does so, he cannot trade securities in the

default period; in any subsequent period, the consumer regains full access to security trade with probability $\mu \in [0,1]$, where μ is an exogenous parameter. The expected duration of market exclusion $1/\mu$ reflects the institutional and legal framework of bankruptcy procedures.[2] The permanent–exclusion case $\mu = 0$ has been studied extensively in the literature; cf. Kehoe and Levine [14,15]. A decentralization with endogenous credit limits has been provided by Alvarez and Jermann [1], and the following equilibrium definition extends theirs to an arbitrary exclusion duration $1/\mu \leq \infty$.

Definition of equilibrium

An *equilibrium with limited commitment and exclusion duration* $1/\mu$ is a list of consumption plans and asset holdings $(c^i(s^t), a^i(s^t))_{(i,s^t)}$, security prices $(q(s^t))_{(s^t)}$, and credit limits $(z(s^t))_{(s^t)}$, such that

(i) For all $i \in [0,1]$, (c^i, a^i) maximizes (1) subject to (2) and to $a^i(s^t) \geq -z(s^t)$, for all $(s^t) \in \{H, L\}^{t+1}$, at given prices and credit limits.

(ii) Markets clear, i.e., for all $t \geq 0$,

$$\int \sum_{s^t} \pi(s^t) c^i(s^t)\, di = 1 \quad \text{and} \quad \int \sum_{s^t} \pi(s^t) a^i(s^t)\, di = 0 \ .$$

(iii) Short–sale constraints prevent default: for any $i \in [0,1]$ and income history s^t, $t \geq 1$, the solvency value from t forward is no smaller than the continuation value from default, that is,

$$U(a^i(s^t), s^t) \geq \bar{U}(s^t) \ , \tag{3}$$

where value functions U and $\overline{U}$ are recursively defined by

$$\begin{aligned} U(a, s^t) &= \max_{c, a_+(H), a_+(L)} \Bigg\{ u(c) + \beta \sum_{s=H,L} \pi(s|s_t) U(a_+(s), (s^t, s)) \\ &\qquad \text{s.t.} \quad c + \sum_{s=H,L} q(s^t, s) a_+(s) \leq y(s_t) + a, \ a_+(s) \geq -z(s^t, s) \Bigg\} \ , \\ \bar{U}(s^t) &= u(y(s_t)) + \beta\mu \sum_{s=H,L} \pi(s|s_t) U(0, (s^t, s)) + \beta(1-\mu) \sum_{s=H,L} \pi(s|s_t) \bar{U}(s^t, s) \ . \end{aligned}$$

(iv) Short–sale constraints are not too tight, i.e., whenever $a^i(s^t) \geq -z(s^t)$ binds in problem (i), the participation constraint (3) is satisfied with equality.

[2] Instead of assuming a deterministic exclusion length, this stochastic formulation helps to characterize equilibrium recursively. In the sovereign debt literature, Arellano [4] and Cuadra, Sanchez and Sapriza [10] also assume that defaulting countries regain access to international credit with some exogenous probability.

Stationary Markov equilibrium

We consider a stationary equilibrium in which, by the law of large numbers, a constant fraction $\varphi_H \equiv \frac{1-\pi_L}{2-\pi_H-\pi_L}$ of consumers has high income and fraction $\varphi_L \equiv 1-\varphi_H$ has low income. We further restrict attention to symmetric stationary Markov equilibria where consumption, security trades and prices depend only on the current income state but are otherwise identical for all agents. We write $q_{ss'}$ for the price that a consumer in income state s must pay to obtain a claim on one unit of consumption in next period's income state s'. Let x be consumption of high–income consumers. Then market clearing implies that low-income agents consume $c_L(x) \equiv 1-(x-1)\frac{1-\pi_L}{1-\pi_H}$.

In any candidate stationary Markov equilibrium, consumption of high-income agents must lie in the interval $x \in [1, \lambda]$. $x = c_L(x) = 1$ corresponds to the symmetric first–best equilibrium with perfect consumption smoothing. At the other extreme, $x = \lambda = y(H) > c_L(x) = y(L)$ describes the autarkic allocation where all asset trades are zero. Values of x below unity or above λ are inconsistent with equilibrium.

In an equilibrium with binding constraints, low–income consumers are constrained in their trade of security $a_{LH} < 0$, whereas all other security trades are unconstrained, as we will see below. Then, security prices q_{HH}, q_{LL} and q_{HL} follow from the Euler equations of unconstrained agents. The price q_{LH} and the corresponding asset trades can be calculated using budget constraints and market–clearing conditions. Given an allocation x, security prices and asset trades in a stationary Markov equilibrium are as follows.

Lemma 1: Let $x \in [1, \lambda]$ be the consumption of high–income agents and $c_L(x) = 1-(x-1)\frac{1-\pi_L}{1-\pi_H}$ be the consumption of low–income agents in a stationary Markov equilibrium. Then security prices are

$$q_{LL} = \beta\pi_L \ , \ q_{HL}(x) = \frac{\beta(1-\pi_H)u'(c_L(x))}{u'(x)} \ ,$$

$$q_{HH} = \beta\pi_H \ , \ q_{LH}(x) = \frac{\beta(1-\pi_L)}{1-\pi_H}\left[\pi_L - \pi_H + (1-\pi_L)\frac{u'(c_L(x))}{u'(x)}\right] \ ,$$

and security trades are $a_{LH} = a_{HH} = -b(x)$, $a_{HL} = a_{LL} = \frac{1-\pi_L}{1-\pi_H}b(x)$ with credit

$$b(x) \equiv \frac{(\lambda - x)}{1-\beta\pi_H + \beta(1-\pi_L)\frac{u'(c_L(x))}{u'(x)}} \ .$$

Proof: Appendix.

Clearly, better consumption smoothing requires larger income transfers across states: b is decreasing in x. At the same time, more credit goes along with higher rates of return (lower security prices): both q_{HL} and q_{LH} are increasing in x. Further, except at the first–best allocation $x = 1$, low–income consumers are credit constrained. Indeed, it is straightforward to verify that

$$u'(c_L(x))q_{LH}(x) > \beta(1-\pi_L)u'(x) \quad \text{if} \quad x > 1 \ .$$

When $x > 1$, we also refer to $b(x)$ as the *credit constraint*.

3 Implementing the first–best allocation

This section identifies conditions under which the symmetric first–best allocation where all agents consume the same, $c_H^i = c_L^i = 1$, is an equilibrium with limited enforcement.[3] In a symmetric first–best equilibrium, all agents achieve perfect consumption smoothing $c_H = c_L = 1$ at security prices $q_{ss} = \beta\pi_s$, $q_{ss'} = \beta(1-\pi_s)$, for $s \neq s' \in \{H, L\}$. In the economy with limited commitment, this allocation is an equilibrium provided that high–income agents have no incentive to deviate into bankruptcy. If the agent stays solvent, his continuation utility is simply $U^* = u(1)/(1-\beta)$. Conversely, if the agent defaults, he obtains value

$$\overline{U}_H = u(\lambda) + \beta(1-\mu)\Big[\pi_H \overline{U}_H + (1-\pi_H)\overline{U}_L\Big] + \beta\mu\Big[\pi_H U_H^0 + (1-\pi_H)U_L^0\Big] \ . \tag{4}$$

Here the first term is utility in the default period where the agent consumes his income $c_H = \lambda$. The second term is expected discounted utility in the event where the agent remains excluded from security trade, which happens with probability $1-\mu$. Here $\overline{U}_L$ satisfies a similar recursion:

$$\overline{U}_L = u(y(L)) + \beta(1-\mu)\Big[\pi_L \overline{U}_L + (1-\pi_L)\overline{U}_H\Big] + \beta\mu\Big[\pi_L U_L^0 + (1-\pi_L)U_H^0\Big] \ . \tag{5}$$

The third terms in equations (4) and (5) are the continuation utilities when the agent regains access to security trade, which occurs with probability μ. In this event, the agent either enters the period with high income, yielding utility value U_H^0, or the agent has low income in the readmission period which yields continuation utility U_L^0.

If the agent regains market access with high income, he has no need to borrow ever again and attains flat consumption

$$c_H^0 = \frac{\lambda(1-\beta) + \beta(2-\pi_H-\pi_L)}{1+\beta(1-\pi_L-\pi_H)} > 1$$

in all subsequent periods.[4] Therefore, the continuation utility in this event is

$$U_H^0 = \frac{1}{1-\beta} u\Big(\frac{\lambda(1-\beta) + \beta(2-\pi_H-\pi_L)}{1+\beta(1-\pi_L-\pi_H)}\Big) \ . \tag{6}$$

If the agent starts trading securities with low income (and zero asset holdings), he can borrow up to a constraint $\overline{b} = z_{LH}$ which is the largest debt limit that prevents default in the subsequent

[3] Clearly, there are many other optimal allocations where agents consume different amounts, depending on their initial asset holdings. However, whenever an asymmetric first-best allocation with consumption $c_H^i = c_L^i \neq 1$ is an equilibrium with limited commitment, the symmetric first best is also an equilibrium after an appropriate redistribution of initial wealth. Hence, non–implementability of a symmetric first–best equilibrium precludes implementability of any asymmetric first–best equilibrium.

[4] To attain this consumption path, the agent saves $a_{HL} = a_{LL} = (\lambda - y(L))/(1 - q_{LL} + q_{HL})$ and does not trade any other security ($a_{LH} = a_{HH} = 0$).

high–income state. This limit is determined endogenously by making the agent indifferent between default and solvency: On the one hand, if the agent enters the next high–income period with wealth $\lambda - \overline{b}$ and stays solvent, he would attain flat consumption $\overline{c} = c_H^0 - \overline{b}(1-\beta)$ in all future periods.[5] On the other hand, if the agent decides to default on the debt $\overline{b}$, his continuation utility would be again $\overline{U}_H$ as defined above. Hence $\overline{b}$ is implicitly defined from

$$u\Big(c_H^0 - \overline{b}(1-\beta)\Big) = (1-\beta)\overline{U}_H \ . \tag{7}$$

When the returning agent has low income, security trades are $a_{LH} = -\overline{b}$ and $a_{LL} = 0$, which yields consumption $c_L^0 = y(L) + q_{LH}\overline{b}$ in the current period as well as in all future subsequent periods with low income.[6] Hence the continuation utility satisfies

$$U_L^0 = u(c_L^0) + \beta\pi_L U_L^0 + \beta(1-\pi_L)\overline{U}_H \ . \tag{8}$$

In the Appendix (proof of Proposition 1) we show that the five equations (4), (5), (6), (7) and (8) have a unique solution $(\overline{U}_H, \overline{U}_L, U_H^0, U_L^0, \overline{b})$.
Given this solution, no agent deviates from the first–best allocation if $U^* \geq \overline{U}_H$. In the Appendix we also prove that this is equivalent to the inequality

$$u(1) \geq \alpha_1 u(\lambda) + \alpha_2 u(y(L)) + \alpha_3 u\Big(\frac{\lambda(1-\beta)+\beta(2-\pi_H-\pi_L)}{1+\beta(1-\pi_L-\pi_H)}\Big) + \alpha_4 u\Big(y(L) + \frac{\beta(1-\pi_L)(\lambda-1)}{1+\beta(1-\pi_L-\pi_H)}\Big) \ , \tag{9}$$

where the coefficients α_i depend on parameters $(\beta, \mu, \pi_H, \pi_L)$ and satisfy $\sum_{i=1}^4 \alpha_i = 1$ (see the Appendix for definitions). We summarize this result as follows:

Proposition 1: The symmetric first–best allocation is an equilibrium with limited commitment and exclusion duration $1/\mu$ if, and only if, condition (9) holds.

The implementability condition can be interpreted as a comparison between two lotteries. The left–hand side is expected utility of the safe lottery which pays one unit of consumption for sure. The right–hand side is expected utility of a risky lottery over four different consumption states, two larger than one, and two smaller than one. As can be verified, expected consumption of the risky lottery is *strictly* larger than one. Intuitively, a defaulting agent obtains a higher net present value of consumption than a solvent agent. It follows that the first best cannot be implemented if risk aversion is sufficiently low; conversely, the first best is an equilibrium if risk aversion is sufficiently strong.

The role of the discount factor

[5] Similar to footnote 4, to attain this consumption path, the agent trades $a_{HL} = a_{LL} = (\lambda - y(L))/(1 - q_{LL} + q_{HL}) - \overline{b}$ and $a_{LH} = a_{HH} = -\overline{b}$.

[6] The assertion $a_{LL} = 0$ follows trivially from the Euler equation $u'(c_L^0)q_{LL} = \beta\pi_L u'(c_L^0)$.

In the limit of permanent market exclusion ($\mu = 0$), condition (9) simplifies to

$$u(1) \geq \frac{1-\beta\pi_L}{1+\beta(1-\pi_H-\pi_L)}u(\lambda) + \frac{\beta(1-\pi_H)}{1+\beta(1-\pi_H-\pi_L)}u(y(L)) \ . \tag{10}$$

This condition is similar to the one in the two–agent, two–state example of Kehoe and Levine [15]. In their economy, as in ours for $\mu = 0$, there is a kind of "folk theorem": for any strictly concave utility function, the first–best allocation is implementable if the discount factor is sufficiently large. Indeed, in the limit $\beta \to 1$, condition (10) compares utility of a safe lottery to utility of a risky lottery with the same mean, and hence the condition is fulfilled, regardless of the degree of risk aversion.

In contrast, if defaulters are readmitted with positive probability, there is no such folk theorem. With lower discounting, agents care less about temporary punishment and more about consumption in the long run. Because the present value of consumption is larger under default than under solvency, agents with low enough risk aversion prefer to default on their debt. Formally,[7]

Proposition 2: For any given $\mu > 0$, there is a class of strictly concave utility functions $\mathcal{U}$ (such as CRRA or CARA with low enough degree of risk aversion) and $\beta_0 < 1$ such that the first best allocation is not implementable for any $u \in \mathcal{U}$ and any $\beta \in [\beta_0, 1)$.

Proof: Appendix.

To illustrate how large risk aversion has to be for the first best to be implementable, we consider a numerical example with CRRA utility $u(c) = c^{1-\sigma}/(1-\sigma)$, $\pi_L = \pi_H = 0.9$ and $\lambda = 1.15$. Table 1 shows the threshold values for σ such that the first best is implementable in the limit $\beta \to 1$ when risk aversion is above σ and not implementable otherwise. With relative risk aversion smaller than two, implementability of the first best requires market exclusion to last more than twenty periods on average.

We also note that implementability of the first best allocation can depend on the discount factor in a non–monotonic way. Particularly, the first best may not be an equilibrium in the limit $\beta \to 1$ although it can be implementable for lower values of β. In the numerical example of Table 1 with $\mu = 1$ and $\sigma = 19$, the first best is implementable for $\beta \in [0.71, 0.94]$ but it is not implementable for higher or for lower values of β. The explanation is that, for intermediate values of β, the temporary loss from punishment has a stronger weight than the long-run gain in expected consumption after market reentry. When β is larger, however, the long-run consumption gain is weighted more heavily, so that default is the more attractive option.

[7] In the limit $\beta \to 1$, condition (9) becomes $u(1) \geq u(1)$, and hence is meaningless. Intuitively, the defaulting consumer attains consumption $c = 1$ in the long run when interest rates are zero, and he does not care about temporary punishment since he is extremely patient. Thus, he is exactly indifferent between solvency and default. The proof of Proposition 2 therefore relies on a limiting argument: we show that the slope of the right-hand side at $\beta = 1$ is strictly negative so that inequality (9) fails for β sufficiently large.

4 Binding debt limits

We now consider stationary Markov equilibria with constrained borrowers. Binding debt limits are equivalent to incomplete consumption smoothing, that is, to $x > 1 > c_L(x)$. Security prices, as well as the credit constraint $b(x)$, are stated in Lemma 1. With low–income agents being constrained in short–selling security a_{LH}, the equilibrium definition entails that the agent must be exactly indifferent between honoring the debt $a_{LH} = -b(x)$ and defaulting. An agent who stays solvent in a high–income period continues to consume x in high–income periods and $c_L(x)$ in low–income periods. Solvency values in high– and low–income periods are then recursively defined by

$$U_H^*(x) = u(x) + \beta\pi_H U_H^*(x) + \beta(1-\pi_H)U_L^*(x) \ , \tag{11}$$

$$U_L^*(x) = u(c_L(x)) + \beta\pi_L U_L^*(x) + \beta(1-\pi_L)U_H^*(x) \ . \tag{12}$$

If the agent instead decides to default, continuation values are

$$\overline{U}_H(x) = u(\lambda)+\beta(1-\mu)\Big[\pi_H\overline{U}_H(x)+(1-\pi_H)\overline{U}_L(x)\Big]+\beta\mu\Big[\pi_H\tilde{U}_H(0,x)+(1-\pi_H)\tilde{U}_L(0,x)\Big] \ , \tag{13}$$

$$\overline{U}_L(x) = u(y(L))+\beta(1-\mu)\Big[\pi_L\overline{U}_L(x)+(1-\pi_L)\overline{U}_H(x)\Big]+\beta\mu\Big[\pi_L\tilde{U}_L(0,x)+(1-\pi_L)\tilde{U}_H(0,x)\Big] \ . \tag{14}$$

These expressions generalize (4) and (5) and they have the same interpretation. The value functions $\tilde{U}_s(a,x)$ are expected discounted utilities for an agent in income state $s = H, L$ with assets a, who trades at security prices $q_{HH} = \beta\pi$, $q_{LL} = \beta(1-\pi)$, $q_{HL}(x)$ and $q_{LH}(x)$, facing constraint $-b(x)$ on short–selling securities a_{LH} and a_{HH}. These value functions are recursively defined by[8]

$$\tilde{U}_s(a,x) = \max_{\substack{a_{sL}\geq -z_L,\\ a_{sH}\geq -b(x)}} u\Big(y(s)+a-q_{sH}(x)a_{sH}-q_{sL}(x)a_{sL}\Big)+\beta\Big[\pi_s\tilde{U}_s(a_{ss},x)+(1-\pi_s)\tilde{U}_{s'}(a_{ss'},x)\Big] \ , \tag{15}$$

for $s \neq s' \in \{H, L\}$. In the previous section, we derive the values $\tilde{U}_s(0,x) = U_s^0$ for the case $x = 1$. In the Appendix (proof of Proposition 3), we calculate these value functions explicitly for values of x close to λ (autarky).

It follows from the previous discussion that any symmetric stationary Markov equilibrium allocation $x \in [1, \lambda]$ is a solution to the following complementary slackness condition:

$$J(x) \equiv U_H^*(x) - \overline{U}_H(x) \geq 0 \quad , \quad x \geq 1 \ . \tag{16}$$

In words, either consumers are credit constrained ($x > 1$) which necessitates a binding participation constraint, $J(x) = 0$; or the participation constraint is slack, credit constraints are not

[8] Formally, the agent also faces a constraint z_L on selling securities a_{LL} and a_{HL}, but since the agent saves for low income realizations, these constraints do not bind.

binding, so that agents can smooth consumption perfectly ($x = 1$). Furthermore, as we demonstrate in the previous section, the implementability condition (9) is equivalent to the requirement $J(1) \geq 0$.

An important insight is that the autarkic solution $x = \lambda$ is an equilibrium with limited commitment for any given exclusion parameter μ. This follows immediately from $U_H^*(\lambda) = \overline{U}_H(\lambda) = \tilde{U}_H(0, \lambda)$ and it extends a result of Alvarez and Jermann [1, Proposition 4.3] for $\mu = 0$ to enforcement with finite exclusion $1/\mu < \infty$. When agents expect asset markets to shut down in all future periods, there is no gain from market participation and the short–sale constraint on security a_{LH} must be zero in the current period. At the same time, security prices q_{HH}, q_{LL} and q_{HL} reflect marginal rates of substitution at the autarkic allocation, so that no agent is willing to trade any of these securities.

Infinite exclusion

The infinite–exclusion case $\mu = 0$ is well understood. Figure 1 graphs

$$J(x) = \frac{1}{(1-\beta)(1+\beta-\beta(\pi_H+\pi_L))}\Big\{(1-\beta\pi_L)\Big[u(x)-u(\lambda)\Big]+\beta(1-\pi_H)\Big[u(c_L(x))-u(y(L))\Big]\Big\}$$

for different levels of β corresponding to three generic outcomes. The dotted curve is an example of a low value of β for which autarky $x = \lambda$ is the unique equilibrium. For larger values of β, there is a unique non–autarkic (trading) equilibrium with constrained agents where $x \in (1, \lambda)$ (dashed curve). When β is even larger, the trading equilibrium is a first–best equilibrium at $x = 1$ (solid curve).

When the economy with infinite exclusion has two stationary equilibria, only the trading equilibrium is robust to the introduction of small bankruptcy costs. To see this, suppose that a defaulter must pay a small cost ε when declaring bankruptcy, so that utility in the default period is $u(\lambda - \varepsilon)$. We say that an equilibrium x with limited commitment in the economy without bankruptcy cost ($\varepsilon = 0$) is *robust* if the economy with a small bankruptcy cost ε has an equilibrium x_ε which converges to x when $\varepsilon \to 0$. Graphically, bankruptcy costs reduce the default value and hence shift the curve $J_\varepsilon(x) = 0$ upwards in Figure 1. Whenever autarky is the unique equilibrium for $\varepsilon = 0$ (dotted curve), the economy with $\varepsilon > 0$ has a unique stationary equilibrium $x_\varepsilon < \lambda$ near autarky which collapses to autarky as the bankruptcy cost becomes negligible. In the other two cases, however, $J'(\lambda) < 0$, and hence $J_\varepsilon(x) = 0$ has no solution $x \leq \lambda$ near λ. Although there is a solution $x > \lambda$, such an allocation is not an equilibrium.[9] Therefore,

[9] Suppose that there is a stationary Markov equilibrium where high–income agents consume $x > \lambda$ and low–income agents consume $c_L < y(L)$. Then, $a_{HL} = a_{LL} < 0$ and $a_{HH} = a_{LH} > 0$. Hence, low–income agents are unconstrained in trading security a_{LH} which has price $q_{LH} = \beta(1-\pi_L)u'(x)/u'(c_L(x))$. It follows from budget constraints and market clearing (similar to the proof of Lemma 1) that the price of security a_{HL} is $q_{HL} = \frac{\beta(1-\pi_H)}{1-\pi_L}[\pi_H - \pi_L + (1-\pi_H)u'(x)/u'(c_L)] < \beta(1-\pi_H)$. But then, the Euler equation of high–income agents $u'(x)q_{HL} \geq \beta(1-\pi_H)u'(c_L)$ implies that $u'(x) > u'(c_L)$, which contradicts $x > 1 > c_L$.

autarky is not a robust equilibrium whenever there exists a trading equilibrium. It follows that the economy with permanent exclusion has a unique robust stationary Markov equilibrium.

Finite exclusion

The situation is different with shorter exclusion periods. Particularly, we prove that autarky is always a robust equilibrium (that is, $J'(\lambda) > 0$) if the readmission probability μ is sufficiently large. Then it follows that the economy can have multiple robust equilibria, which is necessarily the case if the first-best is implementable ($J(1) > 0$).

Proposition 3:

(a) If market exclusion is permanent ($\mu = 0$), there exists at most one stationary trading equilibrium. If such an equilibrium exists, it is the unique robust equilibrium.

(b) If market exclusion is sufficiently short (μ is sufficiently close to 1), autarky is a robust equilibrium. If, in addition, inequality (9) is strict (for example, with CARA or CRRA utility functions and sufficiently large risk aversion), there exist at least three stationary equilibria: (i) a first–best equilibrium with perfect consumption smoothing; (ii) a trading equilibrium with binding constraints; and (iii) an autarkic equilibrium. All these equilibria are robust.

Proof: Appendix.

Part (b) says that the economy with short market exclusion can have multiple robust equilibria. Paralleling Figure 1, Figure 2 shows the same example with larger income fluctuations ($\lambda = 1.3$) and one period exclusion ($\mu = 1$) for changing values of the discount factor (Figure 2(a)) and risk aversion (Figure 2(b)). When β is low or σ is low, autarky is the unique robust equilibrium (dotted curves). For higher values of β and σ, autarky remains a robust equilibrium, but two other equilibria with risk sharing emerge. One of them is the first best at $x = 1$, the other one has constrained borrowers at $x \in (1, \lambda)$. The constrained equilibrium has lower risk sharing if either the discount factor or risk aversion is larger (solid curves).

In these examples, autarky ceases to be a robust equilibrium when the readmission parameter μ is substantially smaller than one. But even when autarky fails to be a robust equilibrium, multiple robust equilibria are still possible. For example, if we set $\mu = 0.85$, $\beta = 0.6$ and $\sigma = 1.32$ (all other parameters as in Figure 2), autarky is not robust, while there are three robust trading equilibria at $x = 1$, $x \approx 1.01$ and $x \approx 1.04$. For many parameter configurations, however, multiplicity seems to disappear when $\mu < 0.9$. We also remark that there are examples with more than three equilibria. With $\lambda = 1.8$, $\pi_H = \pi_L = 0.75$, $\sigma = 2.65$, $\beta = 0.55$ and $\mu = 1$, there are five (robust) equilibria at $x = 1$ (first best), $x \approx 1.125$, $x \approx 1.33$, $x \approx 1.75$ and at $x = 1.8$ (autarky).

To obtain some intuition for equilibrium multiplicity, consider the special case of this model with deterministic income fluctuations where $\pi_H = \pi_L = 0$ so that every agent's income fluctuates periodically between $\lambda > 1$ and $2 - \lambda < 1$. Each period half the agents have high income and the other half has low income, so that aggregate income equals one. Let $x_t \in [1, \lambda]$ denote equilibrium consumption of high-income agents in period t, which implies that low-income agents consume $2 - x_t$. To simplify further, set $\mu = 1$ which implies that a defaulting agent cannot save in the default period but starts borrowing in the subsequent low-income period. We also restrict attention to an equilibrium where credit constraints bind in all periods. Let d_t denote borrowing of low-income agents in period t (which equals savings of high-income agents in the same period) and write R_t for the gross rate of return between periods t and $t+1$.
If an agent defaults in period t, continuation utility is

$$u(\lambda) + \beta u(2 - \lambda + d_{t+1}) + \beta^2 u(x_{t+2}) + \beta^3 u(2 - x_{t+3}) + \cdots \ .$$

In the default period, the agent consumes income λ and cannot save. In period $t+1$, the agent starts to borrow again, up to the same limit d_{t+1} as any solvent agent. Hence, the agent consumes $2 - \lambda + d_{t+1}$ in the period after default. Thereafter, the defaulter achieves the same asset/consumption profile as a solvent agent: consumption is x_{t+k} in periods $t+k$ for $k \geq 2$ even, and $2 - x_{t+k}$ for $k \geq 3$ odd.
If the agent instead stays solvent in period t, continuation utility is

$$u(x_t) + \beta u(2 - x_{t+1}) + \beta^2 u(x_{t+2}) + \beta^3 u(2 - x_{t+3}) + \cdots \ .$$

Binding credit constraints mean that borrowers are exactly indifferent between default and solvency. Hence,

$$\underbrace{u(x_t) + \beta u(2 - x_{t+1})}_{\text{solvency payoff}} = u(\lambda) + \beta u(2 - \lambda + d_{t+1}) = \underbrace{u(\lambda) + \beta u(2 - x_{t+1} - R_t d_t)}_{\text{default payoff}} \ . \tag{17}$$

The second equality uses the budget constraint of a solvent agent with high income

$$x_{t+1} + d_{t+1} = \lambda - R_t d_t \ , \tag{18}$$

which says that income net of debt redemption on the right-hand side is spent for consumption and savings on the left-hand side.[10] Equation (17) shows the key ingredient of temporary market exclusion: the payoff from defaulting on the right-hand side depends negatively on the interest rate R_t since the defaulting agent forgoes the possibility to save between periods t and $t+1$. With higher R_t, default is less attractive, the credit constraint relaxes in period $t-1$, so that

[10] Note our sign convention: $d_t \geq 0$ and $d_{t+1} \geq 0$ denote borrowing (of a low-income agent) and savings (of a high-income agent).

the agent borrows more in $t-1$ and consumes less in period t: thus, equation (17) describes a negative relation between R_t and x_t.

In turn, relaxed credit constraints in period $t-1$ have a direct impact on the market-clearing interest rate in period $t-1$, which follows from the marginal rate of substitution of high-income agents:

$$R_{t-1} = \frac{u'(x_{t-1})}{\beta u'(2-x_t)} \ . \tag{19}$$

To induce lenders to save more in period $t-1$, the equilibrium interest rate must increase: equation (19) describes a positive relation between $2-x_t$ and R_{t-1} (a negative relation between x_t and R_{t-1}). Taken together, equations (17) and (19) describe the dynamic complementarity between future and current interest rates (or between future and current credit constraints) which is the key for equilibrium multiplicity:[11] If agents expect high interest rates to prevail in period t, the cost of default is high, agents are permitted to borrow more in period $t-1$ which then drives up the current equilibrium interest rate. Conversely, if agents expect low interest rates in the future, default is a more attractive option, credit constraints tighten, so that a lower current interest rate is required to clear the market.[12] Note the difference to the environment with permanent market exclusion where the binding participation constraint reads as

$$u(x_t) + \beta u(2-x_{t+1}) = u(\lambda) + \beta u(2-\lambda) \ .$$

In this case, the value of defaulting on the right–hand side is independent of market prices, so that the dynamic complementarity described above is absent.

Although we illustrate these arguments for the deterministic example, a similar intuition is at work in the stochastic economy: High expected security prices impede risk sharing so that the immediate (short run) consequences of default become less threatening. This in turn tightens current constraints and drives up current security prices.

5 Conclusions

We have studied how the sanctions against default on unsecured credit affect the allocation of consumption in a class of stochastic exchange economies with infinitely–lived agents and limited commitment. Strong sanctions, modeled as perpetual exclusion from both sides of all asset markets, are known to deliver "good" results as in Kehoe and Levine [14] and Alvarez and Jermann

[11] To give a numerical example for equilibrium multiplicity in this deterministic economy, set $\lambda = 1.6$, $\beta = 0.95$ and CRRA utility with $\sigma = 2$. Then, besides the autarkic equilibrium at $\overline{x} = \lambda$, there are two other steady state equilibria at $x_1 = 1$ (first best) and at $x_2 = 1.43$ (binding constraints).

[12] This intuition is similar in the three-period life-cycle model of Azariadis and Lambertini [7] who argue that multiplicity requires a high intertemporal complementarity in consumption which is caused by binding constraints, amplifying the impact of future interest rates on the default value.

[1]. Among these is the generic existence of one or two steady states: a constrained efficient outcome, and an inferior autarkic allocation which vanishes when we introduce an arbitrarily small bankruptcy cost. If households are sufficiently patient, the cost of exclusion becomes too high and the constrained efficient outcome coincides with the first best allocation. Strong sanctions deliver the Arrow–Debreu outcome as a unique equilibrium for patient consumers.

It should be no surprise that weaker sanctions, modeled in this paper as *temporary* exclusion from asset markets, cannot accomplish as much as perpetual exclusion. What is surprising is that the Arrow–Debreu outcome cannot be supported as a competitive equilibrium for arbitrarily patient households with moderate degrees of risk aversion. This failure occurs because patient consumers will weigh the short–lived default cost of market exclusion against the long–lived benefit of permanently higher consumption after they resume asset trading as debt–free households.

Weaker sanctions also mean that default must be deterred by short–lived punishments, that is, by debt limits that respond strongly to incomes and prices. Short exclusion then results in a dynamic complementarity between current and future debt limits (equivalently, between current and future asset prices) which permits multiple steady states to co–exist.

Appendix

Proof of Lemma 1:

In a stationary Markov equilibrium, every agent's wealth depends only on the current income state but not on the state history. This necessitates $a_{HH} = a_{LH}$ and $a_{HL} = a_{LL}$. With $a_{HH} = a_{LH} = -b$ and market clearing follows that $a_{HL} = a_{LL} = (1-\pi_L)b/(1-\pi_H)$. Security prices q_{HH} and q_{LL} follow immediately from the Euler equations $q_{HH}u'(x) = \beta\pi_H u'(x)$ and $q_{LL}u'(c_L(x)) = \beta\pi_L u'(c_L(x))$. Furthermore, since high–income agents are unconstrained in their trade of security a_{HL}, $q_{HL}(x)$ follows from the Euler equation $q_{HL}u'(x) = \beta(1-\pi_H)u'(c_L(x))$. With this notation, the budget constraint of a consumer in a high–income state is

$$x - \beta\pi_H b + q_{HL}(x)b\frac{1-\pi_L}{1-\pi_H} = \lambda - b \ , \tag{20}$$

and in a low–income state the budget constraint is

$$c_L(x) - q_{LH}b + \beta\pi_L b\frac{1-\pi_L}{1-\pi_H} = y(L) + b\frac{1-\pi_L}{1-\pi_H} \ . \tag{21}$$

Multiplying (20) by $\varphi_H = \dfrac{1-\pi_L}{1-\pi_H-\pi_L}$ and multiplying (21) by $\varphi_L = 1-\varphi_H$, adding these equations up and using the market–clearing condition $\varphi_H x + \varphi_L c_L(x) = 1$ yields

$$q_{HL}(x)(1-\pi_L)^2 + \beta(1-\pi_L)(1-\pi_H)\pi_L = \beta\pi_H(1-\pi_L)(1-\pi_H) + q_{LH}(1-\pi_H)^2 \ .$$

Solving for q_{LH} yields

$$q_{LH}(x) = \frac{\beta(1-\pi_L)}{1-\pi_H}\Big[\pi_L - \pi_H + (1-\pi_L)\frac{u'(c_L(x))}{u'(x)}\Big] \ .$$

Finally, $b = b(x)$ as stated in the Lemma follows directly from (20). $\square$

Proof of Proposition 1: Combining (4), (5), (6), (7) and (8) shows that $\overline{b}$ is implicitly defined from

$$\frac{u(\overline{c}(\overline{b}))}{1-\beta}\left\{[1-\beta(1-\mu)\pi_H][1-\beta(1-\mu)\pi_L]-\beta^2(1-\mu)^2(1-\pi_H)(1-\pi_L)-\frac{\beta^2\mu(1-\pi_L)(1-\pi_H)}{1-\beta\pi_L}\right\}=$$

$$\Big[1-\beta(1-\mu)\pi_L\Big]u(\lambda)+\Big[\beta(1-\mu)(1-\pi_H)\Big]u(y(L)) \tag{22}$$

$$+\frac{\beta^2\mu(1-\mu)(1-\pi_H-\pi_L)+\beta\mu\pi_H}{1-\beta}u(c_H^0)+\frac{\beta\mu(1-\pi_H)}{1-\beta\pi_L}u\Big(y(L)+\beta(1-\pi_L)\overline{b}\Big)\ ,$$

with

$$\overline{c}(\overline{b})=\frac{\lambda(1-\beta)+\beta(2-\pi_H-\pi_L)}{1+\beta(1-\pi_L-\pi_H)}-\overline{b}(1-\beta)\ .$$

The RHS is strictly increasing and the LHS is strictly decreasing in $\overline{b}$. Moreover, LHS>RHS at $\overline{b}=0$ and LHS<RHS when $\overline{b}$ is sufficiently large. Therefore (22) has a unique solution. Then, utility values $\overline{U}_H$, $\overline{U}_L$, U_H^0, U_L^0 follow uniquely from the linear equations (4), (5), (6), and (8). The agent does not deviate from the first–best allocation $c=1$ if, and only if, $u(1)\geq\overline{U}_H(1-\beta)=u(\overline{c}(\overline{b}))$, which is the same as $\overline{b}\geq b(1)=(\lambda-1)/(1+\beta(1-\pi_L-\pi_H))$. Because of (22) this is true if LHS$\geq$RHS at $\overline{b}=b(1)$. But this inequality is the same as condition (9) with coefficients

$$\begin{aligned}
\alpha_1 &= \gamma(1-\beta)[1-\beta(1-\mu)\pi_L]\ ,\\
\alpha_2 &= \gamma\beta(1-\beta)(1-\mu)(1-\pi_H)\ ,\\
\alpha_3 &= \gamma\Big[\beta^2\mu(1-\mu)(1-\pi_H-\pi_L)+\beta\mu\pi_H\Big]\ ,\\
\alpha_4 &= \frac{\gamma\beta(1-\beta)\mu(1-\pi_H)}{1-\beta\pi_L}\ ,
\end{aligned}$$

where

$$\gamma\equiv\left\{[1-\beta(1-\mu)\pi_H][1-\beta(1-\mu)\pi_L]-\beta^2(1-\mu)^2(1-\pi_H)(1-\pi_L)-\frac{\beta^2\mu(1-\pi_L)(1-\pi_H)}{1-\beta\pi_L}\right\}^{-1}.$$

It is straightforward to verify that $\sum_{i=1}^4\alpha_i=1$. This completes the proof of Proposition 1. $\square$

Proof of Proposition 2: Rewrite the implementability condition (9) as $\Phi(\beta)\geq 0$ with Φ defined by

$$\Phi(\beta)=u(1)\left\{[1-\beta(1-\mu)\pi_H][1-\beta(1-\mu)\pi_L]-\beta^2(1-\mu)^2(1-\pi_H)(1-\pi_L)-\frac{\beta^2\mu(1-\pi_L)(1-\pi_H)}{1-\beta\pi_L}\right\}$$

$$-(1-\beta)\Big[1-\beta(1-\mu)\pi_L\Big]u(\lambda)-\beta(1-\beta)(1-\mu)(1-\pi_H)u(y(L))$$

$$-\Big[\beta^2\mu(1-\mu)(1-\pi_H-\pi_L)+\beta\mu\pi_H\Big]u\Big(\frac{\lambda(1-\beta)+\beta(2-\pi_H-\pi_L)}{1+\beta(1-\pi_L-\pi_H)}\Big)$$
$$-\frac{\beta(1-\beta)\mu(1-\pi_H)}{1-\beta\pi_L}u\Big(y(L)+\frac{\beta(1-\pi_L)(\lambda-1)}{1+\beta(1-\pi_L-\pi_H)}\Big)\ .$$

It is straightforward to verify that $\Phi(1)=0$ and also to calculate the derivative at $\beta=1$:

$$\Phi'(1)=-\frac{(2-\pi_L-\pi_H)(1-(1-\mu)\pi_L)}{1-\pi_L}u(1)$$
$$+\Big[1-(1-\mu)\pi_L\Big]u(\lambda)+(1-\mu)(1-\pi_H)u(y(L))+\frac{\mu(1-\pi_H)}{1-\pi_L}u\Big(y(L)+\frac{(1-\pi_L)(\lambda-1)}{2-\pi_H-\pi_L}\Big)$$
$$+\frac{\mu(\lambda-1)(1-\pi_L-\mu(1-\pi_L-\pi_H))}{2-\pi_H-\pi_L}u'(1)\ .$$

For a linear utility function, it follows after rearranging that

$$\Phi'(1)=\frac{(\lambda-1)\mu\Big[1+(1-\mu)(1-\pi_H-\pi_L)\Big]}{2-\pi_H-\pi_L}\ ,$$

which is strictly positive when the readmission probability is $\mu>0$. Therefore, for all strictly concave utility functions uniformly close to $u(c)=c$ (such as CRRA or CARA with degree of risk aversion close to zero), $\Phi'(1)>0$ remains true. Therefore, with $\mathcal{U}$ being a class of utility functions with low enough degree of risk aversion, $\Phi(\beta)<0$ holds (i.e. the first best is not implementable) for all $\beta<1$ sufficiently close to unity. □

Proof of Proposition 3:

Part (a): Observe that $J(x)=Au(x)+Bu(C-Dx)+E$ for some constants $A>0, B>0, C>0, D>0$ and E. Hence J is strictly concave and it satisfies $J(\lambda)=0$. If $J(1)\geq 0$, it follows that $J(x)>0$ for all $x\in(1,\lambda)$ and $J'(\lambda)<0$. Therefore, the first best $x=1$ is the unique trading equilibrium which is robust, whereas autarky is not robust. If $J(1)<0$ and $J'(\lambda)<0$, there exists a unique $x\in(1,\lambda)$ such that $J(x)=0$ which is the unique (robust) trading equilibrium, and again autarky is not robust. Lastly, if $J(1)<0$ and $J'(\lambda)\geq 0$, $J(x)<0$ for all $x\in(1,\lambda)$. Then autarky is the unique equilibrium and it is robust.

Part (b): Observe first that function $J(.)$ is continuous: Security prices and the constraint $b(x)$ as derived in Lemma 1 are all continuous functions. Hence, the continuation value functions $U_s(a,x)$, as defined recursively in (15), are continuous in x, and so is the utility from default, $\overline{U}_H(.)$. To prove that autarky is a robust equilibrium, we derive $J(x)$ explicitly for x close to λ and then show that $J'(\lambda)>0$ if μ is sufficiently large.

A defaulting agent who reenters asset trade with low income and zero assets immediately hits the credit constraint $b(x)$ and does not trade security a_{LL} (see below); hence consumption is $c_L^0=y(L)+q_{LH}(x)b(x)$. The agent's continuation utility is

$$\tilde{U}_L(0,x)=u(c_L^0)+\beta\pi_L\tilde{U}_L(0,x)+\beta(1-\pi_L)U^*(x)\ . \tag{23}$$

With probability $1-\pi_L$, the agent enters the next period with high income and net assets $-b(x)$. In this event, continuation utility is $U^*(x)$, the same as for any solvent high–income agent with identical net assets. With probability π_L, the agent has low income again in the next period which he enters with zero assets. Therefore, continuation utility is $\tilde{U}_L(0,x)$, the same as in the current period. The agent does not trade security a_{LL} since the price $q_{LL} = \beta(1-\pi)$ induces the agent to enter a subsequent low–income period with the same (zero) asset position (the Euler equation implies that the agent desires a flat consumption profile in all subsequent low–income periods).

Next consider the agent who reenters asset trade with high income and zero assets. This agent buys security a_{HL} to self-insure against a low income realization in the next period. Denote the agent's trade of this security by $\xi(x) \geq 0$. Note that the agent does not trade security a_{HH} in the reentry period because he desires a flat consumption profile when $q_{HH} = \beta(1-\pi_H)$ and hence wishes to enter the next high–income period with the same (zero) asset position. Therefore consumption in the reentry period is $c_H^0 = \lambda - q_{HL}(x)\xi(x)$. When x is sufficiently close to λ (autarky), $q_{HL}(x)$ converges to the agent's MRS $\beta(1-\pi_H)u'(y(L))/u'(\lambda)$, so that $\xi(x)$ converges to zero. In the next period, if the agent has low income and net assets $\xi(x)$, the agent sells security $a_{LH} = -b(x)$ (i.e. the credit constraint binds) and he buys $a_{LL} = \xi(x)$ (once again, the agent desires to enter the next low–income period with the same assets). The credit constraint on short-selling a_{LH} will only bind provided that $\xi(x)$ is sufficiently small (x is sufficiently close to λ).[13] Therefore, the agent's consumption in the subsequent low income period is $c_L^1 \equiv y(L) + \xi(x)[1-\beta\pi_L] + q_{LH}(x)b(x)$. The agent's security trade $\xi(x)$ follows uniquely from the Euler equation

$$u'\Big(\underbrace{\lambda - q_{HL}(x)\xi}_{c_H^0}\Big)q_{HL}(x) = \beta(1-\pi_H)u'\Big(\underbrace{y(L) + \xi[1-\beta\pi_L] + q_{LH}(x)b(x)}_{c_L^1}\Big) \ . \tag{24}$$

Continuation utility in the reentry period is then

$$\tilde{U}_H(0,x) = u(c_H^0) + \beta\pi_H\tilde{U}_H(0,x) + \beta(1-\pi_H)\tilde{U}_L(\xi(x),x) \ , \tag{25}$$

and $\tilde{U}_L(\xi(x),x)$ satisfies

$$\tilde{U}_L(\xi(x),x) = u(c_L^1) + \beta\pi_L\tilde{U}_L(\xi(x),x) + \beta(1-\pi_L)U^*(x) \ . \tag{26}$$

Using (13), (14), (23), (25) and (26), we can calculate the default value $\overline{U}_H(x)$ explicitly. Par-

[13] If x is not close to λ, it can happen that the agent saves so much in the reentry period that the credit constraint in the subsequent period is slack. If this is the case, the agent will alternate between several high–income and low–income periods, before the credit constraint starts to bind at some subsequent date. Conversely, in the "first–best" limit $x \to 1$, the agent remains unconstrained infinitely often and is able to attain flat consumption at c_H^0 (see Section 3 and footnote 4).

ticularly, if $\mu = 1$, we obtain

$$\begin{aligned}\overline{U}_H(x) &= u(\lambda) + \frac{\beta\pi_H}{1-\beta\pi_H}\left\{u(c_H^0) + \frac{\beta(1-\pi_H)}{1-\beta\pi_L}\Big[u(c_L^1) + \beta(1-\pi_L)U_H^*(x)\Big]\right\} \\ &\quad + \frac{\beta(1-\pi_H)}{1-\beta\pi_L}\Big\{u(c_L^0) + \beta(1-\pi_L)U_H^*(x)\Big\} \ . \end{aligned} \tag{27}$$

On the other hand, (11) and (12) imply

$$U_H^*(x) = \frac{1-\beta\pi_L}{(1-\beta\pi_L)(1-\beta\pi_H) - \beta^2(1-\pi_L)(1-\pi_H)}\Big\{u(x) + \frac{\beta(1-\pi_H)}{1-\beta\pi_L}u(c_L(x))\Big\} \ . \tag{28}$$

Substitution of (27) and (28) into $J(x) = U^*(x) - \overline{U}_H(x)$ and some algebra lead to

$$J(x) = \frac{1}{1-\beta\pi_H}\Big[u(x) + \frac{\beta(1-\pi_H)}{1-\beta\pi_L}u(c_L(x))\Big] - u(\lambda) - \frac{\beta\pi_H}{1-\beta\pi_H}u\Big(\lambda - q_{HL}(x)\xi(x)\Big)$$

$$-\frac{\beta^2\pi_H(1-\pi_H)}{(1-\beta\pi_H)(1-\beta\pi_L)}u\Big(y(L)+\xi(x)[1-\beta\pi_L]+q_{LH}(x)b(x)\Big) - \frac{\beta(1-\pi_H)}{1-\beta\pi_L}u\Big(y(L)+q_{LH}(x)b(x)\Big) \ .$$

When we differentiate J at $x = \lambda$, note that all derivatives of $q_{HL}(x)$ and $q_{LH}(x)$ cancel out because of $b(\lambda) = \xi(\lambda) = 0$. Further, derivatives of $\xi(x)$ also cancel out because of (24) (alternatively, since $\xi(x)$ is a utility–maximizing security trade, the envelope theorem implies that the derivative of J with respect to ξ is zero). Therefore, we obtain

$$J'(\lambda) = \frac{1}{1-\beta\pi_H}\Big[u'(\lambda) - \frac{\beta(1-\pi_L)}{1-\beta\pi_L}u'(y(L))\Big] - q_{LH}(\lambda)b'(\lambda)u'(y(L))\frac{\beta(1-\pi_H)}{(1-\beta\pi_L)(1-\beta\pi_H)} \ .$$

Using $b'(\lambda) = -[1-\beta\pi_H + \beta(1-\pi_L)u'(y(L))/u'(\lambda)]^{-1}$ and $q_{LH}(\lambda)$ from Lemma 1 implies that

$$J'(\lambda) = \frac{1}{1-\beta\pi_H}\left\{u'(\lambda) - \frac{\beta(1-\pi_L)}{1-\beta\pi_H + \beta(1-\pi_L)\frac{u'(y(L))}{u'(\lambda)}}u'(y(L))\right\} > 0 \ .$$

This shows that autarky is a robust equilibrium when $\mu = 1$. Because J (and J') depend continuously on parameter μ, autarky is also robust when μ is sufficiently close to unity. Finally, if $J(1) > 0$ holds, continuity of J implies there are at least three equilibria: $x = 1$, $x \in (1, \lambda)$ and $x = \lambda$. Robustness of all equilibria with respect to small bankruptcy costs ε follows because J depends continuously on parameter ε. $J(1) > 1$ is equivalent to the strict inequality (9) which can be expressed as $u(1) > \sum_{i=1}^4 \alpha_i u(c_i)$ such that $\sum_i \alpha_i = 1$, $\sum_i \alpha_i c_i > 1$ and $c_1 < 1$. It follows that this inequality holds for a CARA (CRRA) utility function with absolute (relative) risk aversion parameter sufficiently large (for any given μ). □

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Table 1: Threshold values for relative risk aversion σ for varying exclusion duration $1/\mu$. The symmetric first best allocation is an equilibrium in the limit $\beta \to 1$ only when σ is above the threshold.

$1/\mu$	1	2	3	4	5	10	20	30
σ	19.4	13.2	10.0	8.0	6.7	3.6	1.9	1.3

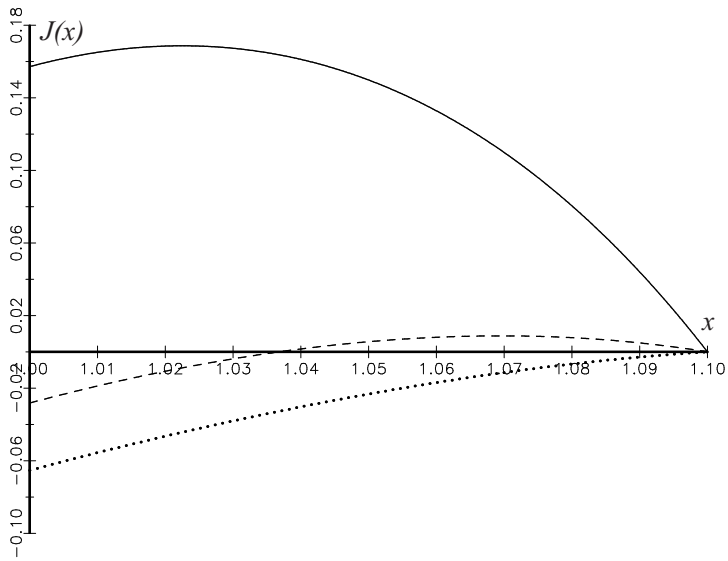

Figure 1: The curve $J(x)$ defining stationary equilibrium when $\mu = 0$, $u(c) = c^{1-\sigma}/(1-\sigma)$, $\sigma = 1.5$, $\lambda = 1.1$, $\pi_L = 0.25$, $\pi_H = 0.75$ and for three different discount factors. Autarky ($x = 1.1$) is always an equilibrium and it is the unique one if $\beta = .5$ (dotted curve). For $\beta = .7$ (dashed) there is a trading equilibrium at $x \approx 1.037$, and for $\beta = .9$ (solid) the trading equilibrium is at $x = 1$ (first best).

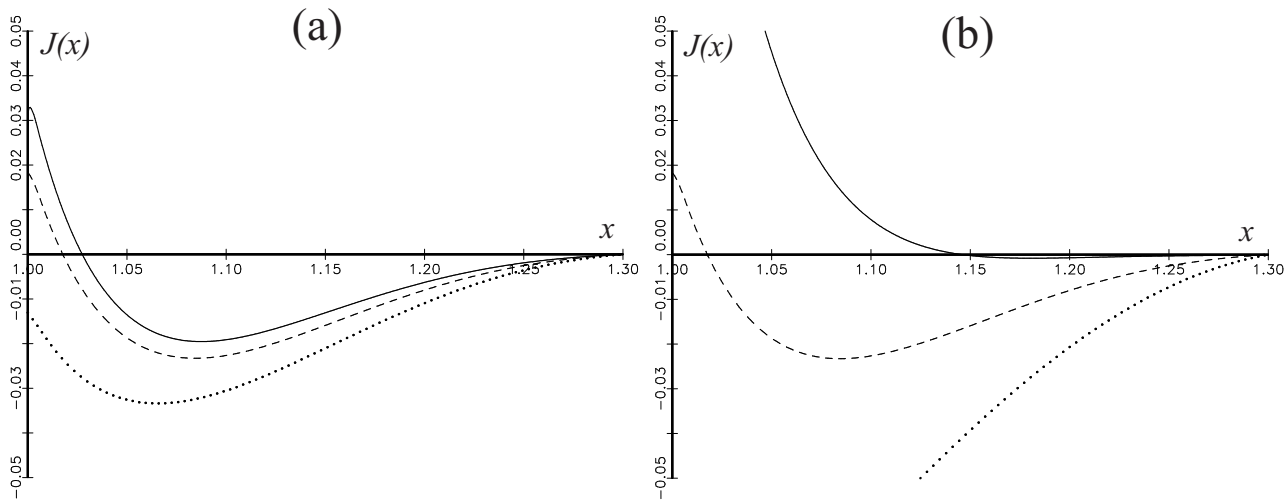

Figure 2: The curve $J(x)$ defining stationary equilibrium when $\mu = 1$ for the same economy as in Figure 1 with $\lambda = 1.3$. Figure (a) has $\sigma = 1.5$ and $\beta = 0.5$ (dotted), $\beta = 0.7$ (dashed), $\beta = 0.9$ (solid). Figure (b) has $\beta = 0.7$ and $\sigma = 1.1$ (dotted), $\sigma = 1.5$ (dashed), $\sigma = 1.9$ (solid).