

Constrained Efficiency with Search and Information Frictions

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Abstract

I characterize the constrained efficient or planner's allocation in a directed (competitive) search model with adverse selection. In this economy, buyers post contracts and sellers with private information observe all postings and direct their search toward their preferred contract. Then buyers and sellers match bilaterally and trade. I define a planner whose objective is to maximize social welfare subject to the information and matching frictions of the environment. I show in my main result that if the market economy fails to achieve the first best, then the planner, using a direct mechanism, achieves strictly higher welfare than the market economy. I also derive conditions under which the planner achieves the first best. I show that the planner can implement the direct mechanism by imposing submarket-specific taxes and subsidies on buyers conditional on trade (sales tax).

In an asset market application, I show that in general the efficient sales tax schedule is non-monotone in the price of assets. This non-monotonicity makes the implementation of the direct mechanism difficult in practice. I show that if in addition to sales tax the planner can use entry tax, submarket-specific taxes and subsidies imposed on buyers conditional on entry to each submarket whether they find a match or not, then the planner can implement the direct mechanism by using monotone tax schedules, increasing sales tax and decreasing entry tax.

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1 Introduction

There are search frictions and private information in asset, labor, housing and other markets. For example consider markets for assets which are traded over the counter (OTC) like mortgage-backed securities, structured credit products and corporate bonds. It is natural to think that sellers in these markets have private information about the value of their assets. Also, they must incur search costs to find buyers for their assets.

Specially after 2008, there has been a lot of discussion about the role of private information in causing the financial crisis and consequently, many policy questions have arisen. One of these questions is whether subsidizing asset purchases is a “good” policy or not from a social point of view. No paper has studied socially efficient policies in this context, although some papers like [Chang \(2012\)](#), [Guerrieri and Shimer \(2014\)](#) and [Chiu and Koepl \(2011\)](#) have studied positive implications of those policies. In particular, [Chang \(2012\)](#) shows that if there are fire sales in the asset market, subsidizing the purchase of low price assets increases the liquidity of all assets in the market. In an application of my model, I contribute in this literature by studying the socially efficient policy in an environment similar to [Chang \(2012\)](#). I characterize the optimal taxation policy in the asset market and show that in general the optimal tax schedule is non-monotone in the price of assets. In particular, I show if there are fire sales in the asset market, then taxing high price assets and subsidizing low price ones is not optimal.

From a theoretical point of view, this paper studies the constrained efficient allocation in economies with directed (competitive) search and adverse selection. My environment is the same as one in [Guerrieri, Shimer and Wright \(2010\)](#) (GSW henceforth). They define and characterize equilibrium and show its existence and uniqueness. I define and characterize the planner’s problem for that environment and show in my main result that if the equilibrium fails to achieve the first best¹, then the equilibrium is generally constrained inefficient². That is, the planner can achieve strictly higher welfare than the equilibrium. I also derive sufficient conditions under which the planner can achieve the first best.

¹The first best allocation is the solution to the planner’s problem when the planner faces only search frictions, but he has complete information about the type of agents.

²In three examples, GSW introduce some pooling or semi-pooling allocations that Pareto dominate the equilibrium allocations. They do not characterize the constrained efficient allocation nor do they define it.

In this economy, there is a large number of homogeneous buyers on one side of the market whose population is endogenously determined through free entry. There is a fixed population of sellers on the other side of the market who have private information about their types. Buyers and sellers match bilaterally and trade in different locations, called submarkets. In each submarket, there are search frictions in the sense that buyers and sellers on both sides get matched generally with probability less than one.

In order to define the planner's problem for this environment, I take a mechanism design approach. The planner's objective is to maximize social welfare and he is subject to the same information and search frictions present in the market economy. That is, the planner cannot observe types of sellers and also cannot force sellers or buyers to participate in the mechanism that the planner designs. In the language of mechanism design, the planner faces incentive compatibility of sellers, participation constraints of sellers and buyers and his own budget-balance condition. That is, the net amount of transfers that the planner makes to agents must be non-positive.

To implement this mechanism, all the planner needs to do is to impose submarket-specific sales taxes and subsidies on buyers in each submarket conditional on trade. The timing of actions are otherwise the same: Having observed the schedule of sales tax, buyers first choose a submarket and then sellers observe all open submarkets (the submarkets that some buyers have selected) and choose where to go. Then buyers³ and sellers trade if they find a match³. Note that the set of open submarkets in the planner's implementation may be different than that in equilibrium. Also, the equilibrium allocation is a feasible allocation for the planner, because the revenue that the planner makes over *each* submarket is zero in the equilibrium allocation.

To understand how the planner can achieve strictly higher welfare than the market economy, I study some examples. In the first one in Section 4, I study an asset market with lemons. Sellers have one indivisible asset which is of two types: high and low. The high-type asset is more valuable to both buyers and sellers. GSW show that there exists a unique separating equilibrium in which different types trade in different submarkets. High-type sellers prefer the higher price submarket with lower probability of matching (submarket two), while low-type sellers are just indifferent between the two submarkets. Low-type sellers are willing to sacrifice price for probability of trade, because they do not want to get stuck with their "lemons". On the other hand, high-type sellers do not want to sacrifice price, because their assets are more valuable to them in case of being unmatched. Probability of matching in this example, in fact, is used as a screening device.

³It is discussed in the paper that if there are not positive gains from trade for some types, lump sum transfers to sellers may be also needed to implement the direct mechanism.

The planner can do better than equilibrium in the following way. Beginning from the equilibrium allocation which is feasible for the planner, the planner subsidizes sellers in submarket one (low-type sellers) by a small amount so that their incentive compatibility constraint for choosing submarket two becomes slack. Now more buyers enter submarket two to get matched with previously unmatched high-type sellers. Therefore, welfare increases due to the formation of new matches. To finance subsidies to the sellers in submarket one (low-type sellers), the planner taxes sellers in submarket two (high-type sellers). The planner keeps subsidizing low-type and taxing high-type sellers until he achieves the first best, in which high-type sellers also get matched with probability one, or participation constraint of high-type sellers binds. The same idea goes through even if there are more than two types.

To understand the nature of inefficiency in the market economy, consider the externalities implied by having one more buyer in a submarket. First, it decreases the probability that other buyers in that submarket are matched. Second, it increases the probability that other sellers are matched in that submarket. In the presence of complete information, it is well established in the literature that buyers entering the market in the directed search setting can internalize these externalities by choosing the “right” price (contract), if sellers’ types are observable and contractible and if buyers can commit to their postings⁴. However, the change in the payoff of sellers following the entry of one more buyer in a submarket has another effect in this environment which is absent in the complete information case. This change will affect the incentive compatibility constraints that buyers who want to attract other types of sellers face, thus affecting the set of feasible submarkets that other buyers can enter to attract those sellers.

Buyers in the market economy do not take into account the effect of their entry on the contracts posted on other submarkets and consequently on the payoff of sellers in other submarkets. The planner internalizes these externalities by imposing appropriate taxes on agents and therefore can do better than the market economy. The extent to which the planner can improve efficiency depends on the details of the environment. In my second result, I derive sufficient conditions under which the planner can eliminate distortions completely to achieve the first best.

In the second example in Section 5, I characterize the constrained efficient allocation

⁴ The efficiency of competitive search equilibrium in the presence of complete information is probably the most important result in this literature. In the random search setting, in contrast, the equilibrium is generally inefficient because the entrants generally fail to internalize the aforementioned externalities. See the following papers for directed search models and their efficiency properties: [Acemoglu and Shimer \(1999\)](#); [Eeckhout and Kircher \(2010\)](#); [Moen \(1997\)](#); [Mortensen and Wright \(2002\)](#); [Peters \(1991\)](#); [Shi \(2001, 2002\)](#); [Shimer \(2005\)](#). See [Mortensen and Pissarides \(1994\)](#) for a random search model.

in a version of the rat race (originally studied by [Akerlof \(1976\)](#)) and compare my results with GSW who solve for the equilibrium allocation in this environment⁵. There are two types of workers. High-type workers incur less cost for working longer hours and generate higher output compared to low-type workers. Also the marginal output with respect to hours of work that high-type workers generate is higher. In equilibrium, high-type workers work inefficiently for longer hours than they would work under complete information and get matched with inefficiently higher probability. The planner, in contrast to the market economy, achieves the first best. He pays low-type workers higher wages and high-type workers lower wages than what they would get under complete information. These subsidies (to low-type workers) and taxes (on high-type workers) are needed to ensure that low-type workers do not have any incentive to apply to the submarket that high-type workers apply to. Moreover, if the share of high-type workers in the population is sufficiently high, the planner's allocation even Pareto dominates the equilibrium allocation.

In the asset market example explained above, the trades which involve high-type (or equivalently high price) assets are taxed and other trades are subsidized. An interesting question is whether this observation can be generalized to more realistic environments or not. To answer this question, I extend the two-type asset market to a continuous type one, which is a static version of [Chang \(2012\)](#), and derive sufficient conditions under which the planner can achieve the first best. The optimal submarket-specific sales tax that implements the optimal mechanism is not generally monotone in the price of assets. This feature makes it hard for the planner to implement this mechanism in the real world, partly because implementing a non-monotone tax schedule requires the planner to have precise information about the details of the the economy, but this requirement is unlikely to be met in the real world applications. For example, with a non-monotone tax schedule little mis-specification of the model by the planner can lead to significant losses in efficiency. Ideally, the tax schedule should be independent of the details of the economy.

In the next step, I show that imposing two types of taxes, not only sales tax but also submarket-specific entry tax, which is imposed on buyers conditional on entry to each submarket regardless of whether they find a match or not, solves the non-monotonicity problem. That is, the planner can always design monotone tax schedules, decreasing entry tax and increasing sales tax, to implement the direct mechanism.

Related Literature. [Guerrieri \(2008\)](#) and [Moen and Rosén \(2011\)](#) study constrained efficient allocation in environments with directed search and private information. [Guerrieri \(2008\)](#) shows that the competitive search equilibrium is constrained inefficient in a dynamic

⁵My paper is also related to the classic adverse selection models like [Akerlof \(1970\)](#) and [Rothschild and Stiglitz \(1976\)](#).

setting, if the economy is not on the steady state path. However in both papers, the agents who search (workers) do not have ex-ante private information. After they get matched with firms, they learn their types which become their private information.

[Goloso et al. \(2013\)](#) studies a model with directed search and private information and show that the equilibrium is constrained inefficient. There are two important differences between their paper and mine. First, the information friction in their paper is moral hazard, because the public cannot observe whether the workers have searched or not and if so, toward which type of firms. In contrast, the information friction in my paper is adverse selection. Second, workers are risk averse in their paper, in contrast to sellers in my paper who are risk neutral. The inefficiency result in their paper relies on the risk aversion assumption. Therefore, the channels through which inefficiency arises in the two papers are different.

[Delacroix and Shi \(2013\)](#) study a model in which sellers with private information post contracts, in contrast to GSW in which the uninformed side of the market posts contracts. They investigate the potentially conflicting roles of prices: the signaling role and the search directing role. Aside from some details⁶, the notion of constrained efficiency defined in this paper and the ideas behind that (that the planner can make transfers across agents or equivalently across submarkets) apply to their model as well, because the environments are similar, although they have a different trading mechanism.

The paper is organized as follows. In Section 2, I develop the environment of the model and define the planner's problem. In Section 3, I characterize the planner's allocation and state my main results. In Section 4, I study a two-type asset market example, characterize the planner's allocation and compare it with the equilibrium allocation. I also explain the nature of inefficiency in the market economy and discuss why and how the planner can allocate resources more efficiently than the market economy. In Section 5, I study a version of the rat race. In Section 6, I study an asset market with a continuous type space to characterize the efficient tax schedule. Section 7 concludes. All proofs appear in the appendix.

2 The Model

2.1 Environment

Consider an economy with two types of agents, buyers and sellers and $n + 1$ goods where $n \in \mathbb{N}$. Goods $1, 2, \dots, n$ are produced by sellers and consumed by buyers, while good $n + 1$ is a

⁶For example in their model, sellers choose the quality of their products. The quality, then, becomes their private information.

numeraire good and is produced and consumed by everyone. Let $a \equiv (a^1, a^2, \dots, a^n) \in \mathbb{A} \subset \mathbb{R}^n$ be a vector where \mathbb{A} is compact, convex and non-empty. Component k of this vector, a^k , denotes the quantity of good k . For example in a labor market, a can be a positive real number denoting the hours of work. When I say an agent produces (or consumes) a , I mean that the agent produces (or consumes) a^1 units of good 1, a^2 units of good 2 and so on.

There is a measure 1 of sellers. A fraction $\pi_i > 0$ of sellers are of type $i \in \{1, 2, \dots, I\}$. Type is seller's private information. On the other side of the market, there is a large continuum of homogenous buyers who can enter the market by incurring cost $k > 0$. After buyers enter the market, buyers and sellers are allocated to different submarkets (described below). Matching is bilateral. After they match, they trade.

There are search frictions in this environment. By search frictions I mean that sellers generally get to match with the buyers they have chosen with probability less than one. Matching occurs in submarkets which are simply some locations for trades. Matching technology determines the probability that sellers and buyers in each submarket get matched. If the ratio of buyers to sellers in one submarket is $\theta \in [0, \infty]$, then the buyers are matched with probability $q(\theta)$. Symmetrically, matching probability for sellers is $m(\theta) \equiv \theta q(\theta)$. As is standard in the literature, I assume that m is non-decreasing and q is non-increasing. Both $m(\cdot)$ and $q(\cdot)$ are continuous.

Sellers' and buyers' payoff functions are quasi-linear in the numeraire good⁷. The payoff of a buyer who enters the market from consuming a and producing $t \in \mathbb{R}$ units of the numeraire good is $v_i(a) - t - k$ if matched with a type i seller and is $-k$ if unmatched. The payoff of a type i seller from producing a and consuming $t \in \mathbb{R}$ units of the numeraire good is $u_i(a) + t$ if matched with a buyer and is 0 otherwise.

2.2 Equilibrium Definition

First, let me briefly describe how the market economy works, the especial case in which the planner does nothing. Then I describe the planner's problem. The definition of equilibrium is taken completely from GSW.

Submarkets in the market economy are characterized by $y \equiv (a, p)$ where $a \in \mathbb{A}$ denotes the vector of goods 1 to n to be produced by sellers in this submarket and $p \in \mathbb{R}$ is the amount of the numeraire good to be transferred from buyers to sellers. No submarket which

⁷The difference between payoff functions in this paper and in GSW is that I assume quasi-linear preferences, while they do not make this assumption. The reason that I impose quasi-linearity assumption is that I want to do welfare analysis and I want to use taxes and subsidies. If the preferences are not quasi-linear, the weight that the planner assigns to buyers and sellers might become important.

would deliver buyers a strictly positive payoff is inactive in the equilibrium. If there was such a submarket, some buyers would have already entered that submarket to exploit that opportunity. On the other side of the market, sellers observe all (a, p) pairs posted in the market, anticipate the market tightness at each submarket and then direct their search toward one which delivers them the highest expected payoff.

Let $\gamma_i(y)$ denote the share of sellers that are type i in the submarket denoted by y , with $\Gamma(y) \equiv \{\gamma_1(y), \dots, \gamma_i(y), \dots, \gamma_I(y)\} \in \Delta^I$ where Δ^I is an I -dimensional simplex, that is, $0 \leq \gamma_i(y) \leq 1$ for all y and $\sum_{i=1}^I \gamma_i(y) = 1$. To make the notation clear for the rest of the paper, the first component of y is denoted by a , rather than y_1 and the second component is denoted by p rather than y_2 . Similarly if the submarket is denoted by y' , the first and second components of y' are denoted by a' and p' .

Definition 1. An equilibrium, $\{Y, \lambda, \theta, \Gamma\}$, is a measure λ on $Y \equiv \mathbb{A} \times \mathbb{R}$ with support Y^P , a function $\theta : Y \rightarrow [0, \infty]$, and a function $\Gamma : Y \rightarrow \Delta^I$ which satisfies the following conditions:

(i) **Buyers' profit maximization and free entry**

For any $y \in Y$,

$$q(\theta(y)) \sum_i \gamma_i(y) (v_i(a) - p) \leq k,$$

with equality if $y \in Y^P$.

(ii) **Sellers' optimal search**

Let $U_i = \max \left\{ 0, \max_{y' \in Y^P} \{m(\theta(y'))(u_i(a') + p')\} \right\}$ and $U_i = 0$ if $Y^P = \emptyset$. Then for any $y \in Y$ and i , $U_i \geq m(\theta(y))(u_i(a) + p)$ with equality if $\theta(y) < \infty$ and $\gamma_i(y) > 0$. Moreover, if $u_i(a) + p < 0$, either $\theta(y) = \infty$ or $\gamma_i(y) > 0$.

(iii) **Market clearing**

For all i , $\int_Y \frac{\gamma_i(y)}{\theta(y)} d\lambda(\{y\}) \leq \pi_i$, with equality if $U_i > 0$.

Let me make a couple of brief comments about the equilibrium definition. For further details, see GSW. Equilibrium condition (i) states that buyers should not earn a strictly positive profit from entering any submarket (on- or off-the-equilibrium-path). That is, there are no opportunities for trade unexploited in the equilibrium. If buyers' expected payoff in one submarket is strictly negative, no buyer enters that submarket. If that is strictly positive, more buyers will enter that submarket and the market tightness will be changed. Therefore, for all markets that the planner wants to be open, buyers must get exactly 0 expected payoff. A buyer has to incur entry cost k if he wants to enter submarket y . Then, he gets matched with a type i seller with probability $\gamma_i(y)$ from which he gets a payoff of $v_i(a)$ in terms of the numeraire good, and pays p units of the numeraire good to the seller.

Equilibrium condition (ii) is composed of two parts. The first part states that among all submarkets in the equilibrium, $y \in Y^P$, sellers choose to go to a submarket which maximizes their payoff. The second part imposes some restrictions on beliefs regarding the market tightness and composition of types for off-the-equilibrium-path, $y \notin Y^P$. The market tightness for off-the-equilibrium-path is set such that sellers who choose to go to those posts do not gain by doing so relative to their equilibrium payoff. Also, this restriction with respect to the composition of types states that if buyers believe that some types would apply to an off-the-equilibrium-path post, then those types should be exactly indifferent between the payoff they get from that post relative to their equilibrium payoff. Equilibrium condition (iii) is straight forward.

2.3 The Planner's Problem

I define a planner whose objective is to maximize the weighted average of the payoff to sellers⁸. The planner faces the same information and search frictions present in the market economy. The planner uses a direct mechanism to allocate resources. In the direct mechanism and thanks to the revelation principle, sellers report their types to the planner and then the planner allocates them to a 4-tuple $(\tilde{a}_i, \tilde{p}_i, \tilde{s}_i, \tilde{\theta}_i)$. Here, \tilde{a}_i is the vector of production of goods 1 to n to be produced by sellers who report type i , \tilde{p}_i is the amount of the numeraire good transferred to them conditional on finding a match, \tilde{s}_i is the amount of the numeraire good transferred to them unconditionally and $\tilde{\theta}_i$ is the average number of buyers assigned to them. The planner maximizes his objective function subject to incentive compatibility of sellers, participation constraint of sellers and his budget-balance condition.

Definition 2. *A feasible mechanism is a set $\{(\tilde{a}_i, \tilde{p}_i, \tilde{s}_i, \tilde{\theta}_i)\}_{i \in \{1, 2, \dots, I\}}$ such that the following conditions hold:*

(1) ***Incentive Compatibility of Sellers***

For all i and j ,

$$U_i \equiv m(\tilde{\theta}_i)(u_i(\tilde{a}_i) + \tilde{p}_i) + \tilde{s}_i \geq m(\tilde{\theta}_j)(u_i(\tilde{a}_j) + \tilde{p}_j) + \tilde{s}_j.$$

(2) ***Participation Constraint of Sellers***

For all i ,

$$U_i \geq 0.$$

(3) ***Planner's Budget-Balance***

$$\sum_{i=1}^I \pi_i [m(\tilde{\theta}_i)(v_i(\tilde{a}_i) - \tilde{p}_i) - k\tilde{\theta}_i - \tilde{s}_i] \geq 0.$$

⁸Note that buyers get payoff 0 either in the market economy or under the planner's allocation.

Two points are worth mentioning about this definition. The first one is that the participation constraint or individual rationality of buyers here is taken implicitly into account by condition (3). Consider the following scenario. The planner charges buyers some participation fee. Once a buyer agrees to participate, the planner assigns the buyer to get matched with one type of sellers according to a uniform distribution. There are $\pi_i \tilde{\theta}_i$ number of buyers who are assigned to type i sellers and overall there are $\sum \pi_j \tilde{\theta}_j$ buyers who participate. Therefore, the probability that a buyer is assigned to type i sellers is $\frac{\pi_i \tilde{\theta}_i}{\sum \pi_j \tilde{\theta}_j}$. Therefore, the expected benefit of the buyer from entering the market and getting matched with type i is $\frac{\pi_i \tilde{\theta}_i}{\sum \pi_j \tilde{\theta}_j} (q(\tilde{\theta}_i) v_i(\tilde{a}_i) - k)$. On the other hand, each type j seller needs to get paid \tilde{p}_j units of the numeraire good conditional on matching and \tilde{s}_j units unconditionally. Therefore, overall $\sum \pi_j (m(\tilde{\theta}_j) \tilde{p}_j + \tilde{s}_j)$ amount of the numeraire good is needed to compensate sellers. Since the planner does not spend any resources from his own pocket, each participating buyer should pay $\frac{\sum \pi_j (m(\tilde{\theta}_j) \tilde{p}_j + \tilde{s}_j)}{\sum \pi_j \tilde{\theta}_j}$. In order for buyers to participate in the direct mechanism, the benefit that each buyer gets ex-ante, $\frac{\pi_i \tilde{\theta}_i}{\sum \pi_j \tilde{\theta}_j} (q(\tilde{\theta}_i) v_i(\tilde{a}_i) - k)$, should weakly exceed the amount of the numeraire good that the buyer should pay, $\frac{\sum \pi_j (m(\tilde{\theta}_j) \tilde{p}_j + \tilde{s}_j)}{\sum \pi_j \tilde{\theta}_j}$. Condition (3) in the definition summarizes this requirement.

The second point is that in this definition, I did not allow the planner to use lotteries. By lotteries I mean that after agents report their types, the planner allocates, say, type i sellers to different 4-tuples, $(\tilde{a}, \tilde{p}, \tilde{s}, \tilde{\theta})$ and $(\tilde{a}', \tilde{p}', \tilde{s}', \tilde{\theta}')$, with positive probability where these 4-tuples may deliver type i sellers different payoffs. If the planner can use lotteries, then the planner may be able to achieve even *higher* welfare than what he achieves in the constrained efficient allocation here⁹, because he would have one more tool¹⁰.

Definition 3. *A constraint efficient mechanism is a feasible mechanism which maximizes the planner's objective function. That is, the planner solves the following problem:*

$$\max_{\{(\tilde{a}_i, \tilde{p}_i, \tilde{s}_i, \tilde{\theta}_i)\}_{i \in \{1, 2, \dots, I\}}}} \sum_i \pi_i U_i$$

s. t. $\{(\tilde{a}_i, \tilde{p}_i, \tilde{s}_i, \tilde{\theta}_i)\}_{i \in \{1, 2, \dots, I\}}$ is a feasible mechanism.

⁹ The lotteries may help the planner to achieve higher welfare if the objective function of the planner is not concave or if the constraint set is not convex.

¹⁰To elaborate, in my first result, I show that if the equilibrium does not achieve the first best, then the planner achieves strictly higher welfare than the equilibrium without using any lotteries. Adding another tool can only make this result stronger. In my second result, I derive conditions under which the planner achieves the first best without using any lotteries. Since the planner achieves the first best, adding another tool does not change this result.

As far as the notation is concerned, whenever a variable has $\tilde{\cdot}$ in the paper, it shows that the variable is an element of a direct mechanism. CE represents the constrained efficient allocation, FB represents the first best allocation and EQ represents the equilibrium allocation.

2.4 Implementation

To implement the direct mechanism, the planner is assumed to have the power to impose taxes and subsidies on agents. It turns out that imposing two types of taxes are sufficient for the planner to implement the direct mechanism discussed above. First, the planner chooses a tax amount for each submarket. This tax will be imposed on buyers conditional on trade, $t(a, p) : \mathbb{A} \times \mathbb{R} \rightarrow \mathbb{R}$. The results will not change if, instead, taxes are imposed on sellers. Second, the planner can make lump sum transfers, $T \in \mathbb{R}_+$ units of the numeraire good¹¹, to sellers¹². Note that any post in the market economy is a special case of this description with $t(y) = 0$ for all submarkets and $T = 0$.

The planner may want agents not to trade in some submarkets, despite the fact that agents in the market economy want to trade in those submarkets. In such a case, the planner can impose sufficiently high amount of tax on trade in those submarkets. Aside from the ability to make these transfers, the market economy and the planner face the same restrictions: Amount of goods to be produced by sellers or payments to be made by buyers cannot be conditioned on the type of sellers. The ex-ante payoff of buyers in both cases should be 0 to ensure that buyers want to participate and also to ensure that there is no excess entry into any submarket. Also in both cases sellers choose submarkets which maximize their expected payoff or stay out. Some sellers' payoffs from entering any open submarket, the submarkets that some buyers choose to go, is non-positive, so they will not apply to any submarket. I call these sellers non-participants. They just receive T .

The planner faces a budget constraint (or a budget-balance condition as called in the mechanism design literature) similar to that in the direct mechanism. This condition states that the net amount of transfers that the planner makes to agents should not exceed 0. Notice that in the market economy, it is not possible to transfer funds (the numeraire good)

¹¹ Without loss of generality, I assume that T must be positive. If T is negative and some types do not participate, i.e. do not apply to any submarket, then sellers' participation constraint is violated. If all types participate, then one can easily incorporate T into prices, that is, one can change p_i to $p_i + \frac{T}{m(\theta_i)}$, to replace negative T by 0. Therefore, it is without loss of generality to assume that T is positive.

¹²It is possible that some types get a negative payoff from active sub-markets, so they prefer not to apply to any submarket. However they distort the allocation for other types via the incentive compatibility constraint. The planner is allowed to pay them T to reduce this distortion.

from one submarket to another. That is, all the surplus generated in any submarket belongs to sellers in that submarket. Under the planner's allocation, on the other hand, sellers might get a higher or lower payoff than the surplus they generate. In short, cross-subsidization across submarkets is possible.

As defined earlier, let $y \equiv (a, p)$ denote a submarket. An allocation $\{\lambda, Y^P, \theta, \Gamma, t, T\}$ is a distribution λ over Y with support Y^P (so Y^P is the set of open submarkets), the ratio of buyers to sellers for each submarket $\theta : Y \rightarrow [0, \infty]$, the distribution of types in each submarket $\Gamma : Y \rightarrow \Delta^I$, the amount of tax (in terms of the numeraire good) to be imposed on buyers at each submarket conditional on trade, $t : Y \rightarrow \mathbb{R}$, and finally the amount of the numeraire good to be transferred to sellers in a lump sum way, $T \in \mathbb{R}_+$. Because the planner faces some constraints, only some allocations are implementable for the planner. The definition of an implementable allocation is given below.

Definition 4. *A planner's allocation $\{\lambda, Y^P, \theta, \Gamma, t, T\}$ is implementable if it satisfies the following conditions:*

(i) **Buyers' maximization and free entry**

For any $y \in Y$,

$$q(\theta(y)) \sum_i \gamma_i(y)(v_i(a) - p - t(y)) \leq k,$$

with equality if $y \in Y^P$.

(ii) **Sellers' maximization**

Let $U_i \equiv \max\{0, \max_{y' \in Y^P} \{m(\theta(y'))(u_i(a') + p')\}\} + T$ and $U_i = T$ if $Y^P = \emptyset$. For any $y \in Y$ and i ,

$$m(\theta(y))(u_i(a) + p) + T \leq U_i,$$

with equality if $\gamma_i(y) > 0$ and $\theta(y) < \infty$. If $u_i(a) + p < 0$, then $\theta(y) = \infty$ or $\gamma_i(y) = 0$.

(iii) **Feasibility or market clearing**

For all i , $\int_{Y^P} \frac{\gamma_i(y)}{\theta(y)} d\lambda(\{y\}) \leq \pi_i$, with equality if $U_i > T$.

(iv) **Planner's budget constraint**

$$\int_{Y^P} q(\theta(y))t(y)d\lambda(\{y\}) \geq T.$$

The definition of implementable allocation is similar to the definition of equilibrium. Regarding condition (i), when buyers want to choose a submarket, they form beliefs regarding market tightness and composition of types at each submarket. The restriction on these beliefs are also exactly the same as those in the definition of equilibrium. Note that here buyers not only need to make payment to sellers but also to the planner. Conditions (ii) and (iii)

are exactly the same as their counterparts in the equilibrium definition. Condition (iv), the budget-balance condition, is self-explanatory.

Condition (ii) summarizes two constraints, sellers' participation constraint and sellers' incentive compatibility constraint. To make the exposition easier, for any given allocation, define X_i as follows: $X_i \equiv \{(\theta(y), a) \mid y \equiv (a, p) \in Y^P, \gamma_i(y) > 0\}$. Denote elements of X_i by (θ_i, a_i) . In words, θ_i is the market tightness of a submarket to which type i applies with strictly positive probability and a_i is the production level of that submarket. Sellers' maximization constraint implies that for any i, j , $(\theta_i, a_i) \in X_i$ and $(\theta_j, a_j) \in X_j$:

$$m(\theta_i)(u_i(a_i) + p_i) \geq m(\theta_j)(u_i(a_j) + p_j) \text{ (IC)}.$$

I call this constraint IC or incentive compatibility constraint. This constraint is equivalent to condition (1) in the definition of feasible mechanism.

Definition 5. *A constrained efficient allocation is an implementable allocation which maximizes welfare among all implementable allocations. That is, a constrained efficient allocation solves the following problem:*

$$\max_{\{\lambda, Y^P, \theta, \Gamma, t, T\}} \sum_i \pi_i U_i$$

s. t. $\{\lambda, Y^P, \theta, \Gamma, t, T\}$ is implementable.

I show in Lemma 1 that the way I define the constrained efficient allocation here is without loss of generality. That is, a planner who uses a direct mechanism as defined earlier achieves the same welfare level as the planner in Definition 5.

Lemma 1. *Given any feasible mechanism, there is an associated implementable allocation under which all types get exactly the same payoff as in the direct mechanism.*

Since implementation of a direct mechanism requires a large amount of communication, which is unrealistic in many economic applications, a proper implementation should be close to the real world applications as much as possible. The way that I have formulated the implementation of the direct mechanism here has this feature, because the elements of the implementable allocation have natural interpretations. For example, t can be interpreted as submarket-specific sales tax. Lemma 1 guarantees that all technical results derived by utilizing the direct mechanisms can be naturally implemented in the real world applications.

Given any equilibrium $\{\lambda, Y, \theta, \Gamma\}$, I construct an allocation called **equilibrium allocation** $\{\lambda^{EQ}, Y^{EQ}, \theta^{EQ}, \Gamma^{EQ}, t^{EQ}, T^{EQ}\}$ where $\lambda^{EQ} = \lambda$, $Y^{EQ} = Y$, $\theta^{EQ}(y) = \theta(y)$, $\Gamma^{EQ}(y) = \Gamma(y)$, $t^{EQ}(y) = 0$ for all y and $T^{EQ} = 0$. The only difference is that I added zero taxes to the definition of equilibrium. The equilibrium allocation is implementable,

because sellers' maximization condition and buyers' profit maximization and zero profit condition are satisfied following their counterparts in the definition of equilibrium. The planner's budget constraint is also trivially satisfied, because $t^{EQ} = 0$ for all $y \in Y^{EQ}$ and $T^{EQ} = 0$. When I say equilibrium allocation, I mean the implementable allocation which is constructed from equilibrium objects as above.

Finally, when I refer to equilibrium in the paper, I mean the notion of equilibrium which was discussed above where the uninformed side of the market posts contracts. I do *not* mean a notion of equilibrium with signaling in which the informed side of the market posts contracts, unless otherwise noted¹³.

3 Characterization

I first study the complete information case as a benchmark and then present my main results.

3.1 Complete Information Allocation or First Best

As a benchmark, consider an otherwise the same environment as introduced above except that the type of sellers is common knowledge. Since buyers have complete information about the type of sellers, the submarkets in the market economy are not only indexed by the level of production and price but also by the type of sellers that the buyers want to meet. The buyers, who contemplate what submarket to enter to attract type i sellers, enter a submarket which maximizes the payoff of type i subject to the free entry condition. (See Moen (1997) for further explanation.) If there is any submarket that would deliver type i sellers a higher payoff, some buyers would enter that submarket and then, sellers would strictly prefer that submarket. Therefore, buyers who attract type i solve the following problem in the market economy with complete information:

$$\begin{aligned} & \max_{\theta, a, p} \{m(\theta)(p + u_i(a))\} \\ & \text{s.t. } q(\theta)(v_i(a) - p) \geq k. \end{aligned}$$

Denote the solution to this problem by $(\theta_i^{FB}, a_i^{FB}, p_i^{FB})$ ¹⁴. It is easy to see that the constraint of the problem must hold with equality, so after eliminating p from the maximization

¹³ I conjecture that my first result regarding the inefficiency of equilibrium will hold even if another notion of equilibrium is considered in which the informed side of the market posts. Of course, one needs to impose some reasonable restrictions on off-the-equilibrium-path beliefs similar to those proposed by Cho and Kreps (1987).

¹⁴All that matters for the first best allocation is the level of production and probability of matching. Since transfers is not part of the first best allocation, having superscript of FB for price in p_i^{FB} is somewhat

problem, one can write the payoff of type i sellers from participating in the market in the complete information case as $\max_{\theta,a}\{m(\theta)(v_i(a) + u_i(a)) - k\theta\}$. Let U_i^{FB} be the payoff of type i in the complete information case. Then U_i^{FB} is calculated as follows:

$$U_i^{FB} = \max_{\theta,a}\{m(\theta)(v_i(a) + u_i(a)) - k\theta\}.$$

Notice that the objective function, $m(\theta)(v_i(a) + u_i(a)) - k\theta$, is exactly equal to the surplus created by a type i seller. Thus, the planner who observes types of sellers solves exactly the same problem as buyers in the market economy with complete information¹⁵. If $U_i^{FB} \geq 0$, the planner wants type i to get matched with probability $m(\theta_i^{FB})$ and to produce a_i^{FB} . (If $U_i^{FB} < 0$, then type i sellers do not participate in the market. The planner does not want them to participate, either.) In this paper, when I say that the planner achieves the complete information allocation or *achieves the first best*, I mean that there exists an implementable allocation in which type i sellers get matched with probability $m(\theta_i^{FB})$ and produce a_i^{FB} for all i .

3.2 Results

As already seen, the equilibrium allocation is feasible for the planner. It is immediately followed that the planner can achieve the level of welfare which is at least as much as that in the market economy. Theorem 1 states that the planner can achieve strictly higher welfare.

Let $\bar{Y} \equiv \cup_i \bar{Y}_i$ where

$$\bar{Y}_i \equiv \{(a, p) \mid (a, p) \in \mathbb{A} \times \mathbb{R}, q(0)(v_i(a) - p) \geq k, \text{ and } u_i(a) + p \geq 0\}.$$

If $(a, p) \notin \bar{Y}$, then no type will be attracted to this submarket in the market economy. Also for the future reference, let \bar{A} be defined as follows:

$$\bar{A} \equiv \{a \mid (a, p) \in \bar{Y} \text{ for some } p \in \mathbb{R}\}.$$

misleading. More precisely, p_i^{FB} is the payment that buyers make to sellers in the market with complete information. I do not want to introduce a new notation for the market with complete information, so I keep p_i with superscript of FB throughout the paper to refer to the payment that buyers make to type i sellers in the market with complete information.

¹⁵This is the core of the argument in the literature which states that the market economy decentralizes the planner's allocation under complete information. As already stated, there are many papers in the literature with different environments but with this common theme that when agents on one side of the market compete with each other in posting contracts and commit to them, then the market decentralizes the planner's allocation, if the contract space is rich enough. See Moen (1997), Acemoglu and Shimer (1999), Shi (2001), Shi (2002), Shimer (2005), Kircher (2009) and Eeckhout and Kircher (2010). If the contract space is not rich enough, the equilibrium might be constrained inefficient, like Galenianos and Kircher (2009)

Assumption 1.

1. *Strict Monotonicity:* For all $a \in \bar{A}$, $v_1(a) < v_2(a) < \dots < v_I(a)$.
2. *Sorting:* For all i , $a \in \bar{A}$ and $\epsilon > 0$, there exists $a' \in B_\epsilon(a) \equiv \{a' \in \mathbb{A} \mid \|a - a'\|_2 < \epsilon\}$ such that

$$u_j(a') - u_j(a) < u_h(a') - u_h(a) \text{ for all } j \text{ and } h \text{ with } j < i \leq h.$$

3. *Technical assumption:* Function $m(\theta)(u_i(a) + v_i(a)) - k\theta$ has only one local maximum on its domain, $(\theta, a) \in \mathbb{R}_+ \times \mathbb{A}$.

Theorem 1 (Result 1). *Suppose Assumption 1 holds. Also assume that all types with positive gains from trade (all i with $U_i^{FB} > 0$) get a strictly positive payoff in the equilibrium. If the equilibrium fails to achieve the first best, then the planner achieves **strictly** higher welfare than the equilibrium.*

Some remarks about the assumptions are in order. A standard single crossing condition states that the indifference curves of different types must cross only once. The sorting assumption here (which is the same as in GSW) is in a sense a local crossing condition, because it allows a' to be greater than a for some a and less than a for other a . Moreover, it is in a sense stronger than single crossing condition, because it states that given any a , there *exists* an a' with such a property. The requirement that all types with positive gains from trade must be active in the equilibrium is satisfied if there are positive gains from trade for *all* types. In an example in Section 4, I will make it clear why this assumption is necessary for this result.

The idea of the proof is as follows. I begin from the equilibrium allocation, propose a direct mechanism which is basically a perturbation of the equilibrium allocation in a particular way and then show that the proposed allocation is feasible and achieves strictly higher welfare than the equilibrium allocation.

We need first to understand how the equilibrium is constructed. Under similar conditions (weak monotonicity and sorting), GSW prove that the equilibrium for type i is characterized by maximizing the payoff of type i , subject to the free entry condition and the incentive compatibility constraint of all lower types. That is, type $j < i$ should not get a higher payoff if he chooses the submarket that type i chooses. They prove that this equilibrium is unique in terms of payoffs.

Let $\{\lambda^{EQ}, Y^{EQ}, \theta^{EQ}, \Gamma^{EQ}, t^{EQ}, T^{EQ}\}$ denote the equilibrium allocation where $Y^{EQ} \equiv \{y_1^{EQ}, y_2^{EQ}, \dots, y_I^{EQ}\}$. Also let U_i^{EQ} denote the utility that type i gets in the equilibrium. In this explanation, assume that all types are active in the equilibrium, $U_i^{EQ} > 0$ (which is the case if there are positive gains from trade for all types). Since the equilibrium does not

achieve the first best, there exists a type, say type i , which creates the surplus that is less than the first best level. It implies that at least one IC constraint in the problem for type i is binding in the equilibrium. For example, suppose type j is indifferent between y_j^{EQ} and y_i^{EQ} with $j < i$.

The planner begins from a direct mechanism in which each type is allocated the same (a, p, θ) as in the equilibrium. Since all types are active in the equilibrium, I assume without loss of generality that the unconditional transfer, s , for all types is initially set equal to 0, that is, $\tilde{s}_l = 0$ for all l . In order to improve welfare, the planner subsidizes all types lower than i identically by a small amount, $\epsilon > 0$. That is, $\tilde{s}_h = \epsilon$ for all $h < i$. Now constraints of the maximization problem for type i become slack, so the planner can find another triple (a', p', θ') such that the surplus generated by type i increases. Therefore, the payoff of type i strictly increases.

Now consider type $i + 1$. The planner solves the maximization problem for type $i + 1$ again. That is, he maximizes the payoff of type $i + 1$ subject to the free entry condition and the incentive compatibility constraint of all lower types. Since all lower types including type i get a strictly higher payoff than the equilibrium allocation now, the maximization problem for type $i + 1$ is now less constrained, so the planner can achieve weakly higher welfare from type $i + 1$ as well. The planner keeps doing the same thing for all types above i and assigns them new (a, p, θ) triples. The welfare of the population has increased so far, because type i has generated strictly higher surplus and all types $i + 1$ to I have generated weakly higher surplus. To satisfy the budget-balance condition, the planner imposes an identical tax on all types so that IC constraints are not affected. Making transfers across agents does not change the welfare of the population, therefore, the welfare level now is strictly higher than that in the equilibrium.

In the next proposition, I provide sufficient conditions for the planner to achieve the first best. Before that, let me introduce some definitions. We say that $a' \geq a$ if $a'^k \geq a^k$ for all $k \in \{1, 2, \dots, n\}$, that is, if a' is greater than a component by component. Function $g : A \times \{1, 2, \dots, I\} \rightarrow \mathbb{R}$ has *increasing differences* in (a, i) if for $a' \geq a$, $g(a', i) - g(a, i)$ is weakly increasing in i . Function $g : A \times \{1, 2, \dots, I\} \rightarrow \mathbb{R}$ is *supermodular* in a if for all $a, b \in A$, $g(a, i) + g(b, i) \leq g(a \vee b, i) + g(a \wedge b, i)$.

Assumption 2. *The following conditions hold:*

1. *Monotonicity of u in i : $u_1(a) \leq u_2(a) \leq \dots \leq u_I(a)$ for all $a \in \bar{A}$.*
2. *u has increasing differences in (a, i) .*
3. *$u + v$ has increasing differences in (a, i) .*

4. Supermodularity of f in a where $f_i(a) \equiv u_i(a) + v_i(a)$ for all $a \in A$ and i .
5. Either (a) holds or (b) and (c) hold:

(a) Monotonicity of v in i : $v_1(a) \leq v_2(a) \leq \dots \leq v_I(a)$ for all $a \in \bar{A}$.

(b) Monotonicity of f in i : $f_1(a) \leq f_2(a) \leq \dots \leq f_I(a)$ for all $a \in \bar{A}$.

(c) Sufficient gains from trade for all types:

$$U_i^{FB} \geq m(\theta_{i-1}^{FB})(u_i(a_{i-1}^{FB}) - u_{i-1}(a_{i-1}^{FB})) \frac{\sum_{j=i}^I \pi_j}{\pi_i}, \text{ for } i > 1 \text{ and } U_1^{FB} \geq 0.$$

Theorem 2 (Result 2). *Under Assumption 2, the planner achieves the first best.*

Part 1 of Assumption 2 simply states that the payoff of higher types is higher than lower types for any given level of production. Part 2 is a standard increasing differences property¹⁶. If u is differentiable, this assumption implies that given a level of production, the marginal *payoff* of higher types with respect to the level of production of good $k \in \{1, 2, \dots, n\}$ is higher than that of lower types¹⁷. Similarly, part 3 states that the marginal *surplus* with respect to the level of production of good k that higher types create is higher than the associated marginal surplus that lower types create. Part 4 is a standard supermodularity condition which states that the marginal surplus created by type i with respect to the level of production of good k is increasing in the level of production of good l ($k \neq l$). In part 5, I require either of the two following conditions. For any given level of production, buyers weakly prefer higher types of sellers. If this assumption is not satisfied, I require that $u_i + v_i$ is increasing in i for any production level (in part 5(b)) and also $\frac{\sum_{j=i}^I \pi_j}{\pi_i}$ is less than some threshold for every $i > 1$.

The proof follows a guess-and-verify approach. I first guess that the planner can achieve the first best under the conditions in Assumption 2. Then I ensure that all conditions for feasibility are satisfied. See Figure 1 for the illustration of the proof.

The planner achieves the first best iff there exists a feasible mechanism in which type i sellers get matched with probability $m(\theta_i^{FB})$ and produce a_i^{FB} . I need to find a set of transfers which together with $(\theta_i^{FB}, a_i^{FB})$ satisfy IC constraints. To find such a set, I show that if part 1 of Assumption 2 holds and if transfers are such that local downward IC constraints are satisfied and are binding, then all IC constraints are satisfied. By local downward IC

¹⁶This property is equivalent to the single crossing condition (which is also called Spence-Mirrlees condition) for a broad class of functions. See [Milgrom and Shannon \(1994\)](#) for a full discussion about these properties and the relationship between them.

¹⁷I do not impose differentiability assumption, though.

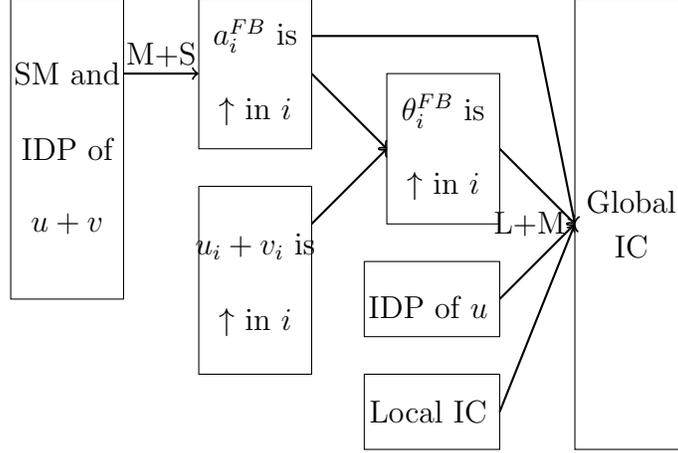


Figure 1: Schematic diagram indicating the proof steps of Theorem 2. SM and IDP refer to supermodularity and increasing differences property respectively. M+S refers to [Milgrom and Shannon \(1994\)](#), L+M refers to [Laffont and Martimort \(2009\)](#)

constraint I mean that type i should not gain by reporting type $i - 1$. Moreover, I show that by this construction method, the amount of transfers to the lowest type (p_1) determines the amount of transfers for all other types. A set of transfers that satisfies local downward IC constraints exists if $(\theta_i^{FB}, a_i^{FB})$ is increasing in i and also if u has increasing differences property in (a, i) (part 2 of Assumption 2). See Theorem 7.1 and 7.3 in [Fudenberg and Tirole \(1991\)](#) or Section 3.1 in [Laffont and Martimort \(2009\)](#) for reference.

According to Theorem 5 in [Milgrom and Shannon \(1994\)](#), if $u_i + v_i$ satisfies parts 3 and 4 of Assumption 2, then $a_i^{FB} \equiv \arg \max_{a \in \mathbb{A}} \{u_i(a) + v_i(a)\}$ is increasing in i , that is, $a_i^{FB} \geq a_{i-1}^{FB}$. If $u_i + v_i$ is increasing in i (part 1 and 5(a) or 5(b) of Assumption 2), then $m(\theta)(u_i(a_i^{FB}) + v_i(a_i^{FB})) - k\theta$ satisfies increasing differences property in (θ, i) . Also, $m(\theta)(u_i(a_i^{FB}) + v_i(a_i^{FB})) - k\theta$ is trivially supermodular in θ because θ is one-dimensional. Therefore, again according to Theorem 5 in [Milgrom and Shannon \(1994\)](#), $\arg \max_{\theta} \{m(\theta)(u_i(a_i^{FB}) + v_i(a_i^{FB})) - k\theta\}$ will be also increasing in i . Hence, $(\theta_i^{FB}, a_i^{FB})$ is increasing in i . Then the planner adjusts \tilde{p}_1 such that all types get a positive payoff. This implies that we can set $\tilde{s}_i = 0$ for all i . Given the transfer scheme (the prices to be paid to sellers), the planner ensures that the budget constraint holds with equality by making identical transfers to all types. In future sections, I will make it clear by a couple of examples the mechanism through which the planner can improve welfare relative to the market economy and how he might achieve the first best.

4 Example 1: Asset Market with Lemons

So far I have considered a general framework. In the following two sections, I study two examples from [Guerrieri et al. \(2010\)](#) and characterize the constrained efficient allocation for them and compare them with the associated equilibrium allocations. At the end of this section, I provide some intuition on how and why the planner can increase welfare by using appropriate transfers. Also, I explain the nature of inefficiency in the models of directed search with adverse selection.

The first example is an asset market with lemons (in the spirit of [Akerlof \(1970\)](#)). There are two types of assets, with value c_i to the seller and h_i to the buyer. Both c_i and h_i are in terms of a numeraire good. The payoff of a buyer matched with a type i seller is $\alpha h_i - t - k$ where α is the probability that the buyer gets the asset from the seller and t is the amount of the numeraire good that he pays (either to the planner or to sellers) in terms of the numeraire good. The payoff of a type i seller matched with a buyer is $-\alpha c_i + t$ where α is the probability that the seller gives the asset to the buyer and t is the amount of the numeraire good he consumes. The buyer's payoff is $-k$ if unmatched. As a special case of the original setting, here: $I = 2$, $n = 1$, $a \equiv \alpha$, $u_i(\alpha) = -\alpha c_i$ and $v_i(\alpha) = \alpha h_i$. The matching function is $m(\theta) = \min\{1, \theta\}$, that is, the short side of the market gets matched for sure. Following GSW, I also make the following assumptions:

Assumption 3. *In the asset market with lemons,*

1. $0 < h_1 < h_2$ and $0 < c_1 < c_2$.
2. For $i = 1, 2$, $c_i < b_i \equiv h_i - k$.

Proposition 1. *In the asset market with lemons, the planner achieves strictly higher welfare than the equilibrium. If $\pi_1 b_1 + \pi_2 b_2 \geq c_2$, then the planner achieves the first best. See full characterization of the constraint efficient allocation in [Table 1](#).*

The first part of this proposition is a special case of [Theorem 1](#) and states that the planner achieves strictly higher welfare than the market economy. Then, in order to fully characterize the constraint efficient allocation, I separate two cases: $\pi_1 b_1 + \pi_2 b_2 \geq c_2$ and $\pi_1 b_1 + \pi_2 b_2 < c_2$. Specially in the first case, I show that the planner achieves the first best. This claim is stronger than [Theorem 2](#), because the requirements are weaker. It is easy to check that in order for [Assumption 2](#) to be satisfied¹⁸, we need $b_2 - c_2 < b_1 - c_1$

¹⁸In order to apply [Theorem 2](#) to this setting, first switch the label of type one and type two. Now see that parts 1 to 4 of [Assumption 2](#) are easily satisfied. It is just left to check [Assumption 2](#) parts 5(b) and 5(c). Part 5(b) is satisfied if $h_2 - c_2 < h_1 - c_1$. Part 5(c) is satisfied if $U_2^{FB} \geq 0$ and $U_1^{FB} \geq \frac{m(1)(-c_1+c_2)}{\pi_1}$.

and $\frac{c_2 - c_1}{\pi_1} \leq b_1 - c_1$, which are stronger requirements than the requirement in Proposition 1 ($\pi_1 b_1 + \pi_2 b_2 \geq c_2$).

In the second and third columns of Table 1 the equilibrium outcomes under complete information and under private information are described respectively. In the fourth and fifth columns, I describe the planner's allocation under different conditions.

Since there are positive gains from trade for both types according to part 2 of Assumption 3, under complete information the planner wants both types to get matched with probability 1 ($\theta_1^{FB} = \theta_2^{FB} = 1$) and also trade with probability 1 ($\alpha_1^{FB} = \alpha_2^{FB} = 1$). As already discussed under complete information, the market decentralizes the first best allocation.

In the equilibrium with private information, different types trade in different submarkets. In submarket one, price is lower, but probability of matching is higher compared to submarket two ($p_1^{EQ} < p_2^{EQ}$). The market tightness is used as a screening device here. The probability of matching for type two is distorted so that type one would not want to apply to submarket two, although the price is higher there. The equilibrium allocation is independent of the distribution of types.

If $\pi_1 b_1 + \pi_2 b_2 \geq c_2$, then the planner achieves the first best through a pooling allocation. See the fourth column of Table 1. In this allocation, the planner does not need to use any transfers. All he needs to do is to restrict the entry of buyers to other submarkets by imposing large taxes on those submarkets and have all sellers trade in a pooling submarket with $p = \pi_1 b_1 + \pi_2 b_2$ and $t = 0$. This allocation cannot be sustained as an equilibrium, because buyers would have incentives to open a new submarket with a higher price to attract only high type sellers from the pool, i.e. cream skimming. But then the probability that high quality sellers get matched will be reduced compared to the first best and the planner does not want that. This is why the planner imposes large taxes on other submarkets.

Now assume that $\pi_1 b_1 + \pi_2 b_2 < c_2$. The planner's allocation in this case is reported in the fifth column of Table 1. Type two would get less than 0 under the pooling allocation, so pooling both types is not feasible. Therefore, the first best is not achievable via a pooling allocation. The first best is not achievable through any separating allocation either, because if $\alpha_1 = \alpha_2 = \theta_1 = \theta_2 = 1$, then the payment to sellers in both submarkets should be the same to satisfy IC condition. If the payments in both submarkets are equal, then this allocation is pooling, but it is already shown that the pooling allocation is not feasible. The same explanation is illustrated via indifference curves in Figure 3.

	Complete information (FB)	Equilibrium	Constrained efficient if $\pi_1 b_1 + \pi_2 b_2 \geq c_2$	Constrained efficient if $\pi_1 b_1 + \pi_2 b_2 < c_2$
α_1, α_2	1	1	1	1
θ_1	1	1	1	1
θ_2	1	$\frac{b_1 - c_1}{b_2 - c_1}$	—	$\frac{\pi_1(b_1 - c_1)}{c_2 - \pi_1 c_1 - \pi_2 b_2}$
p_1	b_1	b_1	$\pi_1 b_1 + \pi_2 b_2$	$\frac{\pi_1 b_1 (c_2 - c_1) + \pi_2 c_1 (c_2 - b_2)}{c_2 - \pi_1 c_1 - \pi_2 b_2}$
p_2	b_2	b_2	—	c_2
t_1	—	—	0	$-\frac{\pi_2 (b_2 - c_2)(b_1 - c_1)}{c_2 - \pi_1 c_1 - \pi_2 b_2}$
t_2	—	—	—	$b_2 - c_2$
U_1	$b_1 - c_1$	$b_1 - c_1$	$\pi_1 b_1 + \pi_2 b_2 - c_1$	$\frac{\pi_1 (b_1 - c_1)(c_2 - c_1)}{c_2 - \pi_1 c_1 - \pi_2 b_2}$
U_2	$b_2 - c_2$	$\frac{b_1 - c_1}{b_2 - c_1} (b_2 - c_2)$	$\pi_1 b_1 + \pi_2 b_2 - c_2$	0

Table 1: Comparison between different allocations in the asset market with lemons. t_1 and t_2 denote the tax amount levied on buyers in submarket one and two in the implementation of the constrained efficient allocation. U_1 and U_2 denote the payoff of type one and two in different allocations. If $\pi_1 b_1 + \pi_2 b_2 \geq c_2$, the planner can achieve the first best through a pooling allocation where both types trade in one submarket with price equal to $\pi_1 b_1 + \pi_2 b_2$. If $\pi_1 b_1 + \pi_2 b_2 < c_2$, the first best is not achievable. The constrained efficient allocation is implemented in the market through a separating allocation.

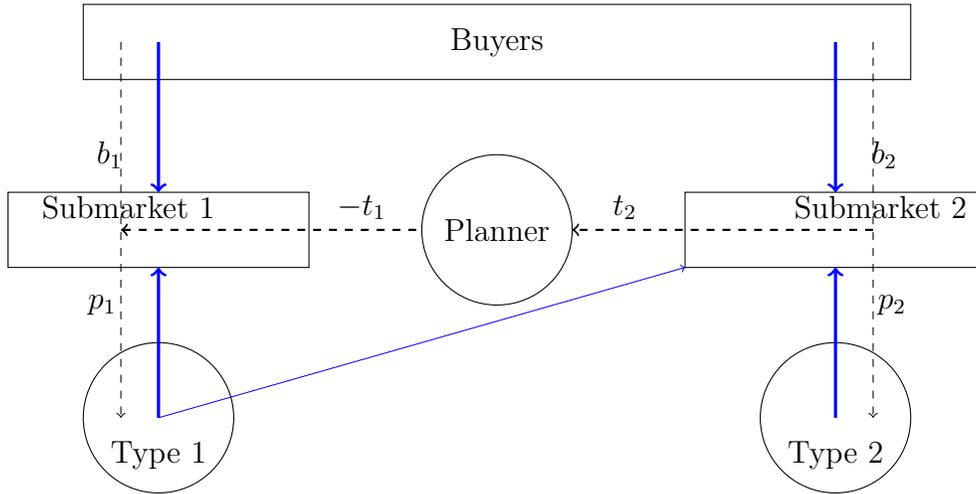


Figure 2: This schematic diagram illustrates how the planner allocates resources. Dashed lines show the flow of funds. In equilibrium, type one is indifferent between two submarkets, while type two strictly prefers submarket two. To improve welfare, the planner taxes sellers in submarket two and subsidizes sellers in submarket one so that type one sellers are discouraged from applying to submarket two (the higher price submarket). Now, more buyers enter submarket two and the outcome approaches the first best.

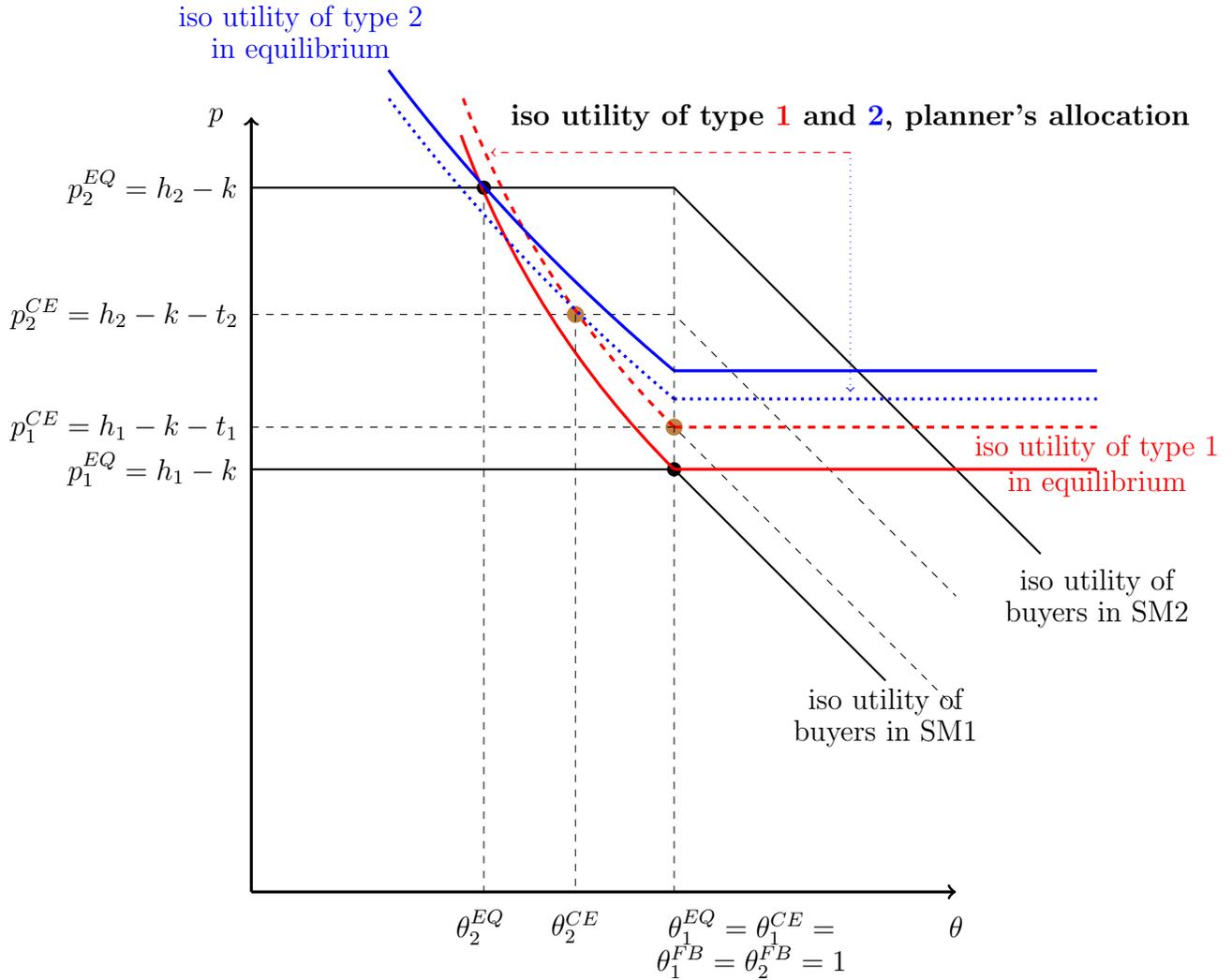


Figure 3: The indifference curves of buyers and sellers are illustrated here when $\pi_1 b_1 + \pi_2 b_2 < c_2$. CE, FB, EQ represent constrained efficient, first best and equilibrium allocations. In the equilibrium allocation, the market tightness for type two is less than 1. Intersection of indifference curve of type one and indifference curve of buyers in submarket two determines θ_2^{EQ} . At this point, type one is indifferent between both submarkets. The planner makes subsidies to buyers at submarket one ($t_1 < 0$), thus pushing buyers' indifference curves in that submarket upward. Because of zero profit condition for buyers, eventually type one sellers get a higher payoff than equilibrium. The planner taxes buyers in submarket two ($t_2 > 0$) to raise funds for subsidies made to type one. Now, the market tightness that the planner assigns to type two is increased compared to that in equilibrium.

4.1 Explanation of the Results

To understand how the planner can achieve strictly higher welfare than the equilibrium in the asset market with lemons, assume as a thought experiment that the planner begins from the equilibrium allocation and wants to increase welfare. We have already seen that the equilibrium allocation is feasible for the planner. In the equilibrium type one is indifferent between choosing submarket one and submarket two. Although some type two sellers are unmatched in submarket two, buyers do not enter submarket two any more, because more entry will make submarket two strictly preferable for type one, thus leading to entry of type one to submarket two. Nevertheless, matching with type one sellers in submarket two with positive probability is not worthwhile for buyers given the high price that buyers need to pay in submarket two. In short, the IC constraints that buyers face do not allow more buyers to enter submarket two.

To improve efficiency, the planner increases the net payment to type one¹⁹, p_1 , so that IC constraint of type one for choosing submarket two becomes slack. That is, type one strictly prefers submarket one over submarket two following this subsidy. Now more buyers have incentives to enter submarket two to get matched with previously unmatched sellers of type two. To finance subsidies to sellers in submarket one (type one sellers), the planner must tax sellers in submarket two (type two sellers). The planner keeps increasing p_1 and decreasing p_2 until one of the following happens. Either he achieves the first best, which is the case in the pooling allocation where both types trade with probability 1, or participation constraints of type two sellers bind, that is, type two sellers get exactly payoff 0. The former happens if $\pi_1 b_1 + \pi_2 b_2 \geq c_2$ and the latter happens if $\pi_1 b_1 + \pi_2 b_2 < c_2$. Figure 2 illustrates this point. Although I explained the main idea through a two-type example, the intuition is the same in the general n-type setting, or even in a continuous type setting which will be discussed in Section 6.

The main difference between planner's allocation and the equilibrium allocation is that in the equilibrium, the payment to sellers is exactly equal to the payment that buyers make. Also, because of the free entry condition, the buyers get 0, so the sellers get the whole surplus in every submarket. However, it is feasible for the planner to give sellers in one submarket more and sellers in other submarkets less than the surplus they generate. The only constraint that the planner faces is the budget constraint over all submarkets. That is, the amount of

¹⁹The way that the planner implements the mechanism is to subsidize buyers in submarket one, that is, $t_1 < 0$. Since there is zero profit condition for buyers in each submarket, buyers in submarket one pays the net amount of b_1 , anyway, which is equal to $p_1 + t_1$. But $t_1 < 0$, so $p_1 > b_1$. In other words, when the planner imposes tax or subsidy on buyers in one submarket, it is as if the planner imposes that tax or subsidy on sellers.

transfers that buyers pay must be equal to amount of transfers that sellers receive over all submarkets.

Entry of more buyers in a submarket creates two types of externalities on others. First, it decreases the probability of matching of other buyers in that submarket. Second, it changes the payoff of sellers in that submarket. In the complete information case, the change in the payoff of sellers in one submarket does not affect the payoff of sellers in other submarkets. In fact, under complete information, the negative externality that entrants impose on other buyers is exactly offset by the amount of positive externalities that they impose on sellers and therefore the equilibrium allocation is efficient. When there is private information, the change in the payoff of one type of sellers alters the IC constraints that other buyers face in other submarkets, thus affecting the set of feasible contracts that those buyers can offer to attract other types of sellers. This, in turn, will affect the payoff of other sellers in other submarkets. The buyers in the market economy does not take this effect into account. The planner, in contrast, internalizes these externalities and therefore is able to increase welfare.

The inability of buyers to internalize the externalities they create on others in equilibrium is similar to the situation in random search models with ex-post bargaining (like [Mortensen and Pissarides \(1994\)](#)) in which the share of the surplus that buyers get is exogenously fixed, so the outcome is generally inefficient. Here, although the division of the surplus to buyers and sellers is not exogenously fixed, it is endogenously pinned down by two constraints that IC and free entry impose on the allocation, so it is generally unlikely that the constrained efficiency is achieved by equilibrium. The planner can internalize these externalities, because he is not constrained by the free entry condition at each submarket, so he can make the buyers' share of the surplus satisfy Hosios condition (See [Hosios \(1990\)](#)).

4.2 What If There Are No Gains from Trade for Some Types?

GSW show that in the asset market with lemons, if there are no gains from trade only for type one, that is, $b_1 - c_1 < 0$ and $b_2 - c_2 > 0$, then the entire market will shut down. I show that in this case the planner cannot help. See Appendix, Page 69, for the proof.

The intuition is as follows. Type two is not active in the equilibrium, so given IC of type one, the highest payoff that type two can get in the market is negative, so type two chooses not to participate in the market. Therefore, both IC are binding (both types get zero payoff anyway.) The trick that worked in the proof of Theorem 1 is not effective here, because any direct subsidies intended for type one equally attracts type two sellers, so type two would also prefer to report to be type one. The takeaway message is that if the distortion is so sever that inactivity of one type in equilibrium leads to inactivity of other types, then the

	Complete information	Equilibrium allocation	Planner's allocation
θ_1	θ_1^{FB}	θ_1^{FB}	θ_1^{FB}
θ_2	θ_2^{FB}	$\theta_2^{EQ} > \theta_2^{FB}$	θ_2^{FB}
a_1	a_1^{FB}	a_1^{FB}	a_1^{FB}
a_2	a_2^{FB}	$a_2^{EQ} > a_2^{FB}$	a_2^{FB}
p_1	$p_1^{FB} \equiv v_1(a_1^{FB}) - \frac{k}{q(\theta_1^{FB})}$	$v_1(a_1^{FB}) - \frac{k}{q(\theta_1^{FB})}$	$p_1^{FB} - t_1$
p_2	$p_2^{FB} \equiv v_2(a_2^{FB}) - \frac{k}{q(\theta_2^{FB})}$	$v_2(a_2^{EQ}) - \frac{k}{q(\theta_2^{EQ})}$	$p_2^{FB} - t_2$
t_1	0	0	$-\frac{\pi_2}{m(\theta_1^{FB})}(U_2^{FB} - U_1^{FB}) + \pi_2(\tau - 1)\phi_2(a_1^{FB})$
t_2	0	0	$\frac{\pi_1}{m(\theta_2^{FB})}(U_2^{FB} - U_1^{FB}) - \pi_1(\tau - 1)\frac{m(\theta_1^{FB})}{m(\theta_2^{FB})}\phi_2(a_1^{FB})$

Table 2: The rat race results. In equilibrium, the probability of finding a match and hours of work of type two workers are distorted upward compared to the first best allocation. The planner subsidizes type one ($t_1 < 0$) and taxes type two ($t_2 < 0$) to correct the distortions and achieve the first best.

planner may not be able to help.

5 Example 2: The Rat Race

In this section I study another example from GSW, the rat race, which was originally discussed in [Akerlof \(1976\)](#). The main reason that I include this example is that the first best here is achievable only through a separating allocation, in contrast to the previous example (asset market with lemons) where the first best was achievable through a pooling allocation (if $\pi_1 b_1 + \pi_2 b_2 \geq c_2$). The planner here achieves the first best by separating different types and using appropriate transfers.

There are two types of workers (as sellers) on one side and firms (as buyers) on the other side of the market. The payoff of a type i worker matched with a firm from a hours of work and consuming t units of the numeraire good is $t - \phi_i(a)$. The worker's payoff is 0 if unmatched. The payoff of a firm matched with a type i worker when the worker works for a hours and the firm pays t units of the numeraire good (either to the worker or to the planner) is $v_i(a) - t - k$. The firm's payoff is $-k$ if unmatched. As a special case of the original setting, here $I = 2$, $n = 1$ and $u_i(a) = -\phi_i(a)$. Matching function $m(\theta)$ is strictly concave and twice differentiable. I make the following assumptions:

Assumption 4. *In the rat race example,*

1. ϕ_i is differentiable, increasing, strictly convex and $\phi_i(0) = \phi_i'(0) = 0$.
2. For all a , $\phi_1(a) = \tau\phi_2(a)$ where $\tau > 1$.

3. v_i is differentiable, increasing and strictly concave.
4. For all a , $v_1(a) \leq v_2(a)$ and $v'_1(a) \leq v'_2(a)$.

Remember from Theorem 2 that Assumption 2 is sufficient for the planner to achieve the first best. I argue here that if Assumption 4 holds, then Assumption 2 is automatically satisfied and therefore I can use that result. Part 1 of Assumption 2 is satisfied because $-\phi_1(a) < -\phi_2(a)$ for all a . Part 2 of Assumption 2 (increasing differences property of $u(\cdot)$) is satisfied because $-\phi_2(a) - (-\phi_1(a))$ is increasing in a due to the assumption that $\tau > 1$. Part 3 of Assumption 2 (increasing differences property of $u(\cdot) + v(\cdot)$) is satisfied because $v_2(a) - \phi_2(a) - (v_1(a) - \phi_1(a))$ is increasing in a . Part 4 of Assumption 2 (supermodularity of $u(\cdot) + v(\cdot)$ in a) is trivially satisfied because a is just one-dimensional. Part 5(a) of Assumption 2 (monotonicity of $v(\cdot)$) is satisfied because $v_2(a) \geq v_1(a)$.

Proposition 2. *If Assumption 4 holds and $U_i^{FB} > 0$ for all i , then the planner achieves the first best. See the fourth column of Table 2 for the full description of the constrained efficient allocation.*

This result is a special case of Theorem 2. The planner subsidizes type one ($\tilde{p}_1^{CE} > p_1^{FB}$) and taxes type two ($\tilde{p}_2^{CE} < p_2^{FB}$) to achieve efficiency. By offering this schedule of transfers, allocating the low type workers higher wage and the high type workers lower wage than their wages in the equilibrium with complete information, the planner discourages type one workers from applying to submarket two, thus reducing the cost of private information.

An interesting point about this result is that the planner achieves the first best regardless of the distribution of types. The intuition is that if the planner sets payments such that type one gets at least payoff 0, then the planner can make positive amount of money over *each* submarket.

GSW make the same assumptions except that they they do not impose $v'_1(a) \leq v'_2(a)$. When $U_2^{FB} - U_1^{FB} \geq (\tau - 1)m(\theta_2^{FB})\phi_2(a_2^{FB})$, then the equilibrium does not achieve the first best. They propose a pooling allocation which Pareto dominates the equilibrium allocation if π_1 is sufficiently small, although the pooling allocation does not achieve the first best. As stated earlier, the planner achieves the first best regardless of π_1 . Moreover, if π_1 is sufficiently small, then the planner's allocation Pareto dominates the equilibrium allocation.

6 Extension: Asset Market with a Continuous Type Space

The model studied in this section is an extension of that in Section 4 to a continuous type space. In Section 4, the efficient tax schedule requires high price assets to be taxed and low price assets to be subsidized. An interesting question is whether the tax schedule which implements the constraint efficient mechanism is generally monotone in the price of assets or not.

This extension is interesting not only because it makes it possible to consider cases in which the value of assets to sellers does not have the same order as the value of assets to buyers, but also because I can answer some relevant policy questions about the optimal taxation in the asset markets²⁰. Also, studying this case makes it possible for us to compare the planner's allocation with the equilibrium allocation in Chang (2012). The setting in this section is basically a static version of Chang's environment, and fortunately the main ideas regarding the equilibrium and the planner's allocation are captured in this static case²¹. Since this environment is not a special case of the original setting in Section 2, I need to define the constrained efficient allocation again. The main ideas discussed so far will be used similarly for this case as well, but the mathematical tools used to characterize the planner's allocation will be different.

6.1 Environment

There is a continuum of measure one of heterogeneous sellers indexed by $z \in Z \equiv [z_L, z_H] \subset \mathbb{R}$, with $F(z)$ denoting the measure of sellers with types below z . F is continuously differentiable and strictly increasing in z and F' is its derivative. Type z is seller's private information. Similar to the original setting, buyers' and sellers' payoffs are quasi-linear. A buyer's payoff who enters the market and gets matched with a type z is $h(z) - t - k$ where

²⁰I could do the same exercise with a discrete type space with more than two types, but the technical analysis with a continuous type space is simpler.

²¹In a dynamic setting, the planner has some intertemporal considerations, because the distribution of types in the population does not necessarily remain the same over time, because some types get matched more quickly than others and exit the market. This observation raises a new and interesting tradeoff, whether the planner wants to have low types find a match early or he wants to have all types together all the way to the end. The analysis of the dynamic setting is beyond the scope of this paper. Since the equilibrium allocation is distribution free, the equilibrium analysis is much easier than the analysis of the planner's problem in the dynamic case. However, if one assumes in the dynamic setting that when sellers sell their assets, they are endowed a new asset with the same quality, the same results can be obtained from the dynamic setting as in the static setting, because the distribution of types does not change over time.

t denotes the amount of a numeraire good that he produces and $h(z)$ is the value of the asset to the buyer in terms of the numeraire good. His payoff is $-k$ if unmatched. The payoff of a type z seller matched with a buyer is $t - c(z)$ where t denotes the amount of the numeraire good that he consumes and $c(z)$ is the value of the asset to the seller in terms of the numeraire good. His payoff is 0 if unmatched. Functions $h : Z \rightarrow R$ and $c : Z \rightarrow R$ are twice continuously differentiable. Matching function $m(\cdot)$ is increasing, strictly concave and twice differentiable with strictly decreasing elasticity. I also assume throughout this section that there are positive gains from trade for all types. Similar to the discrete type space, it turns out that all types will be active both in equilibrium and in the constrained efficient allocation.

6.2 Complete Information Allocation or First Best

Here, I mostly follow the discussion of the complete information case for the discrete type space in Section 3. Consider the market economy with the complete information. The buyers who attempt to attract type z sellers solve the following problem:

$$U^{FB}(z) = \max_{\theta, p} \{m(\theta)(p - c(z))\}$$

$$\text{s.t. } q(\theta)(h(z) - p) \geq k.$$

Let $\theta^{FB}(z)$ and $p^{FB}(z)$ denote the market tightness and the price that solve this problem. I assume that $U^{FB}(z) > 0$ for all z , that is, that there are positive gains from trade for all types. Similar to the discrete type case, $U^{FB}(z) = \max_{\theta} \{m(\theta)(h(z) - c(z)) - k\theta\}$. Also $\theta^{FB}(z)$ solves

$$m'(\theta)(h(z) - c(z)) = k, \tag{1}$$

for both the planner and the market economy with complete information. The left hand side of Equation 1 is the marginal benefit of adding one more buyer to the submarket composed of z sellers. The right hand side is the marginal cost of doing that. The planner keeps adding buyers to each submarket until the marginal cost and marginal benefit become equal²².

²²To verify that the Hosios condition (Hosios (1990)) is satisfied in the market with complete information with directed search, I calculate the share of the surplus that sellers get in equilibrium:

$$\begin{aligned} \frac{p^{FB}(z) - c(z)}{h(z) - c(z)} &= \frac{U^{FB}(z)}{m(\theta^{FB}(z))(h(z) - c(z))} = \frac{m(\theta^{FB}(z))(h(z) - c(z)) - k\theta^{FB}(z)}{m(\theta^{FB}(z))(h(z) - c(z))} \\ &= \frac{m(\theta^{FB}(z))(h(z) - c(z)) - \theta^{FB}(z)m'(\theta^{FB}(z))(h(z) - c(z))}{m(\theta^{FB}(z))(h(z) - c(z))} = -\frac{\theta^{FB}(z)q'(\theta^{FB}(z))}{q(\theta^{FB}(z))} \equiv \eta(\theta^{FB}(z)), \end{aligned}$$

where the third equality follows from Equation 1. Hosios condition states that a necessary condition for the efficiency of any allocation is that the share of the surplus that type z sellers get from the match, $\frac{p-c(z)}{h(z)-c(z)}$,

6.3 Definition of the Planner's Problem

Because our focus in this section is the shape of the optimal tax schedule, whether the optimal tax schedule be monotone in the price of assets or not, the interesting concept to study is the concept of implementable allocation and not direct mechanism. The main definition of implementable allocation in this section is a straightforward modification of definition in Section 2 to allow for a continuous type space. The definition of constrained efficient mechanism and also Lemma 1 can be modified in a straightforward way too. However, I do not repeat them here for the sake of brevity. (See Appendix, Section 8.6.) Specifically, it can be proved in exactly the same fashion as in Lemma 1 that for any feasible direct mechanism, there exists an associated implementable allocation under which all types get exactly the same payoff as in the direct mechanism.

Definition 6. *An implementable allocation, $\{P, G, \theta, \mu, t, T\}$, is a measure G on the set of all possible prices, $\mathbb{P} \equiv \mathbb{R}_+$, with support P , a tightness function, $\theta : \mathbb{P} \rightarrow [0, \infty]$, a conditional density function of buyers' beliefs regarding the type of sellers who would apply to p , $\mu : \mathbb{P} \times \mathbb{Z} \rightarrow [0, 1]$, a tax function denoting the amount of tax to be imposed on buyers at each submarket conditional on trade, $t : \mathbb{P} \rightarrow \mathbb{R}$, and finally the amount of the numeraire good to be transferred to sellers in a lump sum way, $T \in \mathbb{R}_+$ ²³, which satisfies the following conditions:*

(i) **Buyers' profit maximization and free entry**

For any $p \in \mathbb{P}$,

$$q(\theta(p)) \left[\int h(z) \mu(z|p) dz - p - t(p) \right] \leq k,$$

with equality if $p \in P$.

(ii) **Sellers' optimal search**

Let $U(z) = \max \left\{ 0, \max_{p' \in P} \{ m(\theta(p'))(p' - c(z)) \} \right\} + T$ and $U(z) = T$ if $P = \emptyset$. Then for any $p \in \mathbb{P}$ and z , $U(z) \geq m(\theta(p))(p - c(z)) + T$ with equality if $\theta(p) < \infty$ and $\mu(z|p) > 0$. Moreover, if $p - c(z) < 0$, either $\theta(p) = \infty$ or $\mu(z|p) > 0$.

(iii) **Feasibility or market clearing**

For all z , $\int_P \frac{\mu(z|p)}{\theta(p)} dG(p) \leq F'(z)$, with equality if $U(z) > T$.

(iv) **Planner's budget constraint**

$$\int_P q(\theta(p)) t(p) dG(p) \geq T.$$

for any z must be equal to the elasticity of matching function with respect to the number of sellers. As shown above, the equilibrium allocation under complete information satisfies this property.

²³In this section, since I have assumed that there are positive gains from trade for all types, it is easy to check that T is redundant. That is, the welfare level will not be lower without T .

Definition 7. A constrained efficient allocation is an implementable allocation which maximizes welfare among all implementable allocations. That is, a constrained efficient allocation solves the following problem:

$$\begin{aligned} & \max_{\{P, G, \theta, \mu, t, T\}} \int U(z) dF(z) \\ & \text{s. t. } \{P, G, \theta, \mu, t, T\} \text{ is implementable,} \end{aligned}$$

where $U(z)$ is define in part (ii) of Definition 6.

6.4 Characterization

6.4.1 Characterization of the Constrained Efficient Allocation

To find a direct mechanism which solves the planner's problem, I use somewhat a backward approach. I first guess that the planner can achieve the first best. That is, the planner can maximize his objective function for each type separately. Then I find a set of prices such that sellers' IC conditions are satisfied. Given this set of prices, I derive sufficient conditions under which the planner's budget constraint and participation constraint of all types hold simultaneously. To find an implementable allocation which implements this direct mechanism, I calculate taxes in such a way that buyers' maximization and free entry (condition (i) in Definition 6) for on-the-equilibrium-path prices are satisfied. Finally I construct taxes and beliefs for off-the-equilibrium-path prices. Checking for other conditions in Definition 6, then, would be easy.

Assumption 5. For all z , $c'(z) > 0$ and either

1. $h'(z) \leq 0$ for all z , or
2. $h'(z) \leq c'(z)$ and $\psi\left(\frac{k}{h(z)-c(z)}\right) \left[\frac{h(z)-c(z)}{c'(z)}\right] \geq \frac{F(z)}{F'(z)}$ for all z , where $\psi(\cdot) \equiv \eta(m'^{-1}(\cdot))$ and $\eta(\theta) \equiv -\frac{\theta q'(\theta)}{q(\theta)}$.

Proposition 3. If Assumption 5 holds and $U^{FB}(z) > 0$ for all z , then the planner achieves the first best.

Given any implementable allocation, define correspondence $\Omega(z)$ as follows

$$\Omega(z) \equiv \{(p, \theta(p), t(p)) \text{ such that } \mu(z|p) > 0\}.$$

Denote the elements of $\Omega(z)$ by $(\tilde{p}(z), \tilde{\theta}(z), \tilde{t}(z))$ showing the price, market tightness and the tax amount (imposed on buyers) of the submarkets to which type z applies with a strictly

positive probability. Basically, $\tilde{p}(z)$, $\tilde{\theta}(z)$ and $\tilde{t}(z)$ are elements of a direct mechanism²⁴. Similar to the notation in previous sections, $\tilde{x}(z)$ denotes the variable x allocated to type z in a **direct mechanism**, whether the direct mechanism be used for a constraint efficient allocation or an equilibrium allocation.

It is shown in the proof of Proposition 3 that all types trade in submarkets with different market tightness, therefore the allocation is separating and $\tilde{p}^{CE}(z)$, $\tilde{\theta}^{CE}(z)$ and $\tilde{t}^{CE}(z)$ are just functions (as opposed to correspondences) of z . It is also shown in the proof that these variables are given as follows:

$$\begin{aligned} \tilde{\theta}^{CE}(z) &= \theta^{FB}(z) \text{ for all } z, \\ \tilde{p}^{CE}(z) &= c(z) + \frac{U(z_H) + \int_z^{z_H} m(\tilde{\theta}^{CE}(z_0))c'(z_0)dz_0}{m(\tilde{\theta}^{CE}(z))} \text{ for all } z, \end{aligned} \quad (2)$$

where

$$U(z_H) = \int \left[m(\tilde{\theta}^{CE}(z))(h(z) - c(z)) - k\tilde{\theta}^{CE}(z) - m(\tilde{\theta}^{CE}(z))c'(z) \frac{F(z)}{F'(z)} \right] dF(z),$$

and

$$\tilde{t}^{CE}(z) = h(z) - \tilde{p}^{CE}(z) - \frac{k}{q(\tilde{\theta}^{CE}(z))} \text{ for all } z. \quad (3)$$

To have a rough idea how the planner can undo the effects of private information and achieves the first best, I proceed by analyzing the incentive compatibility problem that the planner faces. I assume (without loss of generality) that sellers are allocated to different submarkets through a direct mechanism. That is, if a type z agent reports \hat{z} , his payoff is given by $m(\tilde{\theta}(\hat{z}))(\tilde{p}(\hat{z}) - c(z))$. In a direct mechanism, agents of type z choose a \hat{z} which maximizes their payoff:

$$\max_{\hat{z}} \{m(\tilde{\theta}(\hat{z}))(\tilde{p}(\hat{z}) - c(z))\}. \quad (4)$$

I keep the assumption that $c'(z) > 0$ throughout this section, so the seller's payoff function, $m(\theta)(p(z) - c(z))$, satisfies single crossing condition. (See Theorem 7.3 in [Fudenberg and Tirole \(1991\)](#).) As already discussed in the sketch of the proof of Theorem 2, $\tilde{\theta}(z)$ being decreasing in z implies that there exists a set of transfers to sellers that satisfies IC. Now assume $h'(z) \leq 0$ for all z or $h'(z) \leq c'(z)$ for all z . In either case, $\theta^{FB}(z)$ is decreasing in z according to Equation 1. Therefore, if $\tilde{\theta}(z)$ is set to be equal to $\theta^{FB}(z)$ for all z , one can find such

²⁴Here, having $\tilde{t}(z)$ as a tax amount in the *direct mechanism* is an abuse of notation. This is because I have defined the direct mechanism in such a way that transfers are made only to sellers and the planner just ensures that buyers get an ex-ante payoff 0. Therefore, $\tilde{t}(z)$ should be interpreted as the tax amount that buyers should pay in the implementable allocation if they are matched with type z sellers. The reason that I define it as a function of z is because it makes the analysis simpler and more intuitive.

transfers. Then according to Envelop theorem, one can calculate these transfers as given in Equation 2.

Since $\tilde{\theta}(z)$ is strictly decreasing here (because $\theta^{FB}(z)$ is strictly decreasing), then the associated implementable allocation must be separating, so the amount of tax that should be imposed on buyers in each submarket, $\tilde{t}(z)$, will be a function (not a correspondence) of z and can be easily calculated by buyers' profit maximization and free entry condition. I provide sufficient conditions in the proof such that the planner's budget constraint also holds. If $h'(z) \leq 0$, then the planner has enough resources to distribute among agents regardless of the distribution. If $h'(z) \leq 0$ is not satisfied for some z but $h'(z) - c'(z) \leq 0$ still holds for all z , then I need another condition (in part 2 of Assumption 5) which relates the distribution of types to the payoff and matching functions to ensure that the planner's budget constraint is satisfied.

Proposition 3 analyzes just one possible case for the planner's problem where monotonicity constraint (that $\tilde{\theta}(z)$ should be decreasing in z) and participation constraint for almost all types are not binding. Analyzing other cases where monotonicity constraint or participation constraint is binding does not add much insight to the analysis, so I skip it. As an example, I solve the planner's problem for the case where participation constraint is binding in Appendix, Section 8.6.3. For the case where monotonicity constraint is binding, one can use existing techniques from mechanism design literature to bunch multiple types. The characterization in that case is available upon request.

6.4.2 Characterization of Equilibrium Allocation

I report the results of a static version of Chang (2012) here and compare the equilibrium allocation with the planner's one. Definition of equilibrium is similar to the definition of implementable allocation here, but with the restriction that taxes and transfers must be all equal to 0. I do not repeat the definition of equilibrium here to save space. See Chang (2012) for more details on the equilibrium definition. I study a static model while she studies a dynamic model. To see why I consider a static model, see Footnote 21. Chang assumes that utility of holding the asset until finding a buyer is different across different types of assets. Similarly, I assume sellers with high z values their assets more ($c' > 0$).

Assumption 6. $c'(z) > 0$ and $h'(z) \geq 0$ for all z .

Proposition 4 (Equivalent to Proposition 1 in Chang (2012)). *If Assumption 6 holds and if $U^{FB}(z) > 0$ for all z , then there exists a unique equilibrium. The equilibrium is separating. The market tightness solves the differential equation 6. The initial condition is given by $\tilde{\theta}^{EQ}(z_L) = \theta^{FB}(z_L)$. Prices are given by $\tilde{p}^{EQ}(z) = h(z) - \frac{k}{q(\tilde{\theta}^{EQ}(z))}$.*

Condition	Necessary condition for I.C	Initial conditions
$0 < c'(z)$ and $0 < h'(z)$	$\frac{d\tilde{\theta}^{EQ}}{dz} < 0$	$\tilde{\theta}^{EQ}(z_L) = \theta^{FB}(z_L)$
$0 < c'(z)$ and $0 > h'(z)$	$\frac{d\tilde{\theta}^{EQ}}{dz} < 0$	$\tilde{\theta}^{EQ}(z_H) = \theta^{FB}(z_H)$
$0 > c'(z)$ and $0 < h'(z)$	$\frac{d\tilde{\theta}^{EQ}}{dz} > 0$	$\tilde{\theta}^{EQ}(z_L) = \theta^{FB}(z_L)$
$0 > c'(z)$ and $0 > h'(z)$	$\frac{d\tilde{\theta}^{EQ}}{dz} > 0$	$\tilde{\theta}^{EQ}(z_H) = \theta^{FB}(z_H)$

Table 3: Equilibrium allocation in different cases in the asset market with continuous type space

I explain the logic behind her result. See her paper for the formal proof. First note that the IC constraints that agents face in the market economy are the same as those the planner faces, therefore I can use Equation 4 to describe IC constraints for analyzing equilibrium too. The only difference is that the prices are different in the market economy, because they are pinned down by the free entry condition. Chang shows that any equilibrium under Assumption 6 is separating, so free entry implies that $\tilde{p}^{EQ}(z) = h(z) - \frac{k}{q(\tilde{\theta}^{EQ}(z))}$ for all z . Therefore, the payoff of type z in the market economy, denoted by $U^{EQ}(z)$, is calculated as follows:

$$U^{EQ}(z) = \max_{\hat{z}} \{m(\tilde{\theta}^{EQ}(\hat{z}))(h(\hat{z}) - c(z)) - k\tilde{\theta}^{EQ}(\hat{z})\}, \quad (5)$$

where the objective function is the payoff of type z if he reports type \hat{z} . FOC with respect to \hat{z} (together with the assumption of differentiability of $\tilde{\theta}(z)$) yields

$$\left[m'(\tilde{\theta}^{EQ}(z))(h(z) - c(z)) - k \right] \frac{d\tilde{\theta}^{EQ}(z)}{dz} + m(\tilde{\theta}^{EQ}(z))h'(z) = 0, \quad (6)$$

where I used the fact that at the solution, $\hat{z} = z$ due to IC.

With respect to the initial condition, roughly speaking, the market delivers the complete information payoff to the type which has the most incentive to deviate. For example, when $h' \geq 0$, the lowest type has the most incentive to deviate, so his allocation is set to the complete information level, i.e., $\tilde{\theta}^{EQ}(z_L) = \theta^{FB}(z_L)$. The necessary condition for IC and the initial condition for the differential equation are depicted in Table 3 for different assumptions, where c' and h' are both positive or both negative, or only one of them is positive.

6.4.3 Disagreement in the Ranking of Assets between Buyers and Sellers or Two-dimensional Private Information

Chang (2012) assumes in another part of her paper that sellers have another dimension of private information. Some sellers get liquidity shocks so they need to sell their assets quickly. What is relevant to our discussion is that following this extension, it is possible

that function h has a strict local maximum, keeping the assumption $c'(z) > 0$ fixed. If h has a strict local maximum, she proves that full separation in the market is not possible. Also, she derives some conditions under which an equilibrium with fire sales exists, where many low type sellers and some high type sellers who need liquidity sell their assets with a lower price but very quickly. My characterization, in contrast, shows if $h'(z) \leq c'(z)$ and if part 2 of Assumption 5 or Assumption 6 holds, even if h has a local maximum, then the constrained efficient allocation is separating, that is, the planner wants different types to trade in different submarkets. This case is depicted in Figure 4 where $h'(z)$ is drawn in terms of $c'(z)$ for all z .

Now suppose $h'(z) - c'(z) \leq 0$ is violated for some z . For example, $h - c$ has one local minimum, but $h'(z) \geq 0$ and $c'(z) > 0$ both hold, as depicted in Figure 5. The equilibrium in this case is separating. The planner's allocation, in contrast, involves some pooling, because monotonicity constraint (that $\tilde{\theta}^{CE}(z)$ should be decreasing in z) cannot be satisfied through any separating allocation²⁵. The bottom line is that pooling of types occurs under different conditions in the planner's allocation and the equilibrium allocation.²⁶ The welfare level in the planner's problem is strictly higher than that in the equilibrium by the same argument made in the proof of Theorem 1 even if the planner does not achieve the first best.

6.5 Examples of the Optimal Taxation

In this section, I present two examples in order to compare the first best (FB), equilibrium (EQ) and constrained efficient (CE) allocations and to figure out what types should be taxed and what types should be subsidized.

Example 1. *Model parameters: $m(\theta) = 1 - e^{-\theta}$, $Z = [9, 10.5] \subset R$, $c(z) = z$, $h(z) = 0.04(z - 6.5)(z - 7)(z - 10) + 17$, $k = 1$, and $F(\cdot)$ is uniform.*

Here, $c' > 0$, $h' > 0$ and $h' - c' < 0$. It is easy to check that part 2 of Assumption 5 is satisfied, therefore Proposition 3 holds. Hence, the market tightness at the constrained efficient allocation is given by $\tilde{\theta}^{CE}(z) = \theta^{FB}(z) = m'^{-1}(\frac{h(z)-c(z)}{k}) = \ln(\frac{h(z)-c(z)}{k})$. Then, $\tilde{p}^{CE}(z)$ and $\tilde{t}^{CE}(z)$ are derived from Equation 2 and Equation 3. The net payment that buyers make in the constrained efficient allocation, $\tilde{p}^{CE}(z) + \tilde{t}^{CE}(z)$, is equal to $p^{FB}(z) \equiv h(z) - \frac{k}{q(\theta^{FB}(z))}$. Regarding equilibrium allocation, $\tilde{\theta}^{EQ}(z)$ is derived from differential equation

²⁵Solving explicitly for the planner's allocation in this case does not give us new insights, so I skip its analysis. For example, see the appendix of chapter 7 in Fudenberg and Tirole (1991).

²⁶Roughly speaking, the planner is concerned with the surplus from the match not the value of the match to buyers only, so in the conditions regarding the planner's allocation, usually $h - c$ shows up. The buyers in the equilibrium are concerned with the value of the assets to themselves, so in the conditions regarding the allocation, usually h shows up, not h separately.

6 with the initial condition $\tilde{\theta}^{EQ}(9) = \theta^{FB}(9)$. The price that buyers pay in equilibrium is $\tilde{p}^{EQ}(z) = h(z) - \frac{k}{q(\tilde{\theta}^{EQ}(z))}$.

In Figure 6, $h(z)$ and $c(z)$ in the left graph and $h(z) - c(z)$ in the right graph are depicted. In the upper part of Figure 7, $p(z)$ and $\theta(z)$ for all three cases (FB, EQ and CE) are depicted. In this example, since Assumption 5 holds, $\tilde{\theta}^{CE}$ is equal to θ^{FB} . On the other hand, $\tilde{\theta}^{EQ}$ is less than θ^{FB} , because market tightness is basically the tool that buyers in the market economy use to screen high type sellers. Low type sellers prefer to sell their assets more quickly, because they do not want to get stuck with their “lemons.” Consequently, \tilde{p}^{EQ} is greater than p^{FB} . Also, \tilde{p}^{CE} is higher for lower types and lower for higher types compared to p^{FB} . Since the market tightness is the same in FB and CE, the price that buyers should pay in CE should be the same as that in FB in order for buyers’ profit maximization and free entry condition to be satisfied. On the other hand, \tilde{p}^{CE} is the payment that sellers should receive in CE. Therefore, the amount of tax that buyers should pay, \tilde{t}^{CE} , is just equal to the difference, $p^{FB} - \tilde{p}^{CE}$. In the lower left part of Figure 7, $\tilde{t}^{CE}(z)$ is drawn in terms of z . Also, $\tilde{t}^{CE}(z)$ is drawn in terms of $\tilde{p}^{CE}(z)$ in the lower right part of the figure. In Figure 8, the payoff to sellers of different assets in EQ, FB and CE is depicted. As seen from the figure, the constrained efficient allocation Pareto dominates the equilibrium allocation.

The predictions of this model regarding monotonicity of $\tilde{t}^{CE}(z)$ in terms of z is the same as predictions of the simple two-type example studied in Section 4. As a result, one might think that if buyers and sellers agree on the ranking of assets, the monotonicity of the optimal tax schedule must be a general result. However, this is not true. Specifically, I show in an example that if $h'(z) \geq 0$ for all z and $h'(z_L) = 0$, then the optimal tax schedule is not monotone in the price of assets. Specifically, $\frac{dt^{CE}(p)}{dp} \big|_{p=p_L} < 0$ and $\frac{dt^{CE}(p)}{dp} \big|_{p=p_0} > 0$ for some other p_0 . The proof is in Appendix (Section 8.6.2).

Next, I study another example where h has a local maximum and therefore separation of types in equilibrium is not possible, as explained in the last subsection. Also, the tax schedule will be non-monotone in the price of assets.

Example 2. Model parameters: $m(\theta) = 1 - e^{-\theta}$, $Z = [6, 10.5] \subset \mathbb{R}$, $c(z) = z$, $h(z) = 0.04(z - 6.5)(z - 7)(z - 10) + 17$, $k = 1$, and $F(\cdot)$ is uniform.

Note that functions c and h are both the same as in Example 1. Only the domain (Z) is different. Now, h has a strict local maximum, so the equilibrium will not be separating. Similar to the previous example, Assumption 5 is satisfied, therefore Proposition 3 holds. Hence, the market tightness at the constrained efficient allocation is similarly given by $\tilde{\theta}^{CE}(z) = \theta^{FB}(z) = m'^{-1}\left(\frac{h(z)-c(z)}{k}\right) = \ln\left(\frac{h(z)-c(z)}{k}\right)$.

According to the algorithm proposed by Chang (2012), I calculate one semi-pooling equi-

librium where types $z \in [6, 9)$ trade in a pool with a low price but with high probability. Types $z \in (9, 10.5]$ trade in separating submarkets. Type $z = 9$ is indifferent between the pool and one of the separating submarkets. Prices are calculated similarly as explained in Example 1.

In Figure 9, the value of assets to buyers (left), the value of assets to sellers (middle) and the surplus generated by each type (right) are depicted. In the upper part of Figure 10, price and market tightness for EQ, FB and CE are depicted. Similar to the previous example, market tightness in CE is the same as that in FB. Market tightness in EQ for types $z \in [6, 9)$ is higher than that in FB and is less for other types. Taxes are calculated in the same way as in Example 1. An interesting fact here is that the amount of tax imposed on buyers is neither monotone in the type of sellers that buyers meet (the lower left graph in Figure 10), nor in the price paid to sellers (the lower right graph in the same figure) or buyers (not shown in this figure). Finally in Figure 11, the payoff to sellers of different assets in EQ, FB and CE is depicted similarly to Figure 8.

6.6 Sales Tax and Entry Tax

As shown in the previous subsection, even if buyers and sellers agree on the ranking of assets, it is still possible that the tax schedule imposed on buyers conditional on trade is not monotone in the price of assets. One disadvantage of a non-monotone tax schedule is that it is extremely hard to implement it in the real world. Although it is usually assumed in the literature (including in this paper) that the planner has precise information about the distribution of types and the payoff structure of assets in our models, but ideally one wants to reduce the dependence of what the planner should do on the details of the economy. If the tax schedule is non-monotone in the price of assets, this dependence is crucial. In contrast, if the tax schedule is monotone, possible errors in implementation may cause less inefficiencies. This is because there exists exactly one price with the property that trades with prices above that should be taxed and other trades should be subsidized²⁷.

Given that a monotone tax schedule is desirable, in this subsection I suggest another tax schedule in addition to the tax schedule discussed so far. Therefore, buyers will be subject to two types of taxes, one is conditional on entry to each submarket (entry tax) and the other one conditional on trade (sales tax). The definition of implementable tax schedule should be slightly modified to include both types of taxes. See Appendix, Section 8.7 for

²⁷Of course, that would be more desirable if the optimal tax schedules are linear, because then all the planner needs to determine is just the slope of the tax schedules. Since the payoff structure in our problem is general, one of the two tax schedules suggested in this section to implement the direct mechanism is not linear in general.

the details. I show in the following proposition that in general any feasible mechanism can be implemented by a decreasing entry tax and an increasing sales tax both in the price of assets.

Proposition 5 (Implementation of the direct mechanism with monotone entry and sales tax). *Assume $c'(z) > 0$ for all z . Take any feasible mechanism. Assume that all types get a strictly positive payoff and also that the market tightness allocated to different types is all different. Then there exists an associated implementable allocation with monotone tax schedules in the price of assets, **decreasing** entry tax and **increasing** sales tax, such that all types get the same payoff as what they get in the feasible mechanism.*

To understand why $\tilde{t}(z)$ may not be monotone in z in absence of entry tax and how entry tax may solve this problem, I write the free entry condition as follows, given the fact that the allocation is separating: $q(\tilde{\theta}^{CE}(z))(h(z) - \tilde{p}^{CE}(z) - \tilde{t}^{CE}(z)) = k$. Therefore,

$$\tilde{t}^{CE}(z) = h(z) - \underbrace{\frac{k}{q(\tilde{\theta}^{CE}(z))}}_{\text{decreasing}} - \underbrace{\tilde{p}^{CE}(z)}_{\text{increasing}}.$$

The term $\frac{k}{q(\tilde{\theta}^{CE}(z))}$ is decreasing in z because $\tilde{\theta}^{CE}(z)$ is decreasing in z . I show now that $\tilde{p}^{CE}(z)$ is strictly increasing²⁸ in z . According to Equation 2, one can write:

$$\frac{d[m(\tilde{\theta}^{CE}(z))\tilde{p}^{CE}(z)]}{dz} = m'(\tilde{\theta}^{CE}(z))\frac{d\tilde{\theta}^{CE}(z)}{dz}c(z).$$

Hence

$$\frac{d\tilde{p}^{CE}(z)}{dz} = -\frac{m'(\tilde{\theta}^{CE}(z))}{m(\tilde{\theta}^{CE}(z))}\frac{d\tilde{\theta}^{CE}(z)}{dz}(\tilde{p}^{CE}(z) - c(z)). \quad (7)$$

But $\tilde{p}^{CE}(z) - c(z)$ is strictly positive, for otherwise, type z will be inactive, thus contradicting the assumption that all types are active. Since $\tilde{\theta}^{CE}(z)$ is strictly decreasing, the right hand side of the above equation is strictly positive, that is, $\tilde{p}(z)$ is strictly increasing.

Hence, in general it is not guaranteed that $\tilde{t}^{CE}(z)$ is monotone in z . The idea to make $\tilde{t}^{CE}(z)$ monotone is to add an entry tax for each submarket, $\tilde{t}_e(z)$, so the free entry condition can be written as follows:

$$\tilde{t}(z) = h(z) - \frac{k + \tilde{t}_e(z)}{q(\tilde{\theta}(z))} - \tilde{p}(z).$$

If $\tilde{t}_e(z)$ is constructed to be decreasing sufficiently fast in z , then the effect of $\frac{k + \tilde{t}_e(z)}{q(\tilde{\theta}(z))}$ dominates the effect of $\tilde{p}(z)$ and so $\tilde{t}(z)$ becomes increasing in z .

²⁸The derivation below holds not only for the constrained efficient allocation but also for any allocation that satisfies IC and sellers' participation constraint.

One important point here is that since entry tax will be collected even before buyers get to find a match, it cannot be less than $-k$, for otherwise, buyers would not have incentive to participate in the allocation. In other words, if the entry tax for a submarket is less than $-k$, then buyers pay the tax (basically get this subsidy) and make a positive profit net of the entry cost, $\tilde{t}_e + k$, and then do not participate in the matching stage which gives them a strictly negative payoff. Therefore, $k + \tilde{t}_e(z) \geq 0$ is another constraint for the construction of the optimal tax schedule that I take into account in the proof.

Corollary 1. *Take any constrained efficient mechanism (which is a direct mechanism). Then there exists an associated constrained efficient allocation (which is an implementable allocation) such that all types get exactly the same payoff as in the direct mechanism and the entry tax is decreasing and the sales tax is increasing in the price of assets.*

7 Conclusion

I have characterized the constrained efficient allocation in an environment with directed search and adverse selection. Under similar assumptions that GSW make to characterize the unique equilibrium, the planner can achieve strictly higher welfare than the equilibrium if the equilibrium fails to achieve the first best. Under a different assumption (Assumption 2), the planner can even achieve the first best. The main idea is that the planner tries to use transfers rather than market tightness or production level to have incentive constraints satisfied.

In the market economy, the buyers do not take into account the effect of their entry on the set of feasible submarkets available to buyers who want to attract other types of sellers. Entry of a buyer to a submarket changes the payoff of sellers in that submarket and this in turn, through incentive compatibility constraints, changes the set of feasible contracts that buyers can post in other submarkets and eventually changes the payoff of sellers in other submarkets. The planner takes this externality into account and therefore, he is able to increase welfare by imposing appropriate taxes and subsidies.

I illustrated my results in different examples. In an asset market example in Section 6, I showed that if the value of assets to sellers is increasing and the surplus created by assets is decreasing in the type of assets ($c' > 0$ and $h' - c' \leq 0$), then the planner can achieve the first best by subsidizing low price assets and taxing high prices ones in a large class of environments. The optimal tax schedule, however, is not generally monotone in the price of assets, e.g., when buyers and sellers do not agree on the ranking of assets (which happens if some sellers of high quality assets are financially distressed so they are in an urgent need to

sell their assets).

If directed search with adverse selection is a good framework to capture what happened in OTC markets during the recent financial crisis, as [Chang \(2012\)](#) and [Guerrieri and Shimer \(2014\)](#) use the same framework to analyze these markets, then my results imply that it is not an optimal policy to subsidize the purchase of all low price assets when there are fire sales in asset markets. That is, asset subsidy programs may have not been the best policy from a social point of view (although it may have increased liquidity of assets). Then I showed that if buyers are subject to two types of taxes, not only sales tax but also entry tax, then there exist monotone tax schedules, increasing sales tax and decreasing entry tax, which implement the constrained efficient mechanism.

I have assumed in this paper that agents match bilaterally. An important question is that if one considers a more general framework and allows several sellers to meet with a buyer so that sellers face some competition after meeting a buyer, whether it induces sellers to reveal their types less costly? And importantly, does the equilibrium remain constrained inefficient? In a work in progress, I study a similar environment but with many-on-one meetings. Buyers post mechanisms (which possibly depend on the number of sellers who will show up and on their reports) and commit to them. For example buyers might post second price auctions with reserved prices. I want to characterize both equilibrium and the constrained efficient allocation in such an environment. My conjecture is that the equilibrium will remain constrained inefficient. This is yet to be verified.

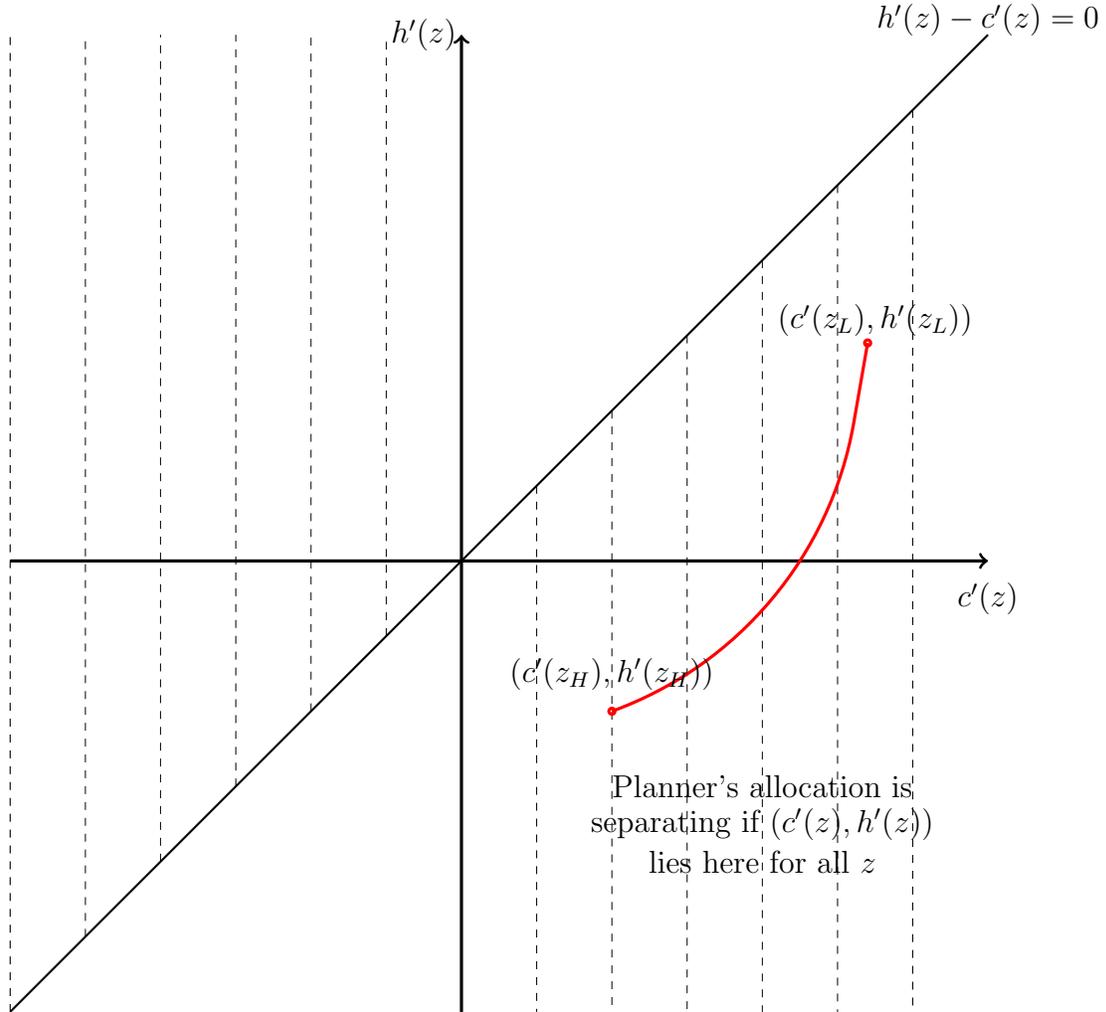


Figure 4: Assume that $h(z)$ has a strict local maximum but $h'(z) - c'(z) \leq 0$ for all z . Also assume that the distribution is such that part 2 of Assumption 5 (or Assumption 7 in Appendix) holds. Because the value of assets to sellers with higher z is not monotone in z , the equilibrium will involve some pooling. Chang (2012) shows this point formally in her Proposition 5. However, the planner's allocation is separating. The planner can actually achieve the first best according to Proposition 3. Symmetrically, if $c'(z) < 0$ and $h'(z) - c'(z) \geq 0$, and if a similar condition to part 2 of Assumption 5 or Assumption 7 holds, then the planner will get a separating allocation.

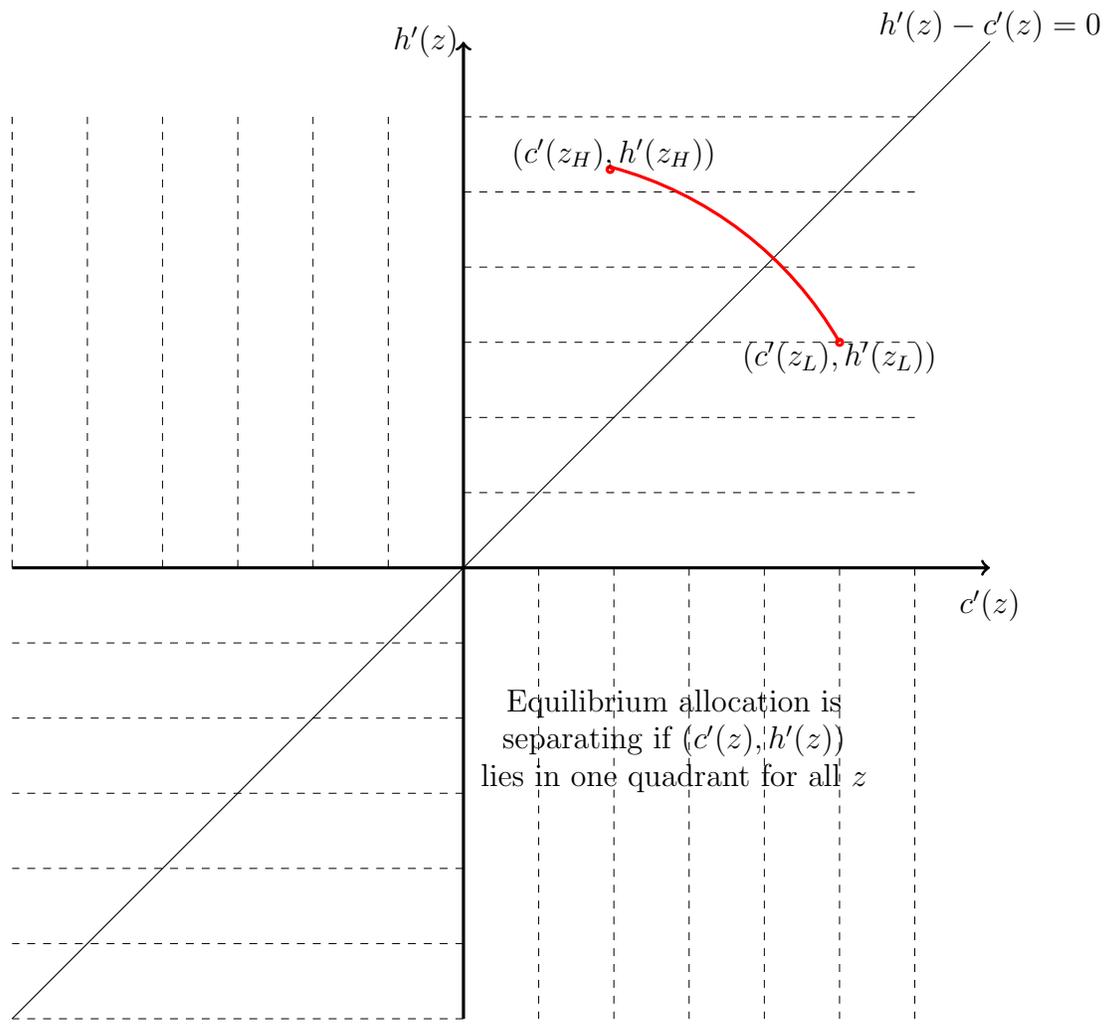


Figure 5: Here, $h - c$ has an interior local minimum and $h'(z) \geq 0$ for all z . Since $h'(z) \geq 0$ for all z , then the equilibrium allocation is separating. However, the planner's allocation is pooling, because monotonicity constraint is not satisfied. Indeed, the planner wants to pool all types higher than a threshold in one submarket.

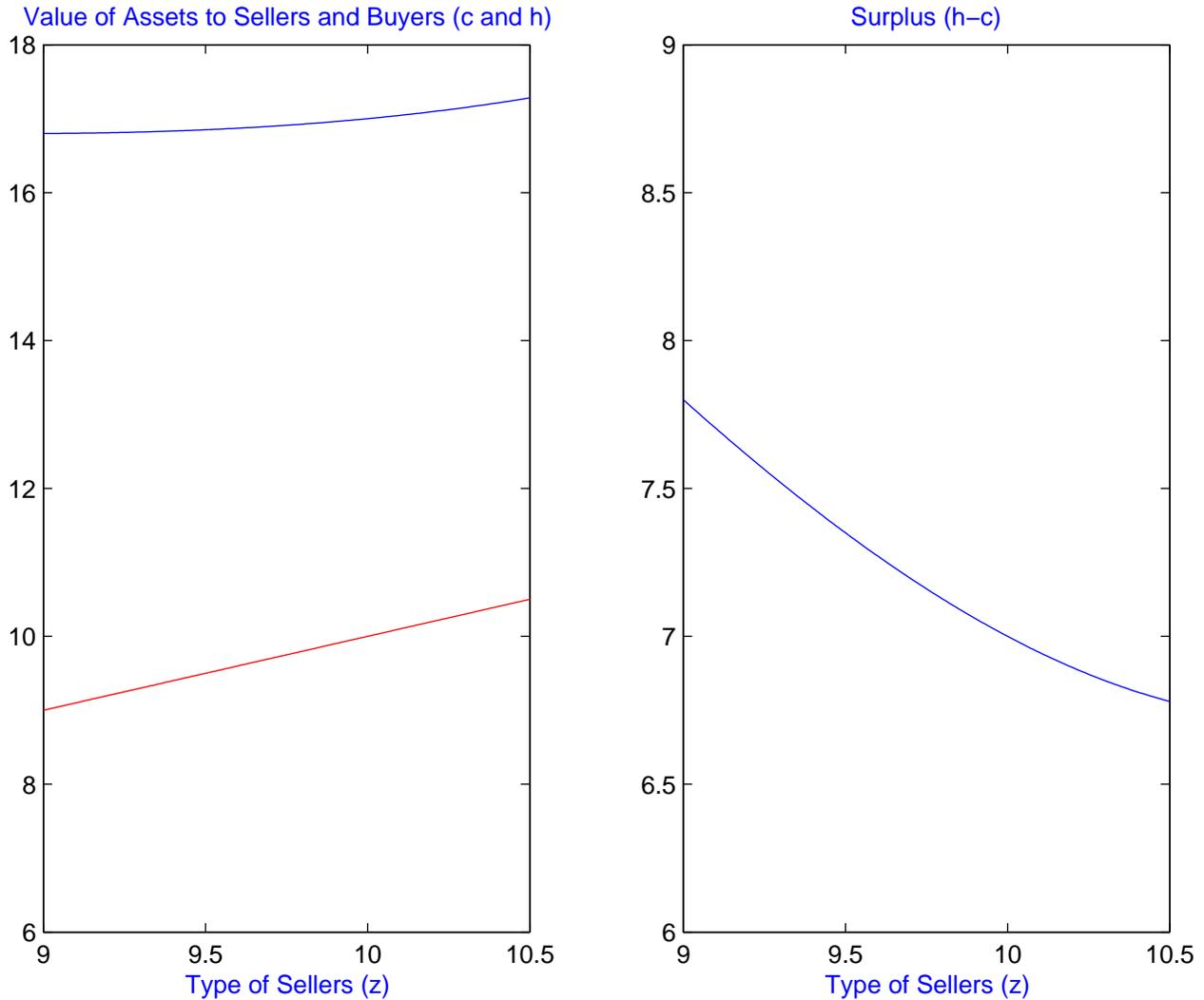


Figure 6: The payoff structure of different assets for the model parameters in Example 1 is depicted here. In the left graph, the value of type z asset to buyers, $h(z)$, (in blue) and the value of type z to sellers, $c(z)$, (in red) are depicted. Gains from trade, $h(z) - c(z)$, is depicted in the right graph.

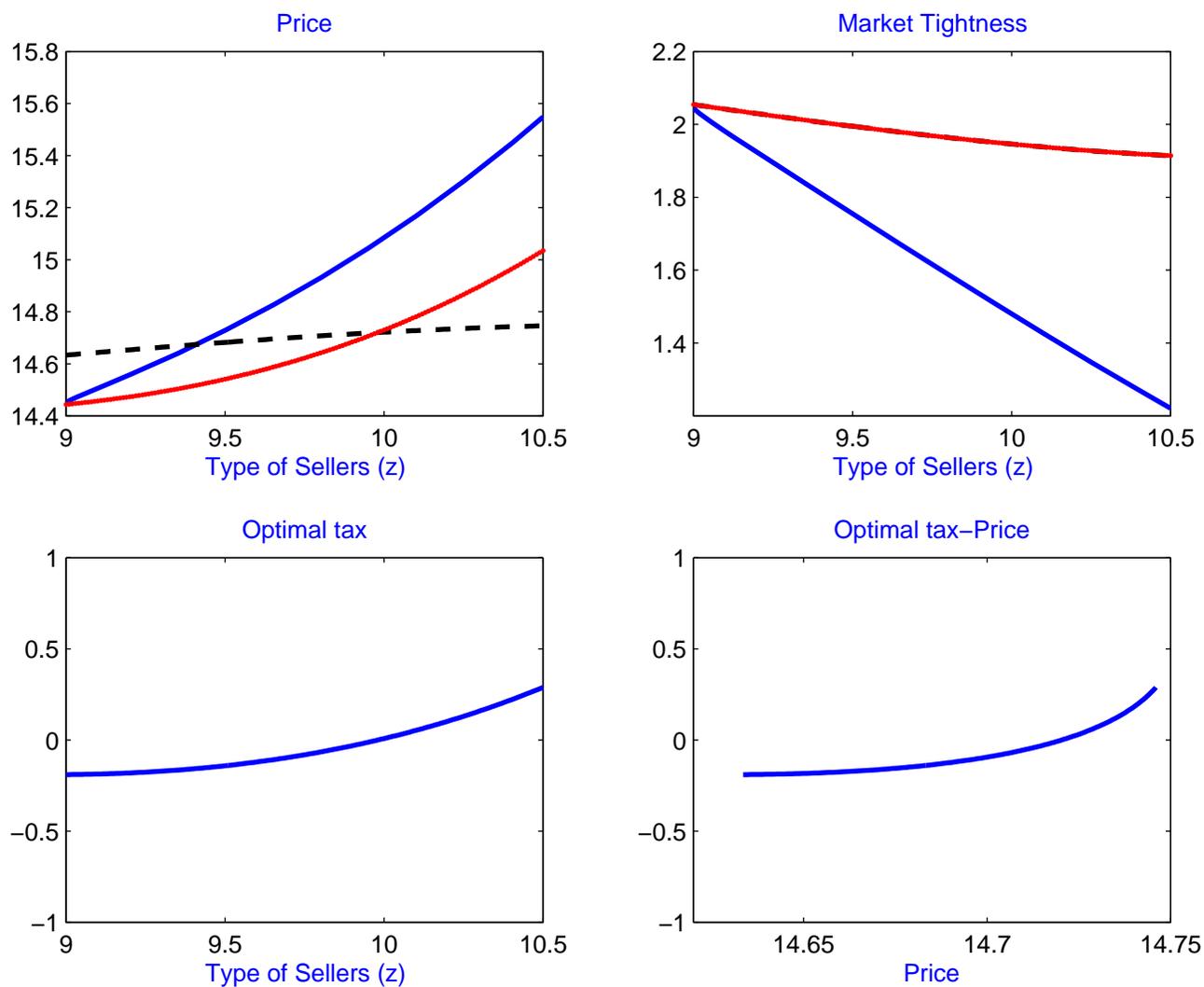


Figure 7: Model parameters are defined in Example 1. In the upper left graph, the price that sellers get in FB (in green), in CE (in red) and in EQ (in dashed blue) are depicted. In the upper right graph, market tightness for each type in FB and CE (in green) and in EQ (in blue) are depicted. In the lower left graph, the optimal level of submarket-specific taxes that buyers should pay, $\tilde{t}^{CE}(z)$, is depicted in terms of z . In the lower right graph $t(z)$ is depicted in terms of the price that sellers get, $\tilde{p}^{CE}(z)$. It is observed here that the efficient tax schedule is monotone in the type or price of assets.

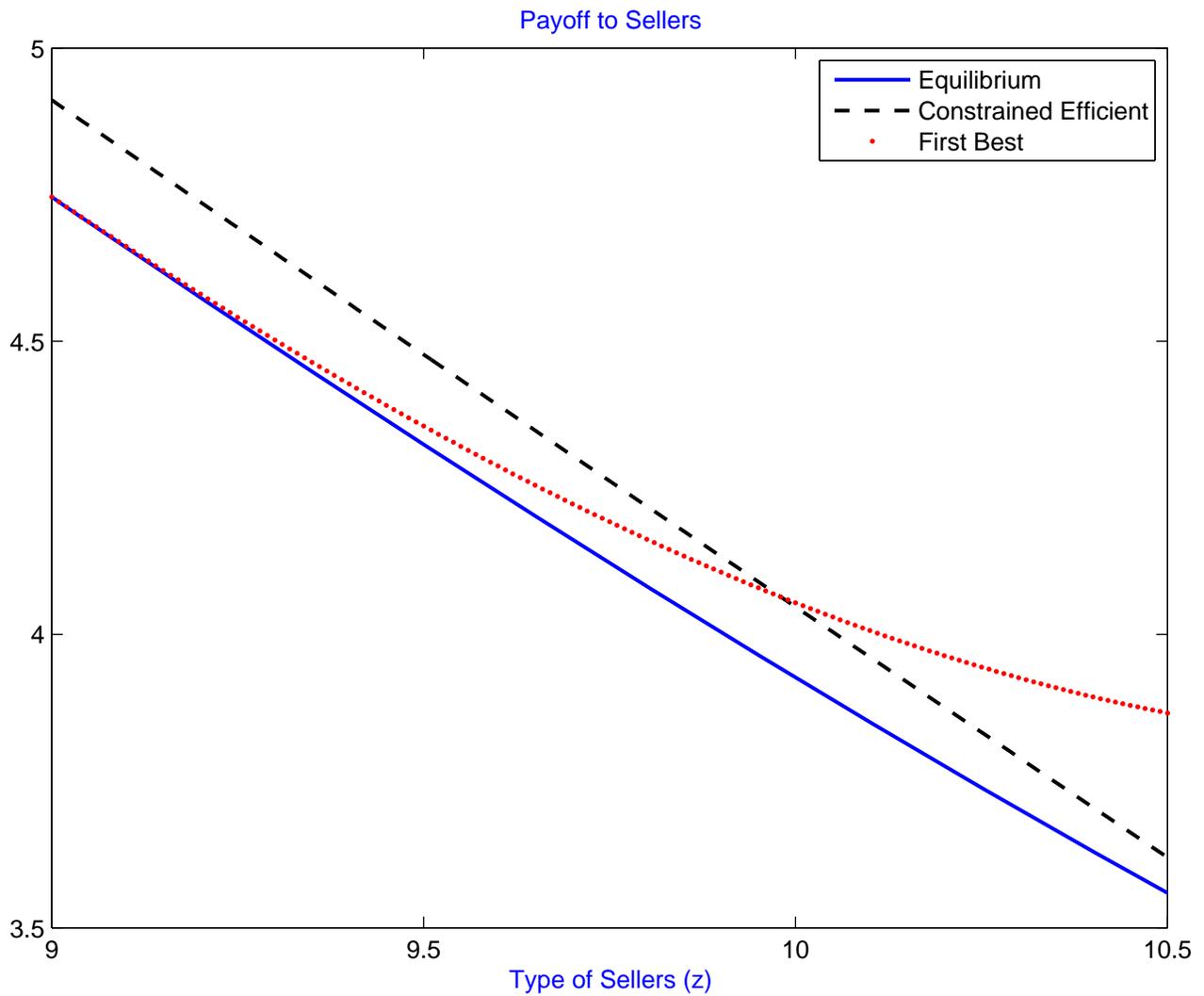


Figure 8: Model parameters are defined in Example 1. The expected payoff to sellers in FB (in green), in CE (in dotted red) and in EQ (in dashed blue) are depicted. CE Pareto dominates EQ in this example.

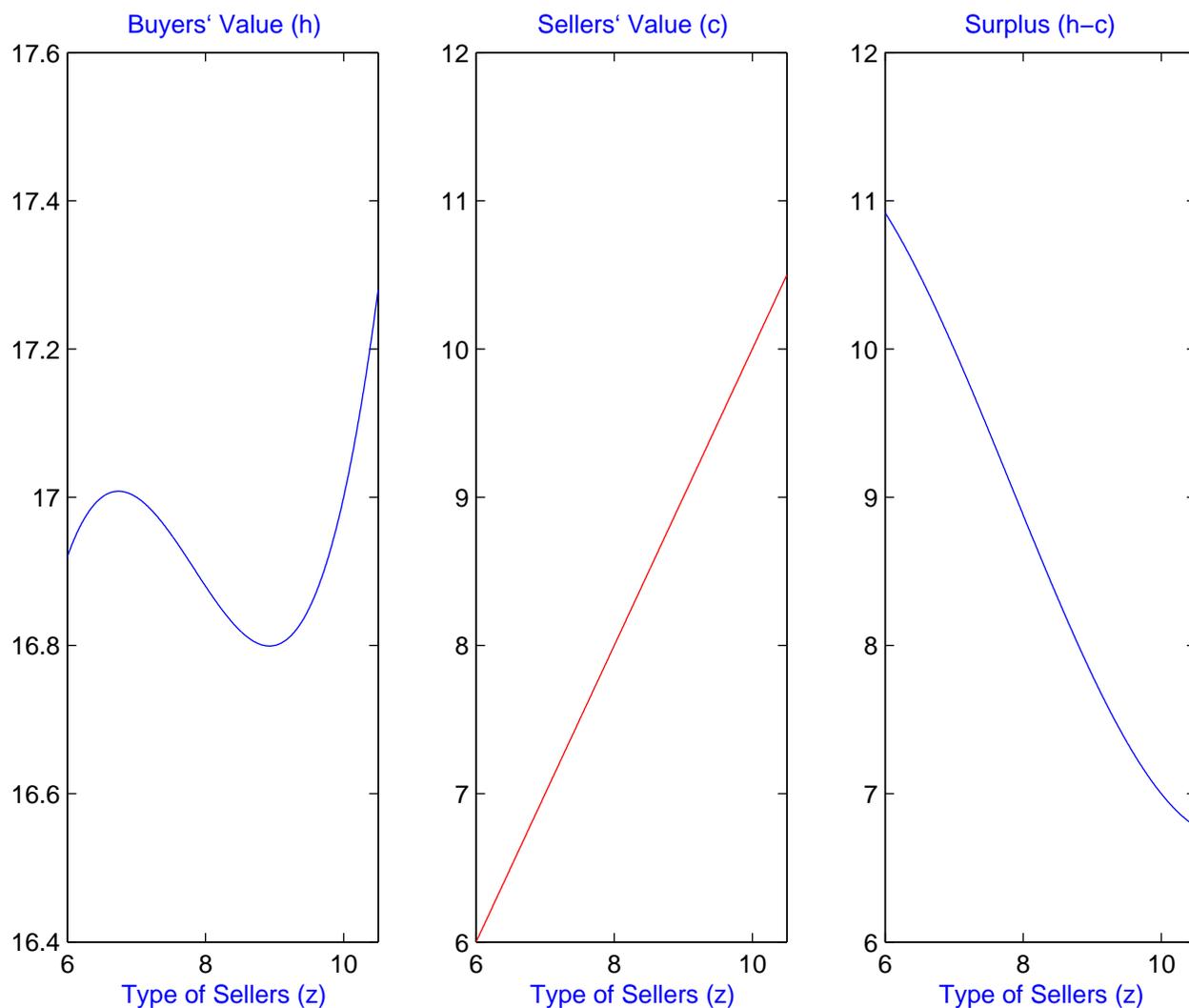


Figure 9: The payoff structure of different assets for the model parameters in Example 2 is depicted here. In the left graph, the value of type z asset to buyers, $h(z)$, in the middle graph the value of type z asset to sellers, $c(z)$, and in the right graph the gains from trade, $h(z) - c(z)$, are depicted.

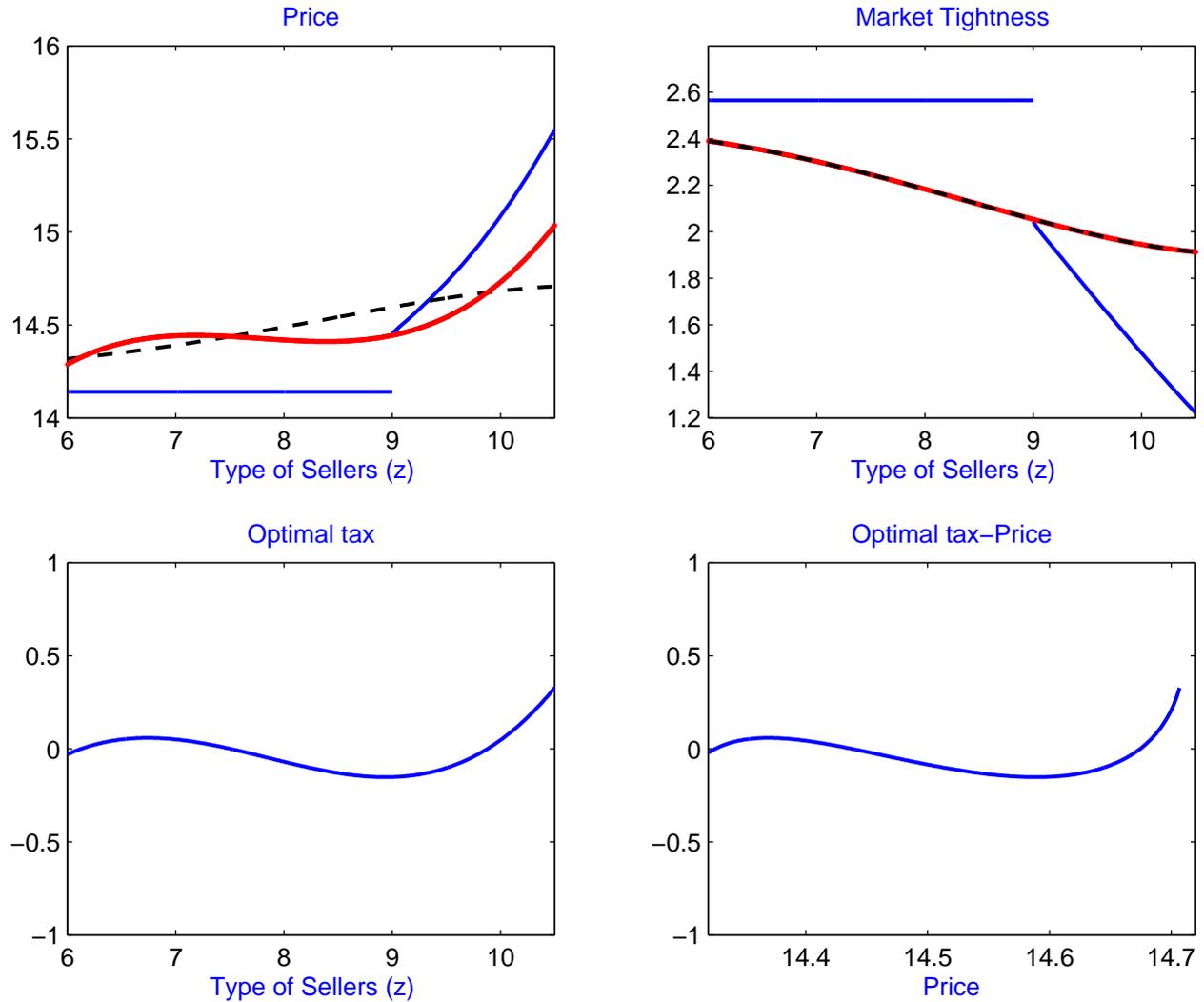


Figure 10: This figure is similar to Figure 7 but with model parameters defined in Example 2. In the upper left graph, the price that sellers get in FB (in green), in CE (in dashed red) and in EQ (in blue) are depicted. In the upper right graph, market tightness for each type in FB and CE (in green) and in EQ (in blue) are depicted. In the lower left graph $t(z)$ is depicted in terms of z and in the lower right graph $\tilde{t}^{CE}(z)$ is depicted in terms of $\tilde{p}^{CE}(z)$. The efficient tax schedule is non-monotone in the type or price of assets.

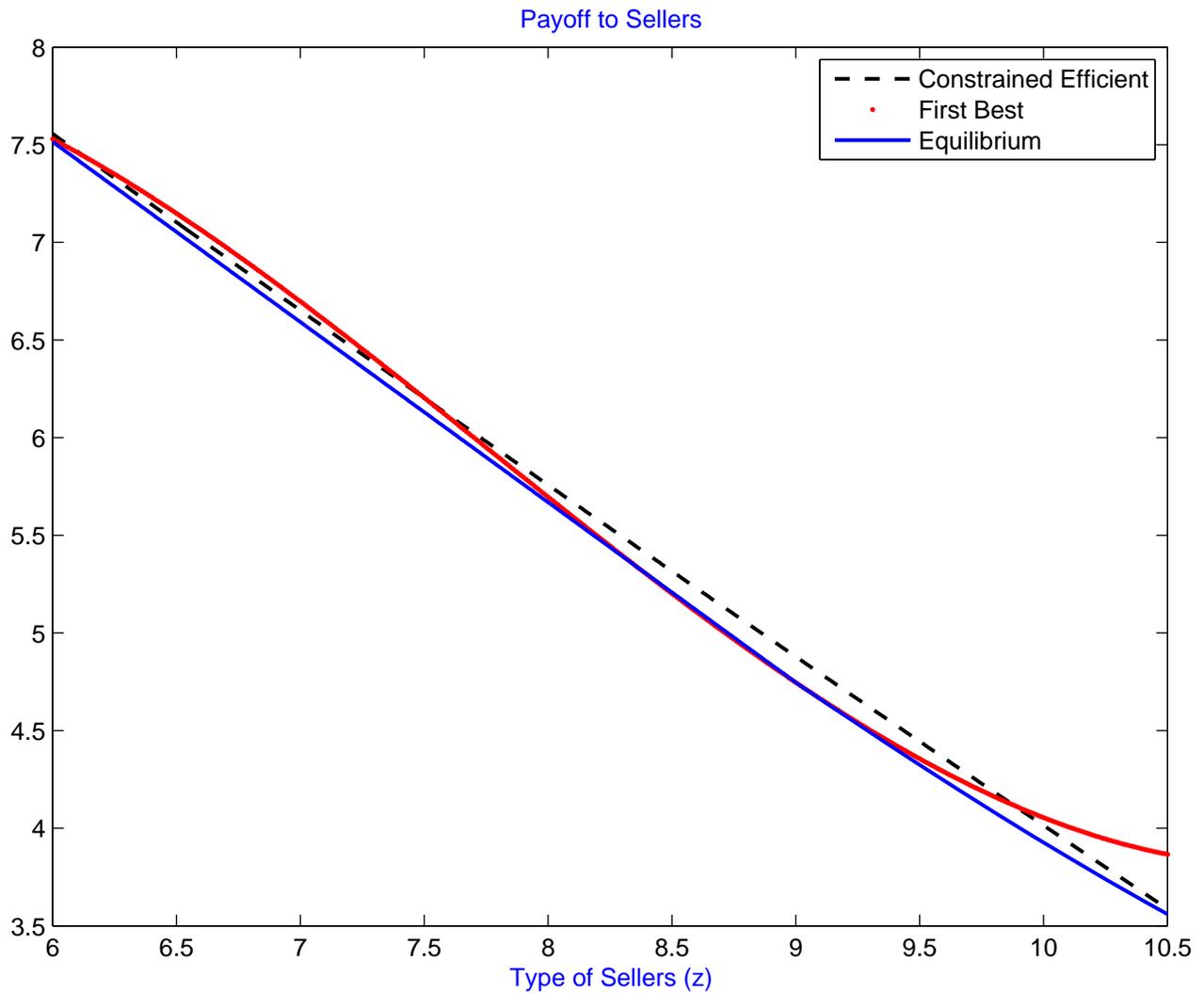


Figure 11: This figure is similar to one in Figure 8 but with model parameters defined in Example 2. The expected payoff to sellers in FB (in green), in CE (in dashed red) and in EQ (in blue) are depicted. CE Pareto dominates EQ in this example, too.

8 Appendix

8.1 Direct Mechanism: Definitions and Proofs

I take a constrained efficient mechanism. I show by construction that there exists a constrained efficient allocation associated with the direct mechanism which delivers the same welfare. Before I get to the results, note that the budget constraint in the constrained efficient mechanism is always binding. Otherwise, one can increase all \tilde{s}_i by an identical small amount. Then all other conditions continue to be met, but welfare strictly increases.

Proof. Proof of Lemma 1

Given a constrained efficient mechanism $\{(\tilde{a}_i, \tilde{p}_i, \tilde{s}_i, \tilde{\theta}_i)\}_{i \in \{1, 2, \dots, I\}}$, I construct an implementable allocation which delivers the same welfare. First, define N to be the set of all types who get matched with strictly positive probability in the direct mechanism, that is, $N \equiv \{i | \tilde{\theta}_i > 0\}$. Second, set

$$T = \begin{cases} \tilde{s}_i & \text{if } \exists i \text{ such that } \tilde{\theta}_i = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Third, for all $i \in N$ construct y_{k_i} as follows:

$$y_{k_i} = (\tilde{a}_i, \tilde{p}_i + \frac{\tilde{s}_i - T}{m(\tilde{\theta}_i)}).$$

and also set $t(y_{k_i}) = v_i(\tilde{a}_i) - \tilde{p}_i - \frac{\tilde{s}_i - T}{m(\tilde{\theta}_i)} - \frac{k}{q(\tilde{\theta}_i)}$ and

$$Y^P = \{y_{k_i} | i \in N\}, \theta(y_{k_i}) = \tilde{\theta}_i, \gamma_i(y_{k_i}) = 1, \lambda(\{y_{k_i}\}) = \pi_i \tilde{\theta}_i, \quad (8)$$

For any other submarket $y \notin Y^P$, define $K(y) = \{j | u_j(a) + p > 0\}$ to denote the types which get a strictly positive payoff by applying to to y . If $K(y) \neq \emptyset$ and $\min_{j \in K(y)} \{\frac{U_j}{u_j(a) + p}\} < \bar{m} \equiv \lim_{\theta \rightarrow \infty} m(\theta)$, then set $\theta(y)$ such that

$$m(\theta(y)) = \min_{j \in K(y)} \left\{ \frac{U_j}{u_j(a) + p} \right\}.$$

If the latter equation holds for several $\theta(y)$, then pick the greatest one. If $K(y) = \emptyset$, or $\min_{j \in K(y)} \{\frac{U_j}{u_j(a) + p}\} \geq \bar{m}$, then set $\theta(y) = \infty$. To define the composition function for $y \notin Y^P$, define $n = \min\{\arg \min_{j \in K(y)} \{\frac{U_j}{u_j(a) + p}\}\}$ and set $\gamma_n(y) = 1$. If $K(y) = \emptyset$, then $\Gamma(y)$ can be chosen arbitrarily, so for example set $\gamma_1(y) = 1$. Also for $y \notin Y^P$, set $t(y) = \max_{i,a} v_i(a) - p$.

T is well-defined, because if there are more than one i with $\tilde{\theta}_i = 0$, then \tilde{s}_i must be the same for all of them, for otherwise, sellers' incentive compatibility constraint in the definition

of feasible mechanism is violated. All y_{k_i} are also well-defined, because $\tilde{\theta}_i$ cannot be equal to 0. Moreover, $t(y)$ is well-defined too, since no $\tilde{\theta}_i$ can be equal to ∞ . If $\tilde{\theta}_i$ goes to ∞ for some i , then the planner's budget-balance condition will be violated, because the planner needs to spend infinite amount of resources to finance entry of buyers (the left hand side of the planner's budget constraint goes to $-\infty$).

If there exist i and j ($i \neq j$) such that $y_{k_i} = y_{k_j}$, I show below that $m(\tilde{\theta}_i) = m(\tilde{\theta}_j)$. Assume without loss of generality that $\tilde{\theta}_i \leq \tilde{\theta}_j$. Then I keep y_{k_i} and remove y_{k_j} and let $\gamma_i(y_{k_i}) = \frac{\pi_i}{\pi_i + \pi_j}$, $\gamma_j(y_{k_i}) = \frac{\pi_j}{\pi_i + \pi_j}$, $\lambda(\{y_{k_i}\}) = (\pi_i + \pi_j)\tilde{\theta}_i$ and $t(y_{k_i}) = \frac{\pi_i v_i(\tilde{a}_i) + \pi_j v_j(\tilde{a}_i)}{\pi_i + \pi_j} - \tilde{p}_i - \frac{\tilde{s}_i - T}{m(\tilde{\theta}_i)} - \frac{k}{q(\tilde{\theta}_i)}$. Now I show that $m(\tilde{\theta}_i) = m(\tilde{\theta}_j)$.

According to sellers' incentive compatibility constraint²⁹ (for type i to report j), one can write: $m(\tilde{\theta}_i)(u_i(\tilde{a}_i) + \tilde{p}_i) + \tilde{s}_i \geq m(\tilde{\theta}_j)(u_i(\tilde{a}_j) + \tilde{p}_j) + \tilde{s}_j = m(\tilde{\theta}_j)(u_i(\tilde{a}_j) + \tilde{p}_j + \frac{\tilde{s}_j - T}{m(\tilde{\theta}_j)}) + T = m(\tilde{\theta}_j)(u_i(\tilde{a}_i) + \tilde{p}_i + \frac{\tilde{s}_i - T}{m(\tilde{\theta}_i)}) + T$ where the second equality follows from the assumption that $y_{k_i} = y_{k_j}$. This implies that $(m(\tilde{\theta}_i) - m(\tilde{\theta}_j))(u_i(\tilde{a}_i) + \tilde{p}_i + \frac{\tilde{s}_i - T}{m(\tilde{\theta}_i)}) \geq 0$. If $\tilde{\theta}_k = 0$ for all k , so $T = 0$ by construction, and due to the participation constraint for type i , $m(\tilde{\theta}_i)(u_i(\tilde{a}_i) + \tilde{p}_i) + \tilde{s}_i \geq 0$. Therefore, $m(\tilde{\theta}_i) \geq m(\tilde{\theta}_j)$. Now assume that there exists k such that $\tilde{\theta}_k = 0$. IC constraint for type i to report k implies that $m(\tilde{\theta}_i)(u_i(\tilde{a}_i) + \tilde{p}_i) + \tilde{s}_i \geq \tilde{s}_k = T$ for all i . Therefore, whether there exists k with $\tilde{\theta}_k = 0$ or not, $u_i(\tilde{a}_i) + \tilde{p}_i + \frac{\tilde{s}_i - T}{m(\tilde{\theta}_i)} \geq 0$. Thus, $m(\tilde{\theta}_i) \geq m(\tilde{\theta}_j)$. Similarly by considering the sellers' incentive compatibility constraint for j to report i , we can get $m(\tilde{\theta}_j) \geq m(\tilde{\theta}_i)$. Therefore, $m(\tilde{\theta}_j) = m(\tilde{\theta}_i)$.

The proposed allocation is implementable because of the following reasons. Regarding **sellers' maximization** condition, I first show that $U_i \geq m(\theta(y))(u_i(a) + p) + T$ for all $y \in Y^P$ and $i \in N$. We know that if $y \in Y^P$, then there exists $i \in N$ such that $y = y_{k_i}$. Therefore I need to show that

$$m(\theta_{k_i})(u_i(a_{k_i}) + p_{k_i}) \geq m(\theta_{k_j})(u_i(a_{k_j}) + p_{k_j}) \text{ for } j \in N, \quad (9)$$

and

$$m(\theta_{k_i})(u_i(a_{k_i}) + p_{k_i}) \geq T. \quad (10)$$

²⁹I stated in the main body of the paper that if one allows the planner to use direct mechanism with lotteries and randomization, then there might be some loss of generality in formulating the problem as formulated in the definition of constrained efficient allocation. This is exactly where we can see why. If the planner uses randomization, then incentive compatibility holds not necessarily for each $(\tilde{a}_i, \tilde{p}_i, \tilde{s}_i, \tilde{\theta}_i)$, but holds in expectation. Therefore, if there are two y_{k_i} and y_{k_j} which are equal and types i and j are allocated to them with positive probability, it might be the case that θ_{k_i} and θ_{k_j} are not equal, thus we cannot construct an implementable allocation from that direct mechanism. As stated earlier, the planner might want to use lotteries if his objective function is not concave or his constraint set is not convex.

To show the above inequalities, note that

$$m(\theta_{k_i})(u_i(a_{k_i}) + p_{k_i}) = m(\tilde{\theta}_i)(u_i(\tilde{a}_i) + \tilde{p}_i) + \tilde{s}_i - T \geq m(\tilde{\theta}_i)(u_i(\tilde{a}_j) + \tilde{p}_j) + \tilde{s}_j - T$$

for all j . The equality follows from the construction of y_{k_i} . The inequality follows from the incentive compatibility condition in the direct mechanism. The right hand side equals to 0 if $j \notin N$ and equals to $m(\theta_{k_j})(u_i(a_{k_j}) + p_{k_j})$ if $j \in N$ due to the construction of y_{k_j} . Thus, Equation 9 is established. For Equation 10, if there exists j such that $j \notin N$, then $\tilde{\theta}_j = 0$, $\tilde{s}_j = T$ and the equation is established. If $N = \{1, 2, \dots, I\}$, then $T = 0$ and $m(\theta_{k_i})(u_i(a_{k_i}) + p_{k_i}) \geq 0$ due to the participation constraint in the direct mechanism.

Now I show that $U_i \geq m(\theta(y))(u_i(a) + p) + T$ for all $y \in Y^P$ and $i \notin N$, so I need to show that

$$m(\theta_{k_j})(u_i(a_{k_j}) + p_{k_j}) + T \leq T \text{ for all } j \in N.$$

But

$$m(\theta_{k_j})(u_i(a_{k_j}) + p_{k_j}) + T = m(\tilde{\theta}_j)(u_i(\tilde{a}_j) + \tilde{p}_j) + \tilde{s}_j \leq m(\tilde{\theta}_i)(u_i(\tilde{a}_i) + \tilde{p}_i) + \tilde{s}_i = \tilde{s}_i = T$$

for all $j \in N$. The first equality follows from the construction of y_{k_j} and $j \in N$. The inequality follows from the incentive compatibility condition in the direct mechanism. The next equality holds because $\tilde{\theta}_i = 0$ since $i \notin N$. The last equality holds due to the construction of T and $i \notin N$.

Now I show that condition (ii) is satisfied. By construction of $\theta(\cdot)$ and $\Gamma(\cdot)$ and as shown above, $U_i \geq m(\theta(y))(u_i(a) + p) + T$ for all $y \in Y^P$ with equality if $\theta(y) < \infty$ and $\gamma_i(y) > 0$. The inequality also holds for $y \notin Y^P$ due to the construction of $\theta(\cdot)$ and $\Gamma(\cdot)$. Also by construction of N , for any $i \in N$, $m(\theta(y))(u_i(a) + p) > 0$. Given y , if $u_i(a) + p < 0$ for some i , then $i \notin K(y)$. Thus, if $K(y) = \emptyset$, then $\theta(y) = \infty$. If $K(y) \neq \emptyset$, then $\gamma_n(y) = 1$ for some $n \in K(y)$, therefore $\gamma_i(y) = 1$.

Buyers' profit maximization and free entry condition holds due to the following reasons. Consider first $y \in Y^P$. Remember that for $y \in Y^P$, there exists $i \in N$ such that $y = y_{k_i}$. But for all $i \in N$, $q(\theta_{k_i})(v_i(a_{k_i}) - p_{k_i} - t_{k_i}) - T = k$. Therefore, the buyers' profit maximization and free entry condition holds with equality for $y \in Y^P$. Now consider $y \notin Y^P$. Then $q(\theta(y))\Sigma\gamma_i(y)(v_i(a) - p) - t(y) < k$ due to the choice of $t(y)$. Therefore condition (i) is satisfied³⁰.

Feasibility condition is obviously satisfied following the construction of λ .

³⁰It is immediately clear from this step of the proof that the restrictions on off-the-equilibrium-path beliefs do not play any role in our analysis. That is, any other off-the-equilibrium-path beliefs would work with the taxes that we chose. This is because the planner does not face any restriction on the tax amount that he can impose.

Planner's budget constraint is satisfied because:

$$\begin{aligned}
\int q(\theta)td\lambda(y) - T &\geq \sum_{i \in N} q(\tilde{\theta}_i)(v_i(\tilde{a}_i) - p_i - \frac{\tilde{s}_i - T}{m(\tilde{\theta}_i)} - \frac{k}{q(\tilde{\theta}_i)})\pi_i\tilde{\theta}_i - T \\
&= \sum_{i \in N} q(\tilde{\theta}_i)(v_i(\tilde{a}_i) - p_i - \frac{\tilde{s}_i}{m(\tilde{\theta}_i)} - \frac{k}{q(\tilde{\theta}_i)})\pi_i\tilde{\theta}_i - \sum_{i \notin N} T \\
&= \sum_{i \in N} [m(\tilde{\theta}_i)(v_i(\tilde{a}_i) - p_i) - \tilde{s}_i - k\tilde{\theta}_i]\pi_i + \sum_{i \notin N} [m(\tilde{\theta}_i)(v_i(\tilde{a}_i) - p_i) - \tilde{s}_i - k\tilde{\theta}_i]\pi_i \geq 0.
\end{aligned}$$

The first inequality holds due to the construction of implementable allocation and due to the following reason. As mentioned earlier, it might be the case that several types have the same y_{k_i} . I showed that $m(\theta_{k_i}) = m(\theta_{k_j})$ and if $\theta_{k_i} \neq \theta_{k_j}$, then I chose the lowest one. The first inequality follows. The second equality holds due to the fact that for $i \notin N$, $\tilde{\theta}_i = 0$ and $\tilde{s}_i = T$. The last inequality holds due to the budget-balance condition in the definition of feasible mechanism.

8.2 Proof of Theorem 1

Proof of Theorem 1. I begin from equilibrium allocation and modify it to improve welfare. Consider a type i seller who gets a strictly positive payoff in the equilibrium and does not produce a_i^{FB} or does not get matched with probability $m(\theta_i^{FB})$. Such a type exists because the equilibrium fails to achieve the first best. Let \bar{i} denote this type. I define a set of problems, similar but not the same as one in GSW, and characterize its solution. From that solution, I construct a feasible mechanism and show that it yields higher welfare for the planner than the equilibrium allocation.

According to Proposition 3 in their paper, GSW show that the following set of problems characterizes the equilibrium.

Problem 1 ($P_i(0)$).

$$\max_{\theta \in [0, \infty], (a, p) \in \bar{Y}} \{m(\theta)(u_i(a) + p)\}$$

subject to

$$q(\theta)(v_i(a) - p) \geq k,$$

$$m(\theta)(u_j(a) + p) \leq \bar{U}_j(0) \text{ for all } j < i.$$

More precisely, define problem $P(0)$ to be the set of problems $P_i(0)$ for all i . Let $\bar{U}_i(0)$ be the value of the objective function in problem $P_i(0)$ given $(\bar{U}_1(0), \bar{U}_2(0), \dots, \bar{U}_{i-1}(0))$ if $\bar{U}_i(0)$ is strictly greater than 0 and $\bar{U}_i(0) = 0$ otherwise. Denote by $I^*(0) \subseteq \{1, 2, \dots, I\}$ the set of types such that the constraint set in $P_i(0)$ is non-empty and $\bar{U}_i(0)$ is strictly greater

than 0. For any $i \in I^*(0)$, let $(\bar{\theta}_i(0), \bar{a}_i(0), \bar{p}_i(0))$ denote the solution to problem $P_i(0)$ given $(\bar{U}_1(0), \bar{U}_2(0), \dots, \bar{U}_{i-1}(0))$. Now, consider another set of problems which is basically a perturbation of the above problem in a specific way. We will see why all variables in the above and below problems are written as functions of 0 and ϵ , respectively.

Problem 2 ($P_i(\epsilon), \epsilon > 0$).

$$\max_{\theta \in [0, \infty], (p, a) \in \bar{Y}} \{m(\theta)(u_i(a) + p) + \delta_i\}$$

subject to

$$q(\theta)(v_i(a) - p) \geq k,$$

and

$$m(\theta)(u_j(a) + p) + \delta_i \leq \bar{U}_j(\epsilon) \text{ for all } j < i,$$

$$\text{where } \delta_i = \begin{cases} \epsilon & \text{if } i < \bar{i} \text{ or } i \notin I^*(0) \\ 0 & \text{otherwise} \end{cases}.$$

Similarly as above, define problem $P(\epsilon)$ to be the set of problems $P_i(\epsilon)$ for all i . Let $\bar{U}_i(\epsilon)$ be the value of the objective function in problem $P_i(\epsilon)$ given $(\bar{U}_1(\epsilon), \bar{U}_2(\epsilon), \dots, \bar{U}_{i-1}(\epsilon))$ if $\bar{U}_i(\epsilon)$ is strictly greater than ϵ and $\bar{U}_i(\epsilon) = \epsilon$ otherwise. Denote by $I^*(\epsilon) \subseteq \{1, 2, \dots, I\}$ the set of types such that the constraint in $P_i(\epsilon)$ is non-empty and $\bar{U}_i(\epsilon)$ is strictly greater than ϵ . For any $i \in I^*(\epsilon)$, let $(\bar{\theta}_i(\epsilon), \bar{a}_i(\epsilon), \bar{p}_i(\epsilon))$ denote the solution to problem $P_i(\epsilon)$ given $(\bar{U}_1(\epsilon), \bar{U}_2(\epsilon), \dots, \bar{U}_{i-1}(\epsilon))$.

Since the constraint of $P_i(\epsilon)$ is exactly the same as $P_i(0)$ for types below \bar{i} , they get exactly $\bar{U}_i(\epsilon) = \bar{U}_i(0) + \epsilon$. (It is easy to see it by induction.) Also for types above \bar{i} who are non-participant, that is, $i > \bar{i}$ and $i \notin I^*(0)$, then $\bar{U}_i(\epsilon) = \bar{U}_i(0) + \epsilon = \epsilon$.

For type \bar{i} , problem $P_i(\epsilon)$ maximizes the objective function given $(\bar{U}_1(\epsilon), \bar{U}_2(\epsilon), \dots, \bar{U}_{i-1}(\epsilon))$. Since type \bar{i} does not achieve the first best in equilibrium, some constraints must be binding at $P_i(0)$, thus Problem $P_i(\epsilon)$ yields strictly higher value of the objective function than problem $P_i(0)$, because those constraints are now relaxed. To elaborate, according to Lemma 2, the first constraint is always binding so we can eliminate p from the problem and rewrite the problem as follows:

$$\max_{\theta \in [0, \infty], (., a) \in \bar{Y}} \{m(\theta)(u_i(a) + v_i(a)) - k\theta\}$$

$$\text{subject to } m(\theta)(u_j(a) + v_i(a)) - k\theta \leq \bar{U}_j(0) \text{ for all } j < i.$$

Due to the assumption that $m(\theta)(u_i(a) + v_i(a)) - k\theta$ has a single peak, locally relaxing the constraint improves welfare if the solution is not at the peak.

For types above \bar{i} who are active in the equilibrium, the objective function is weakly higher than the equilibrium allocation. I show the latter claim by induction. Fix $j > \bar{i}$ and assume that for all k such that $\bar{i} \leq k < j$, then $\bar{U}_k(\epsilon) \geq \bar{U}_k(0)$, that is, the value of the objective functions is higher than that in the equilibrium. This implies that the constraint set in problem $P_j(\epsilon)$ is bigger for all k with $\bar{i} \leq k < j$. On the other hand for $k < \bar{i}$, we already know that the constraint set is bigger by definition of $P_k(\epsilon)$. Hence, the value of the objective function in $P_j(\epsilon)$ should be weakly higher than that in equilibrium.

To summarize, so far I have proved the following:

$$\begin{aligned} \bar{U}_i(\epsilon) &= \bar{U}_i(0) + \epsilon \text{ for all } i < \bar{i} \text{ or } i \notin I^*(0), \\ \bar{U}_{\bar{i}}(\epsilon) &> \bar{U}_{\bar{i}}(0), \\ \text{and } \bar{U}_i(\epsilon) &\geq \bar{U}_i(0) \text{ for all } i > \bar{i} \text{ and } i \in I^*(0). \end{aligned} \quad (11)$$

Set $\epsilon_1 = \min\{U_i(0) | i \geq \bar{i}, i \in I^*(0)\}$. In Lemma 2, I will derive another upper bound for ϵ called ϵ_2 . Let ϵ be in $(0, \min(\epsilon_1, \epsilon_2))$. For example fix

$$\epsilon = \frac{1}{2} \min(\epsilon_1, \epsilon_2). \quad (12)$$

For this ϵ , all types who participate in the allocation ($i \in I^*(0)$) get a strictly positive payoff in the solution to Problem $P_i(\epsilon)$.

Now I propose the following direct mechanism.

$$(\tilde{a}_i, \tilde{p}_i, \tilde{s}_i, \theta_i) = \begin{cases} (\bar{a}_i(0), \bar{p}_i(0), \epsilon - \tilde{\epsilon}, \theta_i(0)) & \text{if } 1 \leq i < \bar{i} \text{ and } i \in I^*(0) \\ (\bar{a}_i(\epsilon), \bar{p}_i(\epsilon) - \frac{\epsilon}{m(\theta_i(\epsilon))}, \epsilon - \tilde{\epsilon}, \theta_i(\epsilon)) & \text{if } \bar{i} \leq i \leq I \text{ and } i \in I^*(0) \\ (\bar{a}_1(0), \bar{p}_1(0), \epsilon - \tilde{\epsilon}, 0) & \text{if } i \notin I^*(0), \end{cases} \quad (13)$$

where $\tilde{\epsilon} \equiv \epsilon(\sum_{i=1}^{\bar{i}-1} \pi_i + \sum_{i=\bar{i}+1, i \notin I^*(0)}^I \pi_i)$. Note that for $i \notin I^*(0)$, \tilde{a}_i and \tilde{p}_i are arbitrary, because θ_i is set to be 0.

It is important to note that $I^*(0) = I^*(\epsilon)$ due to the following reasons. Regarding types below \bar{i} , since they just get lump sum transfers which is common across all types, their incentives to participate or not does not change. Therefore, if their payoff in the equilibrium is less than 0 so they do not apply to any submarket, they remain inactive also under the proposed allocation.

For types \bar{i} and above, if they participate in equilibrium, they want also to participate in the new allocation due to the choice of ϵ . Now, suppose they do not participate in equilibrium. According to the assumption that we made that all types with positive gains from trade will

be active in equilibrium, their non-participation in the equilibrium means that they could not generate a strictly positive payoff. Therefore, relaxing constraints do not help them to generate a strictly positive payoff.

I have made this assumption that all types who get a strictly positive payoff in the complete information case will get a strictly positive payoff in the equilibrium. If there are positive gains from trade for all types, then all types will get a strictly positive payoff in the equilibrium (according to Proposition 4 in GSW). Therefore, the above assumption will be automatically satisfied.

I show below that allocation $\{(\tilde{a}_i, \tilde{p}_i, \tilde{s}_i, \theta_i)\}_{i \in \{1, 2, \dots, I\}}$ is feasible and yields strictly higher welfare than the equilibrium allocation.

Incentive Compatibility of Sellers

$$m(\theta_i)(u_i(\tilde{a}_i) + \tilde{p}_i) + \tilde{s}_i = \begin{cases} m(\bar{\theta}_i(0))(u_i(\bar{a}_i(0)) + \bar{p}_i(0)) + \epsilon - \tilde{\epsilon} & \text{if } 1 \leq i < \bar{i} \text{ and } i \in I^*(0) \\ m(\bar{\theta}_i(\epsilon))(u_i(\bar{a}_i(\epsilon)) + \bar{p}_i(\epsilon)) - \tilde{\epsilon} & \text{if } \bar{i} \leq i \leq I \text{ and } i \in I^*(0) \\ \epsilon - \tilde{\epsilon} & \text{if } i \notin I^*(0), \end{cases}$$

Note that the market tightness and production level that are allocated to agents are the same as those in Problem $P_i(\epsilon)$. The only difference is that all types now get $\tilde{\epsilon}$ less compared to their payoff in Problem $P_i(\epsilon)$. Since all types are taxed by the identical amount of $\tilde{\epsilon}$, incentives are not affected, therefore, in order to check for incentive compatibility in the proposed allocation, we can check whether incentive compatibility holds in Problem $P_i(\epsilon)$ or not.

By construction, all upward IC hold, because they are explicitly taken into account as constraints of Problem $P_i(\epsilon)$. Moreover, in Lemma 2 below, I show that if $\epsilon < \bar{\epsilon}_2$, then all downward IC are also satisfied. Therefore, incentive compatibility of sellers is satisfied.

Participation Constraint of Sellers

The payoff to sellers is summarized as follows:

$$\begin{cases} U_i(0) + \epsilon - \tilde{\epsilon} & \text{if } 1 \leq i < \bar{i} \text{ and } i \in I^*(0) \\ U_i(\epsilon) - \tilde{\epsilon} & \text{if } \bar{i} \leq i \leq I \text{ and } i \in I^*(0) \\ \epsilon - \tilde{\epsilon} & \text{if } i \notin I^*(0) \end{cases} .$$

For $1 \leq i < \bar{i}$ or $i \in I^*(0)$, their payoff will be weakly higher than $\epsilon - \tilde{\epsilon}$ which is obviously positive. For $\bar{i} \leq i \leq I$ and $i \in I^*(0)$, $U_i(\epsilon) \geq \epsilon$ by the choice of ϵ in Equation 12, and because $\epsilon > \tilde{\epsilon}$, therefore, $U_i(\epsilon) \geq \tilde{\epsilon}$.

Planner's Budget Constraint

$$\begin{aligned}
& \sum \pi_i [m(\tilde{\theta}_i)(v_i(\tilde{a}_i) - \tilde{p}_i) - k\tilde{\theta}_i - \tilde{s}_i] \sum \pi_i [m(\tilde{\theta}_i)(v_i(\tilde{a}_i) - \tilde{p}_i) - k\tilde{\theta}_i - \tilde{s}_i] \\
&= \sum_{i \in \{1, 2, \dots, \bar{i}-1\} \cap I^*(0)} \pi_i \bar{\theta}_i(0) \underbrace{[q(\bar{\theta}_i(0))(v_i(\bar{a}_i(0)) - \bar{p}_i(0))]}_{=0} \\
&+ \sum_{i \in \{\bar{i}, \bar{i}+1, \dots, I\} \cap I^*(0)} \pi_i \left[\bar{\theta}_i(\epsilon) \underbrace{[q(\bar{\theta}_i(\epsilon))(v_i(\bar{a}_i(\epsilon)) - \bar{p}_i(\epsilon))]}_{=0} + \epsilon \right] - (\epsilon - \tilde{\epsilon}) = 0
\end{aligned}$$

The two terms with accolades under them are equal to 0 due to Lemma 2. The whole expression equals 0 due to the definition of $\tilde{\epsilon}$.

Last Step: Calculating Welfare

I Calculate welfare from the direct mechanism:

$$\begin{aligned}
\sum \pi_i U_i &= \sum_{i \in \{1, 2, \dots, \bar{i}-1\} \cap I^*(0)} \pi_i (U_i(0) + \epsilon - \tilde{\epsilon}) + \sum_{i \in \{\bar{i}, \bar{i}+1, \dots, I\} \cap I^*(0)} \pi_i (U_i(\epsilon) - \tilde{\epsilon}) + \sum_{i \notin I^*(0)} \pi_i (\epsilon - \tilde{\epsilon}) \\
> \sum_{i \in \{1, 2, \dots, \bar{i}-1\} \cap I^*(0)} \pi_i (U_i(0) + \epsilon - \tilde{\epsilon}) + \sum_{i \in \{\bar{i}, \bar{i}+1, \dots, I\} \cap I^*(0)} \pi_i (U_i(0) - \tilde{\epsilon}) + \sum_{i \notin I^*(0)} \pi_i (\epsilon - \tilde{\epsilon}) &= \sum \pi_i U_i^{EQ}.
\end{aligned}$$

The inequality follows from Equation 13. The proof is now complete, because we have found a feasible direct mechanism that yields higher welfare than the equilibrium. Of course, this allocation is implementable due to Lemma 1. \square

I prove in the following lemma that at the solution to Problem $P(\epsilon)$, the first constraint in $P_i(\epsilon)$ should be binding. Also, I show that sellers are not attracted to submarkets designed for higher types, if ϵ is chosen sufficiently small. This lemma is similar to Lemma 1 in GSW, but I prove a stronger claim. I prove that higher types are *strictly* worse off if they apply to submarkets designed for lower types. That is, downward IC cannot be binding. The reason that I get a stronger result is that I assume strict monotonicity for $v_i(a)$ in i for every a with $a \in \bar{A}$, while they just assume weak monotonicity.

Lemma 2. *There exist $I^*(\epsilon) \subseteq \{1, 2, \dots, I\}$, $\{\bar{U}_i(\epsilon)\}_{i \in \{1, 2, \dots, I\}}$ and $\{(\bar{\theta}_i(\epsilon), \bar{a}_i(\epsilon), \bar{p}_i(\epsilon))\}_{i \in I^*(\epsilon)}$ that solve problem $P(\epsilon)$. Also, there exists $\bar{\epsilon}_2 > 0$ such that for every $\epsilon \in [0, \bar{\epsilon}_2)$, the following holds at any solution for Problem $P_i(\epsilon)$ for $i \in I^*(\epsilon)$:*

$$\begin{aligned}
& q(\bar{\theta}_i(\epsilon))(v_i(\bar{a}_i(\epsilon)) - \bar{p}_i(\epsilon)) = k, \\
& m(\bar{\theta}_i(\epsilon))(u_j(\bar{a}_i(\epsilon)) + \bar{p}_i(\epsilon)) + \delta_i < \bar{U}_j(\epsilon) \text{ for all } j > i,
\end{aligned}$$

$$\text{where } \delta_i = \begin{cases} \epsilon & \text{if } i < \bar{i} \text{ or } i \notin I^*(0) \\ 0 & \text{otherwise} \end{cases}.$$

Proof.

Part 1: Existence of a solution to Problem $P(\epsilon)$

First fix ϵ and set $I^*(\epsilon) = \emptyset$. For $i = 1$, the objective function is continuous and the constraint set is compact. If the constraint is empty, set $\bar{U}_1(\epsilon) = \epsilon$. Otherwise, since the objective function is continuous and the constraint set is compact, $P_1(\epsilon)$ has a solution and a unique maximum. If the value of the maximum is less than ϵ , again set $\bar{U}_1(\epsilon) = \epsilon$. Otherwise, denote by $(\bar{\theta}_1(\epsilon), \bar{a}_1(\epsilon), \bar{p}_1(\epsilon))$ one of the maximizers and add 1 to the set $I^*(\epsilon)$.

I proceed by induction. By induction hypothesis, I have found $(\bar{U}_1(\epsilon), \bar{U}_2(\epsilon), \dots, \bar{U}_{i-1}(\epsilon))$ and also $(\bar{\theta}_i(\epsilon), \bar{a}_i(\epsilon), \bar{p}_i(\epsilon))$ for all $i \in I^*(\epsilon)$. Again, if the constraint is empty, set $\bar{U}_i(\epsilon) = \epsilon$. Otherwise, $\bar{U}_i(\epsilon)$ is well-defined and unique. If the value of the maximum is less than ϵ , again set $\bar{U}_i(\epsilon) = \epsilon$. Otherwise, denote by $(\bar{\theta}_i(\epsilon), \bar{a}_i(\epsilon), \bar{p}_i(\epsilon))$ one of the maximizers and add i to the set $I^*(\epsilon)$.

Part 2: The first constraint in $P_i(\epsilon)$ (the free entry condition) is binding

Assume by way of contradiction that the constraint is not binding for some $i \in I^*(\epsilon)$. First note that $\bar{\theta}_i(\epsilon) > 0$ because $\bar{U}_j(\epsilon) > \delta_j$ for all $j \in I^*(\epsilon)$. According to part 2 of Assumption 1 (sorting), for every $\tau > 0$, there exists an $a' \in B_\tau(\bar{a}_i(\epsilon))$ such that

$$u_i(a') > u_i(\bar{a}_i(\epsilon)) \tag{14}$$

$$\text{and } u_j(a') < u_j(\bar{a}_i(\epsilon)) \text{ for all } j < i. \tag{15}$$

Set $\tau > 0$ sufficiently small such that $q(\bar{\theta}_i(\epsilon))(v_i(a') - \bar{p}_i(\epsilon)) \geq k$ for all $B_\tau(\bar{a}_i(\epsilon))$. Now consider $(\bar{\theta}_i(\epsilon), a', \bar{p}_i(\epsilon))$. The first constraint in Problem $P_i(\epsilon)$ is satisfied following the choice of τ and other constraints are satisfied because $m(\bar{\theta}_i(\epsilon))(u_j(a') + \bar{p}_i(\epsilon)) + \delta_i < m(\bar{\theta}_i(\epsilon))(u_j(\bar{a}_i(\epsilon)) + \bar{p}_i(\epsilon)) + \delta_i \leq \bar{U}_j(\epsilon)$ for all $j < i$. But the value of the objective function is now higher: $m(\bar{\theta}_i(\epsilon))(u_i(a') + \bar{p}_i(\epsilon)) + \delta_i > m(\bar{\theta}_i(\epsilon))(u_i(\bar{a}_i(\epsilon)) + \bar{p}_i(\epsilon)) + \delta_i$, which is a contradiction with $(\bar{\theta}_i(\epsilon), \bar{a}_i(\epsilon), \bar{p}_i(\epsilon))$ being a solution to problem $P_i(\epsilon)$.

Part 3: Incentive compatibility for all types when $\epsilon = 0$

Fix i such that $i \in I^*(\epsilon)$. In this part, I show that incentive compatibility holds at $\epsilon = 0$, that is,

$$m(\bar{\theta}_i(0))(u_j(\bar{a}_i(0)) + \bar{p}_i(0)) < \bar{U}_j(0) \text{ for all } i \in I^*(0), j > i.$$

Assume by way of contradiction that there exists n such that $n > i$ and $m(\bar{\theta}_i(0))(u_n(\bar{a}_i(0)) + \bar{p}_i(0)) \geq \bar{U}_n(0)$. Denote the smallest such n by h . That is,

$$m(\bar{\theta}_i(0))(u_j(\bar{a}_i(0)) + \bar{p}_i(0)) < \bar{U}_j(0) \text{ for all } i \leq j < h, \quad (16)$$

$$m(\bar{\theta}_i(0))(u_h(\bar{a}_i(0)) + \bar{p}_i(0)) \geq \bar{U}_h(0). \quad (17)$$

Now I show that $(\bar{\theta}_i(0), \bar{a}_i(0), \bar{p}_i(0))$ is feasible for problem $P_h(0)$. The first constraint in problem $P_h(0)$ is satisfied because $q(\bar{\theta}_i(0))(v_h(\bar{a}_i(0)) - \bar{p}_i(0)) > q(\bar{\theta}_i(0))(v_i(\bar{a}_i(0)) - \bar{p}_i(0)) \geq k$, where the first inequality follows from part 1 of Assumption 1 (strict monotonicity)³¹. Also, $m(\bar{\theta}_i(0))(u_j(\bar{a}_i(0)) + \bar{p}_i(0)) \leq \bar{U}_j(0)$ holds true for any j with $i < j < h$ according to Equation 16, and holds true for any j with $j \leq i$, because $(\bar{\theta}_i(0), \bar{a}_i(0), \bar{p}_i(0))$ is feasible for problem $P_i(0)$.

According to part 2 of Assumption 1 (sorting), there exists $b \in \bar{A}_i(0)$ sufficiently close to a_i , such that $q(\bar{\theta}_i(0))(v_h(b) - \bar{p}_i(0)) \geq k$, and

$$u_h(b) > u_h(\bar{a}_i(0)), \quad (18)$$

$$\text{and } u_j(b) < u_j(\bar{a}_i(0)) \text{ for all } j < h. \quad (19)$$

Now, the claim is that $(\bar{\theta}_i(0), b, \bar{p}_i(0))$ is feasible for problem $P_h(0)$ but delivers strictly higher utility for type h . First, the first constraint is satisfied by choice of b . Second, all incentive compatibility constraints are satisfied because $m(\bar{\theta}_i(0))(u_j(b) + \bar{p}_i(0)) < m(\bar{\theta}_i(0))(u_j(\bar{a}_i(0)) + \bar{p}_i(0)) \leq \bar{U}_j(0)$ for all $j < i$, where the weak inequality follows from the fact that $(\bar{\theta}_i(0), \bar{a}_i(0), \bar{p}_i(0))$ is feasible for problem $P_i(0)$. The value of the objective function is greater than $\bar{U}_h(0)$ because $m(\bar{\theta}_i(0))(u_h(b) + \bar{p}_i(0)) > m(\bar{\theta}_i(0))(u_h(\bar{a}_i(0)) + \bar{p}_i(0)) \geq \bar{U}_h(0)$, which contradicts with $\bar{U}_h(0)$ being a maximizer of $P_h(0)$.

Part 4: Existence of a neighborhood $[0, \epsilon_2]$ such that incentive compatibility for all types is satisfied

First of all, it is easy to see that similar to the argument in previous part, the first constraint in Problem $P_i(\epsilon)$ must be binding. Therefore, we can eliminate $\bar{p}_i(\epsilon)$ to write the problem in the following form:

$$\max_{\theta \in [0, \infty), (\cdot, a) \in \bar{Y}} \{m(\theta)(u_i(a) + v_i(a)) - k\theta + \delta_i\}$$

³¹I show here that $a_i(0) \in \bar{A}$ so we can use strict monotonicity of v_i . Since $(\bar{\theta}_i(0), \bar{a}_i(0), \bar{p}_i(0))$ is feasible for Problem $P_i(0)$, so $q(\bar{\theta}_i(0))(v_i(\bar{a}_i(0)) - \bar{p}_i(0)) \geq k$. But $\bar{\theta}_i(0) \geq 0$ so $q(0)(v_i(\bar{a}_i(0)) - \bar{p}_i(0)) \geq k$. Also $m(\bar{\theta}_i(0))(u_i(\bar{a}_i(0)) + \bar{p}_i(0)) + \delta_i = \bar{U}_i(0)$, but $\bar{U}_i(0) > \delta_i$ by construction of $I^*(0)$, therefore $u_i(\bar{a}_i(0)) + \bar{p}_i(0) \geq 0$. Hence $\bar{a}_i(0) \in \bar{A}$.

$$\text{s. t. } m(\theta)(u_j(a) + v_i(a)) - k\theta + \delta_i \leq \bar{U}_j(\epsilon) \text{ for all } j < i.$$

Our goal here is to apply theorem of the maximum to this problem and show that $\bar{U}_h(\epsilon)$ is continuous in ϵ and $\bar{a}_h(\epsilon)$, $\bar{\theta}_h(\epsilon)$ and $\bar{p}_h(\epsilon)$ are all upper hemi-continuous in ϵ . I proceed by induction on h .

For $h = 1$, it is trivial because the constraint set is independent of ϵ . Also the constraint set is compact, due to the following reasons. With respect to a , note that $a \in \bar{A}$ and \bar{A} is compact. Regarding θ , I show that we can assume without loss of generality that θ lies in a closed interval which is a subset of R_+ . Suppose by way of contradiction that θ is unbounded. Because $m(\theta) \leq 1$ and $u_i(a) + v_i(a)$ is bounded and $k > 0$, the objective function goes to $-\infty$. Therefore, we can restrict our attention to an interval $[0, M]$ for some $M \in R_+$. As a result, we can assume without loss of generality that $(\theta, a) \in [0, M] \times \bar{A}$.

The objective function is continuous in (θ, a) and ϵ . Therefore, $\bar{U}_1(\epsilon)$ is continuous in ϵ . and $\bar{a}_1(\epsilon)$, $\bar{\theta}_1(\epsilon)$ and $\bar{p}_1(\epsilon)$ are all upper hemi-continuous in ϵ .

Now consider $h > 1$. By induction hypothesis, $\bar{U}_j(\epsilon)$ are continuous in ϵ for all $j < h$, therefore, the constraint set is continuous in ϵ too. With a similar argument as above, we can conclude that the constraint set is a compact valued and continuous correspondence in ϵ . Therefore, $\bar{U}_h(\epsilon)$ is continuous in ϵ and $\bar{a}_h(\epsilon)$, $\bar{\theta}_h(\epsilon)$ and $\bar{p}_h(\epsilon)$ are all upper hemi-continuous in ϵ for all h . Both $\bar{\theta}_h(\epsilon)$ and $\bar{a}_h(\epsilon)$ are UHC in ϵ , and $m(\cdot)$ and $u_i(\cdot)$ are continuous functions, therefore $m(\bar{\theta}_h(\epsilon))$ and $u_i(\bar{a}_h(\epsilon))$ are UHC in ϵ . (See [Aliprantis and Border \(1986\)](#), Theorem 17.23.)

Define $e_{k,i}(\epsilon)$ as follows: $e_{k,i}(\epsilon) = \bar{U}_k(\epsilon) - m(\bar{\theta}_i(\epsilon))(u_k(\bar{a}_i(\epsilon)) + p_i(\epsilon)) - \delta_i$. Since $e_{k,i}(\epsilon)$ is just sum of some UHC correspondences, $e_{k,i}(\epsilon)$ itself is also UHC. But $e_{k,i}(\epsilon)|_{\epsilon=0} > 0$ according to part 3. I show below that because $e_{k,i}(\epsilon)$ is UHC in an interval close to 0 and its value at 0 is strictly positive, there must exist a neighborhood $[0, \epsilon_{k,i}]$ for some $\epsilon_{k,i} > 0$ such that $e_{k,i}(\epsilon)$ is strictly positive, too. Now, set

$$\bar{\epsilon}_2 = \min_{i,k>i} \epsilon_{k,i}.$$

That is, there exists a neighborhood $[0, \bar{\epsilon}_2]$ around 0 such that higher types are strictly worse off, if they report a lower type.

To show that for any i and $k > i$ there must exist a neighborhood $[0, \epsilon_{k,i}]$ for some $\epsilon_{k,i} > 0$ such that $e_{k,i}(\epsilon)$ is strictly positive, suppose by way of contradiction that there does not exist such a neighborhood. That is, there exists i and $k > i$ such that for any $\epsilon > 0$, there exists function $\tilde{\epsilon}(\epsilon) \in e_{k,i}(\epsilon)$ with $\tilde{\epsilon}(\epsilon) \leq 0$. Consider $\{\epsilon_n\}_{n \in \mathbb{N}}$ where $\epsilon_n = \frac{1}{n}$. Since $e_{k,i}(\cdot)$ is UHC and because $\epsilon_n \rightarrow 0$, there exists a convergent sub-sequence $\{\tilde{\epsilon}_n\}_{n \in \mathbb{N}}$ of $\{\tilde{\epsilon}(\epsilon_n)\}_{n \in \mathbb{N}}$ such that its limit point is in $e_{k,i}(0)$. This is a contradiction, because $e_{k,i}(0) > 0$ but $\tilde{\epsilon}_n \leq 0$ for all n , so its limit point cannot be a strictly positive number.

The proof is complete because I have shown that there exists a $\bar{\epsilon}_2$ such that $e(\epsilon) > 0$ for any $\epsilon \in [0, \bar{\epsilon}_2]$. Therefore, all incentive compatibility constraints are satisfied. □

8.3 Proof of Theorem 2

Proof. I will construct a feasible direct mechanism in which type i sellers get matched with probability $m(\theta_i^{FB})$ and produce a_i^{FB} . Under part 5 (a) or 5 (b) of Assumption 2, it can be easily shown that U_i^{FB} is increasing in i . Let \hat{i} denote the highest type of sellers without gains from trade. Then all types $1, 2, \dots, \hat{i}$ are inactive, that is, they are matched with probability 0. Given this observation, I assume that there are positive gains from trade for all types and then I construct a feasible direct mechanism that achieves the first best for all these types. If there are not positive gains from trade for some types, the same construction method with little adjustments can be used to establish the proof.

Consider the following direct mechanism:

$$\{(a_i^{FB}, \tilde{p}_i, \tilde{s}_i, \theta_i^{FB})\}_{i \in \{1, 2, \dots, I\}},$$

where $\tilde{p}_1 = -u_1(a_1^{FB})$ and \tilde{p}_i is defined for $i \geq 2$ recursively as follows:

$$m(\theta_i^{FB})(\tilde{p}_i + u_i(a_i^{FB})) = m(\theta_{i-1}^{FB})(\tilde{p}_{i-1} + u_i(a_{i-1}^{FB})). \quad (20)$$

Also, $\tilde{s}_i = \tilde{s} \equiv \sum_{j=1}^I \pi_j [m(\theta_j^{FB})(v_j(a_j^{FB}) - \tilde{p}_j) - k\theta_j^{FB}]$ for all i . I just need to show that conditions for feasibility are satisfied.

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I prove that this condition is satisfied in four steps:

Step 1: $(\theta_i^{FB}, a_i^{FB})$ is increasing in i .

I use Assumption 2. First, a_i^{FB} is increasing in i because $u_i(a) + v_i(a)$ satisfies increasing differences property in (a, i) and also because $u_i(a) + v_i(a)$ is supermodular in a . (See Theorem 5 in [Milgrom and Shannon \(1994\)](#)). Furthermore, $m(\theta)(u_i(a_i^{FB}) + v_i(a_i^{FB})) - k\theta$ satisfies increasing differences property in (θ, i) , because m is increasing, because $u_i(a) + v_i(a)$ is increasing in i and because a_i^{FB} is increasing in i . Hence, θ_i^{FB} is increasing in i .

Step 2: Local IC constraints are satisfied.

In equation 20, \tilde{p}_i is set such that all local downward incentive compatibility constraints are satisfied and binding. That is, for all $i \geq 2$ type i is indifferent between reporting i and $i - 1$. Now, I show that sellers' maximization constraint is satisfied.

First, I show that type $i - 1$ weakly prefers to report $i - 1$ over i (local upward incentive compatibility). That is,

$$m(\theta_i^{FB})(\tilde{p}_i + u_{i-1}(a_i^{FB})) \leq m(\theta_{i-1}^{FB})(\tilde{p}_{i-1} + u_{i-1}(a_{i-1}^{FB})). \quad (21)$$

I begin from the left hand side:

$$\begin{aligned} m(\theta_i^{FB})(\tilde{p}_i + u_{i-1}(a_i^{FB})) &= m(\theta_i^{FB})(\tilde{p}_i + u_i(a_i^{FB})) - m(\theta_i^{FB})(u_i(a_i^{FB}) - u_{i-1}(a_i^{FB})) \\ &= m(\theta_{i-1}^{FB})(\tilde{p}_{i-1} + u_{i-1}(a_{i-1}^{FB})) + m(\theta_{i-1}^{FB})(u_i(a_{i-1}^{FB}) - u_{i-1}(a_{i-1}^{FB})) - m(\theta_i^{FB})(u_i(a_i^{FB}) - u_{i-1}(a_i^{FB})) \\ &\leq m(\theta_{i-1}^{FB})(\tilde{p}_{i-1} + u_{i-1}(a_{i-1}^{FB})) + m(\theta_i^{FB})(u_i(a_{i-1}^{FB}) - u_{i-1}(a_{i-1}^{FB}) - u_i(a_i^{FB}) + u_{i-1}(a_i^{FB})) \\ &\leq m(\theta_{i-1}^{FB})(\tilde{p}_{i-1} + u_{i-1}(a_{i-1}^{FB})) \end{aligned}$$

The first equality follows from the construction of \tilde{p}_i (Equation 20). The first inequality follows from the fact that θ_i and $u_i(\cdot)$ are both increasing in i . The second inequality follows from increasing differences property of u in (a, i) and also from the fact that $a_{i-1}^{FB} \leq a_i^{FB}$ (component by component).

Second, I calculate \tilde{p}_i in terms of \tilde{p}_1 :

$$\begin{aligned} m(\theta_i^{FB})(\tilde{p}_i + u_i(a_i^{FB})) &= m(\theta_{i-1}^{FB})(\tilde{p}_{i-1} + u_i(a_{i-1}^{FB})) \\ &= m(\theta_{i-1}^{FB})(\tilde{p}_{i-1} + u_{i-1}(a_{i-1}^{FB})) + m(\theta_{i-1}^{FB})(u_i(a_{i-1}^{FB}) - u_{i-1}(a_{i-1}^{FB})) \\ &= m(\theta_{i-1}^{FB})(\tilde{p}_{i-1} + u_{i-1}(a_{i-1}^{FB})) + K_i(\theta_{i-1}^{FB}, a_{i-1}^{FB}) - K_{i-1}(\theta_{i-1}^{FB}, a_{i-1}^{FB}), \end{aligned} \quad (22)$$

where $K_i(\theta, a)$ is defined as follows:

$$K_i(\theta, a) \equiv m(\theta)u_i(a).$$

Using telescoping technique yields the following equation for all $i \geq 2$:

$$m(\theta_i^{FB})(\tilde{p}_i + u_i(a_i^{FB})) = m(\theta_1^{FB})(\tilde{p}_1 + u_1(a_1^{FB})) + \sum_{j=2}^i [K_j(\theta_{j-1}^{FB}, a_{j-1}^{FB}) - K_{j-1}(\theta_{j-1}^{FB}, a_{j-1}^{FB})]. \quad (23)$$

Step 3: Other upward IC constraints are satisfied.

Now, I show that for all i and k with $k \leq i - 1$, type k does not gain by reporting i , that is,

$$m(\theta_k^{FB})(\tilde{p}_k + u_k(a_k^{FB})) \geq m(\theta_i^{FB})(\tilde{p}_i + u_k(a_i^{FB})).$$

Note that

$$m(\theta_k^{FB})(\tilde{p}_k + u_k(a_k^{FB})) - m(\theta_i^{FB})(\tilde{p}_i + u_k(a_i^{FB}))$$

$$\begin{aligned}
&= \sum_{j=k}^{i-1} \left[m(\theta_j^{FB})(\tilde{p}_j + u_j(a_j^{FB})) - m(\theta_{j+1}^{FB})(\tilde{p}_{j+1} + u_j(a_{j+1}^{FB})) \right. \\
&\quad \left. + m(\theta_{j+1}^{FB})u_j(a_{j+1}^{FB}) - m(\theta_j^{FB})u_j(a_j^{FB}) \right. \\
&\quad \left. - m(\theta_{j+1}^{FB})u_k(a_{j+1}^{FB}) + m(\theta_j^{FB})u_k(a_j^{FB}) \right] \\
&\geq \sum_{j=k}^{i-1} \left[m(\theta_{j+1}^{FB})u_j(a_{j+1}^{FB}) - m(\theta_j^{FB})u_j(a_j^{FB}) \right. \\
&\quad \left. - m(\theta_{j+1}^{FB})u_k(a_{j+1}^{FB}) + m(\theta_j^{FB})u_k(a_j^{FB}) \right] \\
&\geq \sum_{j=k}^{i-1} \left[m(\theta_j^{FB})(u_j(a_{j+1}^{FB}) - u_k(a_{j+1}^{FB}) - u_j(a_j^{FB}) + u_k(a_j^{FB})) \right] \geq 0.
\end{aligned}$$

The first equality is derived by doing some algebra and using telescoping technique. The first inequality uses the fact that type $i - 1$ weakly prefers to report $i - 1$ over i (See equation 21). The second inequality uses $\theta_{j+1}^{FB} \geq \theta_j^{FB}$ and also the fact that u_i is increasing in i for $a \in \bar{A}$. The last inequality is the implication of the increasing differences property of u (part 1 of Assumption 1) and the fact that $a_{j+1}^{FB} \geq a_j^{FB}$.

Step 4: Other downward IC constraints are satisfied.

Again, I show that type k does not gain by reporting i . If $i + 1 \leq k$, I use the same technique as above:

$$\begin{aligned}
&m(\theta_k^{FB})(\tilde{p}_k + u_k(a_k^{FB})) - m(\theta_i^{FB})(\tilde{p}_i + u_k(a_i^{FB})) \\
&= \sum_{j=i+1}^k \left[m(\theta_j^{FB})(\tilde{p}_j + u_j(a_j^{FB})) - m(\theta_{j-1}^{FB})(\tilde{p}_{j-1} + u_j(a_{j-1}^{FB})) \right. \\
&\quad \left. + m(\theta_{j-1}^{FB})u_j(a_{j-1}^{FB}) - m(\theta_j^{FB})u_j(a_j^{FB}) \right. \\
&\quad \left. - m(\theta_{j-1}^{FB})u_k(a_{j-1}^{FB}) + m(\theta_j^{FB})u_k(a_j^{FB}) \right] \\
&\geq \sum_{j=i+1}^k \left[m(\theta_{j-1}^{FB})u_j(a_{j-1}^{FB}) - m(\theta_j^{FB})u_j(a_j^{FB}) \right. \\
&\quad \left. - m(\theta_{j-1}^{FB})u_k(a_{j-1}^{FB}) + m(\theta_j^{FB})u_k(a_j^{FB}) \right] \\
&\geq \sum_{j=i+1}^k \left[m(\theta_{j-1}^{FB})(u_j(a_{j-1}^{FB}) - u_k(a_{j-1}^{FB}) - u_j(a_j^{FB}) + u_k(a_j^{FB})) \right] \geq 0.
\end{aligned}$$

The first equality is again derived by using telescoping technique. The first inequality follows from construction of \tilde{p}_i . The second inequality uses $\theta_{j+1}^{FB} \geq \theta_j^{FB}$ and also the fact that u_i is increasing in i for every a . The last inequality is the implication of increasing differences property of u in (a, i) and the fact that $a_{j+1}^{FB} \geq a_j^{FB}$.

Participation constraint

To show $U_i \equiv m(\theta_i^{FB})(\tilde{p}_i + u_i(a_i^{FB})) + \tilde{s}_i \geq 0$ for all i , consider Equation 23. The first term in the right hand side is zero following the construction of \tilde{p}_1 . The summation is positive following the assumption that u_i is increasing in i for every a . Also, I show below that \tilde{s} is always positive.

$$\begin{aligned}
\tilde{s} &= \sum \pi_i \left[m(\theta_i^{FB})(v_i(a_i^{FB}) - \tilde{p}_i) - k\theta_i^{FB} \right] \\
&= \sum_{i=1}^I \pi_i \left[m(\theta_i^{FB})(v_i(a_i^{FB}) + u_i(a_i^{FB})) - k\theta_i^{FB} \right] - \sum_{i=1}^I \pi_i \left[m(\theta_i^{FB})(\tilde{p}_i + u_i(a_i^{FB})) \right] \\
&= \sum_{i=1}^I \pi_i \left[m(\theta_i^{FB})(v_i(a_i^{FB}) + u_i(a_i^{FB})) - k\theta_i^{FB} \right] - \sum_{i=1}^I \pi_i \left[\sum_{j=2}^i [K_j(\theta_{j-1}^{FB}, a_{j-1}) - K_{j-1}(\theta_{j-1}^{FB}, a_{j-1})] \right] \\
&\quad - m(\theta_1)(\tilde{p}_1 + u_1(a_1)). \tag{24}
\end{aligned}$$

For the fourth equality, I used the definition of \tilde{p}_i from Equation 23.

First, suppose part 5(a) of Assumption 2 holds.

To prove that the participation constraint is satisfied, it is sufficient to show that the right hand side of Equation 24 is positive for all i . I proceed with induction on i . If $i = 1$, the right hand side of the equation is equal to 0 by the choice of \tilde{p}_1 . For $i = 2$ ³²:

$$\begin{aligned}
m(\theta_2)(v_2(a_2) + u_2(a_2)) - k\theta_2 &\geq m(\theta_1)(v_2(a_1) + u_2(a_1)) - k\theta_1 \\
&\geq m(\theta_1)(u_2(a_1) - u_1(a_1)) + m(\theta_1)(v_2(a_1) + u_1(a_1)) - k\theta_1 \\
&\geq m(\theta_1)(u_2(a_1) - u_1(a_1)) + m(\theta_1)(v_1(a_1) + u_1(a_1)) - k\theta_1 + \\
&\quad \geq m(\theta_1)(u_2(a_1) - u_1(a_1)) + m(\theta_1)(u_2(a_1) + \tilde{p}_1)
\end{aligned}$$

The first inequality holds true due to the fact that θ_1 and a_1 are feasible for the second type maximization problem ($\max_{\theta, a} \{m(\theta)(v_2(a) + u_2(a)) - k\theta\}$). The second inequality holds because $v_i(\cdot)$ is increasing in i . The last inequality holds due to the construction of \tilde{p}_1 .

³²Establishing the claim for $i = 2$ is redundant, but I just do it here to make clear the main idea used in the general case (for $i > 1$).

Now assume that the induction hypothesis for type $i-1$ is correct. Then, I show that the hypothesis will be correct for type i as well. Let me remind you that the induction hypothesis states that $m(\theta_i^{FB})(v_i(a_i^{FB}) + u_i(a_i^{FB})) - k\theta_i^{FB} \geq m(\theta_1)(\tilde{p}_1 + u_1(a_1)) + \sum_{j=2}^i [K_j(\theta_{j-1}^{FB}, a_{j-1}) - K_{j-1}(\theta_{j-1}^{FB}, a_{j-1})]$.

$$\begin{aligned}
& m(\theta_i^{FB})(v_i(a_i^{FB}) + u_i(a_i^{FB})) - k\theta_i^{FB} \geq m(\theta_{i-1}^{FB})(v_i(a_{i-1}^{FB}) + u_i(a_{i-1}^{FB})) - k\theta_{i-1}^{FB} \\
& = m(\theta_{i-1}^{FB})(v_i(a_{i-1}^{FB}) + u_{i-1}(a_{i-1}^{FB})) - k\theta_{i-1}^{FB} + m(\theta_{i-1}^{FB})(u_i(a_{i-1}^{FB}) - u_{i-1}(a_{i-1}^{FB})) \\
& \geq m(\theta_{i-1}^{FB})(v_{i-1}(a_{i-1}^{FB}) + u_{i-1}(a_{i-1}^{FB})) - k\theta_{i-1}^{FB} + m(\theta_{i-1}^{FB})(u_i(a_{i-1}^{FB}) - u_{i-1}(a_{i-1}^{FB})) \\
& \geq m(\theta_1)(\tilde{p}_1 + u_1(a_1)) + \sum_{j=2}^{i-1} [K_j(\theta_{j-1}^{FB}, a_{j-1}) - K_{j-1}(\theta_{j-1}^{FB}, a_{j-1})] + K_i(\theta_{i-1}^{FB}, a_{i-1}^{FB}) - K_{i-1}(\theta_{i-1}^{FB}, a_{i-1}^{FB}) \\
& = m(\theta_1)(\tilde{p}_1 + u_1(a_1)) + \sum_{j=2}^i [K_j(\theta_{j-1}^{FB}, a_{j-1}) - K_{j-1}(\theta_{j-1}^{FB}, a_{j-1})]
\end{aligned}$$

Similar to the case for $i = 2$, the first inequality holds because θ_{i-1}^{FB} and a_{i-1}^{FB} are feasible for type i maximization problem ($\max_{\theta, a} \{m(\theta)(v_i(a) + u_i(a)) - k\theta\}$). The second inequality holds because v_i is increasing in i . The last inequality holds due to the induction hypothesis.

Second, suppose parts 5(b) and 5(c) of Assumption 2 hold, instead.

Here, I cannot show that the terms inside the sigma in Equation 24 are positive for each i . Rather, I need to algebraically simplify the right hand side of Equation 24 as follows. To simplify the notation, I use $\Delta_i \equiv \sum_{k=i}^I \pi_k$.

$$\begin{aligned}
\tilde{s} &= \sum \pi_i \left[m(\theta_i^{FB})(v_i(a_i^{FB}) - \tilde{p}_i) - k\theta_i^{FB} \right] \\
&= \sum_{i=1}^I \pi_i \left[m(\theta_i^{FB})(v_i(a_i^{FB}) + u_i(a_i^{FB})) - k\theta_i^{FB} \right] - \sum_{i=1}^I \pi_i \left[m(\theta_i^{FB})(\tilde{p}_i + u_i(a_i^{FB})) \right] \\
&= \sum_{i=1}^I \pi_i \left[m(\theta_i^{FB})(v_i(a_i^{FB}) + u_i(a_i^{FB})) - k\theta_i^{FB} \right] - \sum_{i=1}^I \pi_i \left[\sum_{j=2}^i [K_j(\theta_{j-1}^{FB}, a_{j-1}) - K_{j-1}(\theta_{j-1}^{FB}, a_{j-1})] \right] \\
&\quad - m(\theta_1)(\tilde{p}_1 + u_1(a_1)) \\
&= \sum_{i=1}^I \pi_i \left[m(\theta_i^{FB})(v_i(a_i^{FB}) + u_i(a_i^{FB})) - k\theta_i^{FB} \right] - \sum_{j=2}^I \left[\pi_j [K_j(\theta_{j-1}, a_{j-1}) - K_{j-1}(\theta_{j-1}, a_{j-1})] \frac{\Delta_j}{\pi_j} \right] \\
&\quad - m(\theta_1)(\tilde{p}_1 + u_1(a_1)) \\
&= \sum_{i=2}^I \pi_i \left[m(\theta_i^{FB})(v_i(a_i^{FB}) + u_i(a_i^{FB})) - k\theta_i^{FB} - [K_j(\theta_{j-1}, a_{j-1}) - K_{j-1}(\theta_{j-1}, a_{j-1})] \frac{\Delta_i}{\pi_i} \right]
\end{aligned}$$

$$+\pi_1 U_1^{FB} - m(\theta_1)(\tilde{p}_1 + u_1(a_1)) \geq 0.$$

For the second equality, I changed the order of summations for the double sigma term and then used the definition of Δ_i . I used Assumption 2 part 3 and choice of \tilde{p}_1 to establish the last inequality.

Budget constraint

This condition is trivially satisfied due to the construction of \tilde{s} .

The proposed allocation achieves the maximum among all feasible allocations, because the level of θ and a assigned to every type is exactly equal to that under the first best, so it is not possible to increase the value of the objective function any more. This concludes the proof. \square

8.4 Proofs of Asset Market with Lemons

Proof of Asset market with lemons if $\pi_1 b_1 + \pi_2 b_2 \geq c_2$.

Consider the following direct mechanism: $\{(\alpha_i, p_i, s_i, \theta_i)\}_{i \in \{1,2\}}$ with $\alpha_i = 1$, $p_i = \pi_1 b_1 + \pi_2 b_2$, $\tilde{s}_i = 0$, $\theta_i = 1$ for all i . I suppress \sim for the proofs in the asset market with lemons to make the notation simpler.

Incentive compatibility of sellers is clearly satisfied, because both types get the same $(\alpha_i, p_i, s_i, \theta_i)$. Also, both types get a positive payoff, so participation constraint of sellers is also obviously satisfied. Planner's budget-balance is also trivially satisfied. The objective function is maximized because the θ and α allocated to both types is the same as what they get under complete information. The proof is complete. \square

Proof of Asset market with lemons when $\pi_1 b_1 + \pi_2 b_2 < c_2$.

Here the first best is not achievable through a pooling allocation, because type two gets a strictly negative payoff in the pooling allocation, therefore, pooling allocation is not feasible. If $b_2 - c_2$ is greater than $b_1 - c_1$, part 5(b) of Assumption 2 is violated. If $b_2 - c_2$ is less than or equal to $b_1 - c_1$, then it is easy to check that although part 5(b) is satisfied, part 5(c) is violated. Therefore, it is not possible to use Theorem 2. Hence, I need to solve the planner's problem completely by taking all constraints into account. I proceed in 6 steps. In the first step, I use a direct mechanism to write down the planner's problem. In the second step, I show that the market tightness for both types must be strictly positive. In the third step, I show that the market tightness for both types must be less than or equal to 1. In the fourth step, I show that α (probability that the seller gives the asset to the buyer) for both types must be equal to 1. In the fifth step, I show that market tightness for type one must be

equal to 1. In the last step, I calculate the market tightness for type two. This will conclude the characterization of the constrained efficient mechanism.

Step 1: Formulating the problem and simplifying it

Let $\{(\alpha_i, p_i, s_i, \theta_i)\}_{i \in \{1,2\}}$ denote the allocation with the direct mechanism. For now, I assume $s_i = 0$ for all i . In the proof, since there are positive gains from trade for both types, I can show that both types must be active. Therefore, the assumption that $s_i = 0$ is without loss of generality. That is, if $s_i \neq 0$ for some i , we can change p_i to $p_i + \frac{s_i}{m(\theta_i)}$ and set $s_i = 0$. Therefore, the planner's problem can be written as follows:

Problem 3 (Asset market with lemons, 1).

$$\max_{\{(\alpha_i, p_i, \theta_i)\}_{i \in \{1,2\}}} \sum_{i=1}^2 \pi_i \min\{\theta_i, 1\} (p_i - \alpha_i c_i),$$

subject to

$$\min\{\theta_1, 1\} (p_1 - \alpha_1 c_1) \geq \min\{\theta_2, 1\} (p_2 - \alpha_2 c_1) \quad (IC-12),$$

$$\min\{\theta_2, 1\} (p_2 - \alpha_2 c_2) \geq \min\{\theta_1, 1\} (p_1 - \alpha_1 c_2) \quad (IC-21),$$

$$\min\{\theta_1, 1\} (p_1 - \alpha_1 c_1) \geq 0 \quad (IR-1),$$

$$\min\{\theta_2, 1\} (p_2 - \alpha_2 c_2) \geq 0 \quad (IR-2),$$

$$\sum_{i=1}^2 \pi_i [\min\{\theta_i, 1\} (\alpha_i h_i - p_i) - k \theta_i] \geq 0 \quad (BB).$$

In the first (second) line, I ensure that type one (two) does not want to report type two (one). I call this constraint IC-12 (IC-21). In the third (fourth) line, I ensure that type one (two) gets a strictly positive payoff. I call this constraint IR-1 (IR-2). The last line is planner's budget constraint.

Notice that the planner's budget constraint must be binding. If not binding, we can distribute the extra resources in a lump sum way and identically among both types to increase the value of the objective function, while keeping all other constraints satisfied. Therefore, we can write from BB that $\sum_{i=1}^2 \pi_i \min\{\theta_i, 1\} p_i = \sum_{i=1}^2 \pi_i [\min\{\theta_i, 1\} \alpha_i h_i - k \theta_i]$. Hence we can write the objective function as $\sum_{i=1}^2 \pi_i \min\{\theta_i, 1\} \alpha_i (h_i - c_i) - k \theta_i$.

Step 2: $\theta_1 > 0$ and $\theta_2 > 0$

If both θ_1 and θ_2 are 0 then the welfare level equals 0. But this is not possible because we know at least that equilibrium allocation is feasible and delivers strictly positive utility. To rule out the case that one of them is 0, note that IC-12 and IC-21 together imply that:

$$(m(\theta_1)\alpha_1 - m(\theta_2)\alpha_2)c_1 \leq m(\theta_1)p_1 - m(\theta_2)p_2 \leq (m(\theta_1)\alpha_1 - m(\theta_2)\alpha_2)c_2. \quad (25)$$

But $c_1 < c_2$, therefore,

$$m(\theta_1)\alpha_1 \geq m(\theta_2)\alpha_2. \quad (26)$$

If $\theta_1 = 0$, then θ_2 must be 0 as well, and this leads to 0 level of welfare. Nevertheless, this cannot be part of a planner's allocation, given the fact that the equilibrium allocation is feasible and delivers strictly positive welfare. Thus $\theta_1 > 0$. If $\theta_2 = 0$, then it is easy to check that the maximum possible welfare in this case (even if $\theta_1 = 1$) is less than the level of welfare under the proposed solution. Therefore, $\theta_2 > 0$.

Let $r_i \equiv \min\{\theta_i, 1\}p_i$ for all i . For any θ_i and $r_i \in \mathbb{R}$, we can find a unique $p_i \in \mathbb{R}$ which solves the maximization problem. From now on, we work with r_i instead of p_i because it simplifies the analysis. Therefore, we can rewrite the problem as follows:

Problem 4 (Asset market with lemons, 2).

$$\max_{\{(\theta_i, \alpha_i, r_i)\}_{i=1,2}} \sum_{i=1}^2 \pi_i (\min\{\theta_i, 1\} \alpha_i (h_i - c_i) - k\theta_i),$$

subject to

$$r_1 - \min\{\theta_1, 1\} \alpha_1 c_1 \geq r_2 - \min\{\theta_2, 1\} \alpha_2 c_1 \quad (IC-12),$$

$$r_2 - \min\{\theta_2, 1\} \alpha_2 c_2 \geq r_1 - \min\{\theta_1, 1\} \alpha_1 c_2 \quad (IC-21),$$

$$r_1 - \min\{\theta_1, 1\} \alpha_1 c_1 \geq 0 \quad (IR-1)$$

$$r_2 - \min\{\theta_2, 1\} \alpha_2 c_2 \geq 0 \quad (IR-2) \text{ and}$$

$$\sum_{i=1}^2 \pi_i (\min\{\theta_i, 1\} \alpha_i h_i - k\theta_i - r_i) = 0 \quad (BB).$$

Step 3: $\theta_1 \leq 1$ and $\theta_2 \leq 1$

Suppose $\theta_i > 1$ for some i . Then consider the following: $\theta'_i = 1$, $r'_i = r_i + k(\theta_i - 1)\pi_i$ and $r'_j = r_j + k(\theta_i - 1)\pi_i$ where $j \neq i$. Therefore, if I replace θ_i , r_1 and r_2 by $\theta'_i = 1$, r'_1 and r'_2 respectively, I can increase the value of the objective function by $k(\theta_i - 1)$. Also, the new solution satisfies all the constraints because of the following: Obviously, IC-12 and IC-21 are still satisfied, because the change in r_1 is the same as the change in r_2 and also $\min\{\theta_1, 1\}$ and $\min\{\theta_2, 1\}$ have not changed. IR-1 and IR-2 are satisfied because $r'_1 > r_1$ and $r'_2 > r_2$. BB is also satisfied by construction of r'_1 and r'_2 . A contradiction. Therefore, for all $i \in \{1, 2\}$, $\theta_i \leq 1$.

Step 4: $\alpha_1 = \alpha_2 = 1$

Suppose $\alpha_i < 1$ for some i . Let α'_i be defined such that $\alpha'_i(\theta_i - \epsilon)$ equals $\alpha_i\theta_i$, where $0 < \epsilon < \theta_i(1 - \alpha_i)$. Fix ϵ and consider the following: $\theta'_i = \theta_i - \epsilon$, $r'_i = r_i + k\epsilon\pi_i$ and $r'_j = r_j + k\epsilon\pi_i$ where $j \neq i$.

Now, if I replace α_i, θ_i, r_1 and r_2 by $\alpha'_i, \theta'_i, r'_1$ and r'_2 respectively, I can increase the value of the objective function by $k\epsilon$. I show that the new solution satisfies all the constraints because of the following: Obviously, IC-12 and IC-21 are still satisfied, because $\min\{\theta_i, 1\}\alpha_i = \min\{\theta'_i, 1\}\alpha'_i$. IR-1 and IR-2 are satisfied because $r'_1 > r_1$ and $r'_2 > r_2$. BB is also satisfied by construction of r'_1 and r'_2 . A contradiction. Therefore, for all $i \in \{1, 2\}$, $\alpha_i = 1$.

For simplicity, I write the planner's problem again incorporating the results so far:

Problem 5 (Asset market with lemons, 3).

$$\max_{\{(\theta_i, r_i)\}_{i=1,2}} \sum_1^2 \pi_i(\theta_i(h_i - c_i) - k\theta_i),$$

subject to

$$r_1 - \theta_1 c_1 \geq r_2 - \theta_2 c_1 \quad (IC-12),$$

$$r_2 - \theta_2 c_2 \geq r_1 - \theta_1 c_2 \quad (IC-21),$$

$$r_1 - \theta_1 c_1 \geq 0 \quad (IR-1),$$

$$r_2 - \theta_2 c_2 \geq 0 \quad (IR-2),$$

$$\sum_1^2 \pi_i(\theta_i h_i - k\theta_i - r_i) = 0 \quad (BB).$$

Step 5: $\theta_1 = 1$

First note that $\theta_1 \geq \theta_2$ following Equation 26 and because $\alpha_1 = \alpha_2 = 1$ according to step 4. By way of contradiction, assume that $\theta_1 < 1$ at a solution. I consider two cases. First, assume that IR-2 is not binding. I propose the following: $\theta'_i = \theta_i + \epsilon$ for all i where $\epsilon \in (0, \min\{1 - \theta_1, \frac{r_2 - \theta_2 c_2}{c_2 - \pi_1 b_1 - \pi_2 b_2}\})$. Fix ϵ and let $r'_i = r_i + (\pi_1 b_1 + \pi_2 b_2)\epsilon$ for all i . It is easy to check that all constraints are satisfied, but the value of the objective function now has increased by $(\pi_1(b_1 - c_1) + \pi_2(b_2 - c_2))\epsilon$, a contradiction. Note that I used Equation 26 to ensure that $\theta_2 + \epsilon < 1$.

Second, assume that IR-2 is binding. I propose the following: $\theta'_1 = \theta_1 + \epsilon$ where $\epsilon < 1 - \theta_1$, $r'_1 = r_1 + b_1\epsilon$. It is again easy to check that all constraints are satisfied. The only tricky thing here is to check that IC-21 is satisfied. But the LHS in IC-21 is fixed. The RHS increases by $\epsilon(b_1 - c_2)$ which is a negative number, so IC-21 is not violated. (Note that $b_1 - c_2 < 0$, otherwise $\pi_1 b_1 + \pi_2 b_2 > \pi_1 c_2 + \pi_2 c_2 = c_2$ which contradicts the initial assumption

that $\pi_1 b_1 + \pi_2 b_2 < c_2$). But the value of the objective function now has increased by $\pi_1 b_1 \epsilon$, a contradiction.

Step 6: Calculating θ_2 and the rest of unknowns

I write r_1 from the budget constraint in terms of other variables, specially r_2 :

$$r_1 = b_1 + \frac{\pi_2}{\pi_1} \theta_2 b_2 - \frac{\pi_2}{\pi_1} r_2 \quad (27)$$

Now, one can write Equation 25 as follows after replacing r_1 from the above equation:

$$(1 - \theta_2)c_1 \leq b_1 + \frac{\pi_2}{\pi_1} \theta_2 b_2 - \frac{r_2}{\pi_1} \leq (1 - \theta_2)c_2. \quad (28)$$

First, note that IR-1 is implied by IC-12 and IR-2.

Second, I argue that IR-2 must be binding at the solution. By way of contradiction, suppose not. Then only Equation 28 is sufficient to determine θ_2 . But in order to maximize the objective function, I need to choose the highest possible θ_2 consistent with Equation 28, which is $\theta_2 = 1$. But according to equation 28, $r_2 = \pi_1 b_1 + \pi_2 b_2$ and $r_2 > c_2$ from IR-2, which is a contradiction with $\pi_1 b_1 + \pi_2 b_2 < c_2$. Therefore, IR-2 is binding.

Third, since IR-2 is binding, I replace r_2 by $\theta_2 c_2$ and rewrite equation 28 again:

$$(1 - \theta_2)c_1 \leq b_1 + \frac{\theta_2}{\pi_1} (\pi_2 b_2 - c_2) \leq (1 - \theta_2)c_2. \quad (29)$$

Now, it is easy to see that the right inequality in 29 is satisfied for any $\theta_2 \in [0, 1]$, because $b_1 < c_2$. In order to maximize the objective function, I need to find the maximum value for θ_2 under which the left inequality in Equation 29 is satisfied ($(1 - \theta_2)c_1 \leq b_1 + \frac{\theta_2}{\pi_1} (\pi_2 b_2 - c_2)$). This implies that

$$\theta_2 = \frac{\pi_1 (b_1 - c_1)}{c_2 - \pi_2 b_2 - \pi_1 c_1}.$$

The proof is complete, because I have found the values for α_i , θ_i and r_i . I can calculate values of p_i from θ_i and r_i and check that they are the same as them in Table 1. Note that t_i in Table 1 is calculated such that buyers' free entry and zero profit condition is satisfied for each submarket in the decentralized economy.

□

What if there are no gains from trade for some types?

Because the proof is similar to the the previous proof up to step 5, I do not repeat those steps here, so I begin from Problem 5. I want to show that $\theta_1 = \theta_2 = 0$ at the solution. First note that Equation 26, implies that $\theta_1 \geq \theta_2$. Also note that IR-1 is implied by IC-12

and IR-2, so we ignore IR-1. If $\theta_1 = 0$, then $0 \leq \theta_2 \leq \theta_1 = 0$ and the proof is complete. Therefore, by way of contradiction assume that $\theta_1 > 0$. In the first step below, I show that $\theta_2 < \theta_1$ (with strict inequality). In the second step, I show that IC-12 is not binding. Then I propose a new set of $\{(\theta_1, r_1), (\theta_2, r_2)\}$ such that all constraints are satisfied, but the value of the objective function is increased.

Step 1: $\theta_2 < \theta_1$

Suppose to the contrary that $\theta_2 \geq \theta_1$, but θ_2 cannot exceed θ_1 as mentioned above, so $\theta_2 = \theta_1$. IC-12 and IC-21 together imply that $r_1 = r_2$. Then, BB implies that $r_2 = (\pi_1 b_1 + \pi_2 b_2)\theta_2$. The latter together with IR-2 implies that $(\pi_1 b_1 + \pi_2 b_2 - c_2)\theta_2 \geq 0$. But $\pi_1 b_1 + \pi_2 b_2 - c_2 < 0$, therefore $\theta_2 = 0$ and so $\theta_1 = 0$. This is a contradiction with $\theta_1 > 0$.

Step 2: IC-21 is binding

By way of contradiction, suppose IC-21 is not binding. Consider $\theta'_1 = \theta_1 - \epsilon$ and $r'_1 = r_1 - b_1 \epsilon$ with $\epsilon > 0$. Since $\theta_1 > 0$ and IC-21 is not binding, we can find a sufficiently small ϵ such that $\theta'_1 > 0$ and IC-21 still holds. Now, it is easy to check that $\{(\theta'_1, r'_1), (\theta_2, r_2)\}$ is feasible for Problem 5, but it leads to higher value for the objective function than $\{(\theta_1, r_1), (\theta_2, r_2)\}$. Notice that we used $b_1 - c_1 < 0$ to check that IC-12 is satisfied.

Step 3: IC-12 is not binding

Suppose by way of contradiction that IC-12 is binding, then following step 2 (stating that IC-21 is binding), it is easy to check that $r_1 = r_2$ and $\theta_1 = \theta_2$. Then BB implies that $r_2 = (\pi_1 b_1 + \pi_2 b_2)\theta_2$. According to IR-2, $(\pi_1 b_1 + \pi_2 b_2 - c_2)\theta_2 \geq 0$. But $\pi_1 b_1 + \pi_2 b_2 - c_2 < 0$, so $\theta_1 = \theta_2 = 0$. This is a contradiction, so IC-12 is not binding.

Step 4: $\theta_1 = 0$

We have assumed $\theta_1 > 0$. Now, we want to get a contradiction. Now, consider $\{(\theta'_1, r'_1), (\theta_2, r'_2)\}$ where $\theta'_1 = \theta_1 - \epsilon$ with $\epsilon > 0$, $r'_1 = r_1 - (\pi_1 b_1 + \pi_2 b_2)\epsilon$ and $r'_2 = r_2 + \pi_1(c_2 - b_1)\epsilon$. Since $\theta_1 > 0$ and IC-12 is not binding, we can find a sufficiently small ϵ such that $\theta'_1 > 0$ and IC-12 still holds. Now it is easy to check that $\{(\theta'_1, r'_1), (\theta_2, r'_2)\}$ is feasible for Problem 5, but it leads to higher value for the objective function than $\{(\theta_1, r_1), (\theta_2, r_2)\}$. A contradiction, so the proof is complete.

□

8.5 Proof of the Rat Race

Proof. This proposition is basically a special case of Theorem 2. It is straight forward to check that all conditions are satisfied. Specially note that, part 5(a) of Assumption 2 is satisfied, therefore, we do not need any assumption on the distribution of types. \square

8.6 Asset Market with Continuous Type Space

Here I define feasible mechanism which is exactly similar to its counterpart with discrete type space (Definition 2). The planner allocates each (reported) type a market tightness, $\tilde{\theta} : Z \rightarrow \mathbb{R}_+$, a transfer conditional on finding a match, $\tilde{p} : Z \rightarrow \mathbb{R}$, and an unconditional transfer, $\tilde{s} : Z \rightarrow \mathbb{R}$.

Definition 8. *A feasible mechanism is a set $\{(\tilde{p}(\cdot), \tilde{s}(\cdot), \tilde{\theta}(\cdot))\}$ such that the following conditions hold:*

1. *(Incentive Compatibility of Sellers) For all z and \hat{z} ,*

$$U(z) \equiv m(\tilde{\theta}(z))(\tilde{p}(z) - c(z)) + \tilde{s}(z) \geq U(z, \hat{z}) \equiv m(\tilde{\theta}(\hat{z}))(\tilde{p}(\hat{z}) - c(z)) + \tilde{s}(\hat{z}).$$

2. *(Participation Constraint of Sellers) For all z ,*

$$U(z) \geq 0.$$

3. *(Planner's Budget-Balance)*

$$\int [m(\tilde{\theta}(z))(h(z) - \tilde{p}(z)) - k\tilde{\theta}(z) - \tilde{s}(z)]dF(z) \geq 0.$$

Definition 9. *A constraint efficient mechanism is a feasible direct mechanism which maximizes the planner's objective function.*

The ideas used here are the same as those used in the discrete type space and specially similar to the asset market with lemons. However, mathematical tools that I use here are different, because the state space is continuous.

One way of proving Proposition 3 is to take a Guess-And-Verify approach. I guess that the first best is achievable. Then I check whether conditions for feasibility are satisfied. One problem is that if the first best is not achievable (like conditions in Proposition 6), this approach does not work, because checking for feasibility is not sufficient, since there might be other implementable allocations which deliver a higher value of the objective function for the planner. Therefore, in order to be able to use a general solution method, I first characterize the incentive compatible schemes, as is common in the mechanism design literature. Then,

I work with a modified problem in which sellers' maximization condition has been replaced by some other constraints (monotonicity and Envelope condition).

In the first step, note that similar to the discrete type space, the budget-balance constraint must be satisfied with equality at the constrained efficient mechanism. Otherwise, the planner can distribute extra resources identically among all types. No constraint will be changed, but all types get a strictly higher payoff and therefore the planner can improve welfare. Now I write the planner's problem into the following form.

Problem 6.

$$\begin{aligned} & \max_{\theta(z), p(z)} \int \left[m(\theta(z))(h(z) - c(z)) - k\theta(z) \right] dF \\ & \text{s.t. } z \in \arg \max_{\hat{z}} U(z, \hat{z}), \quad (IC) \\ & \quad U(z, z) \geq 0 \quad (IR), \\ & \text{and } \int \left[m(\theta(z))(h(z) - p(z)) - k\theta(z) - ss(\hat{z}) \right] dF = 0 \quad (BB), \\ & \text{in which } U(z, \hat{z}) \equiv m(\theta(\hat{z}))(p(\hat{z}) - c(z)) + ss(\hat{z}). \end{aligned}$$

Note that no transfer appears in the objective function, because we have assumed that all types participate in the mechanism and also we have replaced $\int [m(\theta(z))p(z)]dF$ by $\int [m(\theta(z))h(z) - k\theta(z)]dF$ from the budget-balance condition.

Characterizing the Incentive Compatible Schemes

By assumption $c(z)$ is strictly monotone in z . The first two parts of the following lemma state that $c'(z) \frac{d\tilde{\theta}(z)}{dz} \leq 0$ is necessary and sufficient for any allocation which satisfies IC. Necessity is clear. Sufficiency means that there exists transfer schedules $\tilde{p}(\cdot)$ and $\tilde{s}(\cdot)$ such that the direct mechanism, $\{(\tilde{p}(\cdot), \tilde{s}(\cdot), \tilde{\theta}(\cdot))\}$, satisfies IC. The third part characterizes $U(z)$ for any allocation which satisfies IC.

Lemma 3 (Necessary and sufficient condition for $\tilde{\theta}(z)$ to be implementable). *Assume that $c(z)$ is strictly monotone in z .*

1. *Take any mechanism $\{(\tilde{p}(\cdot), \tilde{s}(\cdot), \tilde{\theta}(\cdot))\}$ that satisfies IC. If $\tilde{\theta}(z)$ is a piecewise C^1 function, then $c'(z) \frac{d\tilde{\theta}(z)}{dz} \leq 0$ wherever $\tilde{\theta}(z)$ is differentiable at z .*
2. *Consider any piecewise C^1 function $\tilde{\theta}(z)$ satisfying $c'(z) \frac{d\tilde{\theta}(z)}{dz} \leq 0$. Then there exists transfer schedules $\tilde{p}(\cdot)$ and $\tilde{s}(\cdot)$ such that the mechanism $\{(\tilde{p}(\cdot), \tilde{s}(\cdot), \tilde{\theta}(\cdot))\}$ satisfies IC.*
3. *If mechanism $\{(\tilde{p}(\cdot), \tilde{s}(\cdot), \tilde{\theta}(\cdot))\}$ satisfies IC, then $U(z)$ must satisfy*

$$U(z) = U(z_H) + \int_z^{z_H} m(\tilde{\theta}(z_0))c'(z_0)dz_0.$$

Proof. Define $V(X, R, z) \equiv Xc(z) + R$, $x(z) \equiv -m(\theta(z))$ and $r(z) \equiv m(\tilde{\theta}(z))\tilde{p}(z) + \tilde{s}(z)$. Obviously, $U(z, \hat{z}) = V(x(\hat{z}), r(\hat{z}), z)$.

Following [Fudenberg and Tirole \(1991\)](#), theorem 7.1, a **necessary** condition for $x(\cdot)$ to satisfy IC is

$$\frac{\partial}{\partial z} \left[\frac{\frac{\partial V}{\partial X}}{\frac{\partial V}{\partial R}} \right] \frac{dx}{dz} \geq 0,$$

whenever $x(\cdot)$ is differentiable at z . But $\frac{\partial}{\partial z} \left[\frac{\frac{\partial V}{\partial X}}{\frac{\partial V}{\partial R}} \right] \frac{dx}{dz} = \frac{\partial}{\partial z} \left(\frac{c(z)}{1} \right) (-m'(\tilde{\theta}(z))) \frac{d\tilde{\theta}(z)}{dz}$. Also $c'(\cdot) > 0$ and $m'(\cdot) \geq 0$, therefore, the necessary condition is equivalent to

$$c'(z) \frac{d\tilde{\theta}(z)}{dz} \leq 0. \quad (30)$$

According to [Fudenberg and Tirole \(1991\)](#) theorem 7.3, a **sufficient** condition for $x(\cdot)$ to satisfy IC is that $\frac{dx(z)}{dz} \geq 0$, or equivalently, $c'(z) \frac{d\tilde{\theta}(z)}{dz} \leq 0$ ³³.

For the third part of the lemma, I use corollary 1 from [Milgrom and Segal \(2002\)](#). This result states that if $\tilde{\theta}(z)$ satisfies IC, then $U(\cdot)$ can be written as follows:

$$U(z) = U(z_H) - \int_z^{z_H} \frac{\partial U(z_0, z_0)}{\partial z} dz_0 = U(z_H) + \int_z^{z_H} m(\tilde{\theta}(z_0))c'(z_0)dz_0. \quad (31)$$

This equation is derived from the envelope theorem and is standard in mechanism design literature. The requirements of the result of [Milgrom and Segal \(2002\)](#) that we need to check are as follows:

1. $U(z, \hat{z})$ is differentiable and absolutely continuous in z .

This is satisfied because c is assumed to be twice differentiable.

2. $\sup_{\hat{z}} \left| \frac{\partial U(z, \hat{z})}{\partial z} \right|$ is integrable.

This is satisfied because $\sup_{\hat{z}} \left| \frac{\partial U(z, \hat{z})}{\partial z} \right| \leq |c'(z)| < M$ for some $M \in \mathbb{R}$, because $c'(\cdot)$ is continuous and is defined over a compact set $[z_L, z_H]$.

3. $\tilde{\theta}(z)$ is obviously non-empty. □

We know from IC that $U(z) = m(\tilde{\theta}(z))(\tilde{p}(z) - c(z)) + \tilde{s}(z)$ for all z . All types are active, because as we will verify it later, the first best is achievable and there are positive gains from trade for all types. Therefore, from now on we can assume without loss of generality that $\tilde{s}(z) = 0$ for all types. (Otherwise, we can change $\tilde{p}(z)$ to $\tilde{p}(z) + \frac{\tilde{s}(z)}{m(\tilde{\theta}(z)})$.) I substitute $U(\cdot)$ from Equation 31 into $U(z) = m(\tilde{\theta}(z))(\tilde{p}(z) - c(z))$ to derive transfers:

$$\tilde{p}(z) = c(z) + \frac{U(z_H) + \int_z^{z_H} m(\tilde{\theta}(z_0))c'(z_0)dz_0}{m(\tilde{\theta}(z))}. \quad (32)$$

³³Briefly, the idea of the proof for necessity is that the second order condition for IC maximization problem ($\max_{\hat{z}} U(z, \hat{z})$) should hold. For sufficiency, the proof goes by contradiction. The proof of this lemma is standard in mechanism design literature thus omitted from here.

Now, I use budget-balance condition to derive $U(z_H)$:

$$\begin{aligned}
0 &= \int \left[m(\tilde{\theta}(z))[h(z) - p(z)] - k\tilde{\theta}(z) \right] F'(z) dz \\
&= \int \left[m(\tilde{\theta}(z))[h(z) - c(z)] - k\tilde{\theta}(z) - m(\tilde{\theta}(z))(\tilde{p}(z) - c(z)) \right] F'(z) dz \\
&= \int \left[m(\tilde{\theta}(z))(h(z) - c(z)) - k\tilde{\theta}(z) - \int_z^{z_H} m(\tilde{\theta}(z_0))c'(z_0)dz_0 - U(z_H) \right] F'(z) dz \\
&= \int \left[m(\tilde{\theta}(z))(h(z) - c(z)) - k\tilde{\theta}(z) - m(\tilde{\theta}(z))c'(z)\frac{F(z)}{F'(z)} \right] F'(z) dz - U(z_H)
\end{aligned}$$

The third equality follows from Equation 32. The fourth equality uses the relationship between $U(z)$ and $\tilde{p}(z)$ and also Equation 31. The fifth equality is established using integration by parts³⁴. Therefore,

$$U(z_H) = \int \left[m(\tilde{\theta}(z))(h(z) - c(z) - c'(z)\frac{F(z)}{F'(z)}) - k\tilde{\theta}(z) \right] F'(z) dz \quad (33)$$

According to Equation 31 and because $c'(\cdot) > 0$, if $U(z_H) \geq 0$, then $U(z) \geq 0$ for all z . Hence, the following inequality,

$$\int \left[m(\tilde{\theta}(z))(h(z) - c(z) - c'(z)\frac{F(z)}{F'(z)}) - k\tilde{\theta}(z) \right] F'(z) dz \geq 0, \quad (34)$$

implies that planner's budget constraint and participation constraint of all types are satisfied.

So far I have reduced IC constraint in the planner's problem to two conditions 30 and 32. Planner's budget constraint and participation constraint of all types are also summarized in Equation 34. Therefore, thanks to Lemma 3, I can rewrite the planner's problem as follows to derive $\tilde{\theta}(z)$ and $\tilde{p}(z)$.

Problem 7. *Planner's problem*

$$\begin{aligned}
&\max_{\theta(z), p(z)} \int \left[m(\theta(z))[h(z) - c(z)] - k\theta(z) \right] F'(z) dz \\
s. \quad &t. \quad c'(z)\frac{d\theta(z)}{dz} \leq 0, \quad U(z_H) \equiv \int \left[m(\tilde{\theta}(z))(h(z) - c(z) - c'(z)\frac{F(z)}{F'(z)}) - k\tilde{\theta}(z) \right] F'(z) dz \geq 0
\end{aligned}$$

From now one, I work with this problem and characterize the solution to this problem.

³⁴For any differentiable functions F and G , if $G(z_H) = 0$, and $F(z_L) = 0$ we will have: $\int_{z_L}^{z_H} F'(z)G(z)dz = -\int_z^{z_H} F(z)G'(z)dz$ using integration by parts. In the above equality, set $\int_z^{z_H} m(\tilde{\theta}(z_0))c'(z_0)dz_0$.

8.6.1 Proof of Proposition 3

I propose the following direct mechanism as a solution to this problem (for Proposition 3):

$$\tilde{\theta}^{CE}(z) = \theta^{FB}(z), \quad (35)$$

$$\tilde{p}^{CE}(z) = c(z) + \frac{U(z_H) + \int_z^{z_H} m(\theta^{FB}(z_0))c'(z_0)dz_0}{m(\theta^{FB}(z))}, \quad (36)$$

where $U(z_H) = \int \left[m(\theta^{FB}(z))(h(z) - c(z)) - m(\theta^{FB}(z))c'(z)\frac{F(z)}{F'(z)} \right] F'(z)dz$ and

$$\tilde{s}^{CE}(z) = 0 \text{ for all } z.$$

Later on, I construct an associated implementable allocation with the above market tightness and transfers. Specifically, I construct off-the-equilibrium-path beliefs for that.

Proof of Proposition 3 under Part 1 of Assumption 5

Proof. I prove Proposition 3 under the first assumption, $h'(\cdot) \leq 0$, using somewhat a backward approach. I first guess that the planner can achieve the first best. That is, the planner can maximize his objective function point-wise (for each z , separately). What I need to do then is to check that the two constraints of Problem 7, monotonicity constraint and $U(z_H) \geq 0$, are satisfied.

The first best level of market tightness, $\theta^{FB}(z)$, is given by

$$m'(\theta^{FB}(z))(h(z) - c(z)) - k = 0. \quad (37)$$

By differentiating it with respect to z , one yields

$$\frac{d\theta^{FB}(z)}{dz} = -\frac{k(h'(z) - c'(z))}{m''(\theta^{FB}(z))(h(z) - c(z))^2}. \quad (38)$$

By assumption, $h'(\cdot) - c'(\cdot) \leq 0$ and $m''(\cdot) \leq 0$, so $\frac{d\theta^{FB}(z)}{dz}$ is negative. Hence, $c'(z)\frac{d\theta(z)}{dz} \leq 0$ constraint in problem 7 is satisfied. Now I calculate $U(z_H)$ and show that it is positive. From equation 33, one can write

$$\begin{aligned} U(z_H) &= \int \left[m(\tilde{\theta}(z))(h(z) - c(z) - c'(z)\frac{F(z)}{F'(z)}) - k\tilde{\theta}(z) \right] F'(z)dz \\ &= \int \left[-\int_z^{z_H} m(\theta^{FB}(z))(h'(z) - c'(z))dz + U^{FB}(z_H) \right] - m(\theta^{FB}(z))c'(z)\frac{F(z)}{F'(z)} \Big] F'(z)dz \\ &= -\int \left[m(\theta^{FB}(z))(h'(z) - c'(z)) + m(\theta^{FB}(z))c'(z) \right] F(z)dz + U^{FB}(z_H) \end{aligned}$$

$$= - \int m(\theta^{FB}(z))h'(z)F(z)dz + U^{FB}(z_H) \geq 0$$

The second equality uses the fact that $\tilde{\theta}(z) = \theta^{FB}(z)$ and also the fact that

$$\frac{d[\max_{\theta}(m(\theta)(h(z) - c(z)) - k\theta)]}{dz} = m(\theta^{FB})(h'(z) - c'(z)).$$

The third equality is derived by using integration by parts. The inequality holds because $h'(z) < 0$ by assumption and $U^{FB}(z_H) \geq 0$ because there are positive gains from trade for all types. Both constraints in Problem 7 are satisfied. Also, because the proposed allocation for the solution is the first best allocation, we do not need to check that any other allocation achieves higher welfare, because this is the highest possible welfare. This completes the proof that the first best is achievable by a feasible mechanism³⁵. \square

Note that in order to show that $\theta^{FB}(z)$ is decreasing, it was sufficient to have $h'(\cdot) - c'(\cdot) \leq 0$ (according to Equation 38). In the next part of the proposition, I replace the assumption $h'(\cdot) \leq 0$ with a weaker assumption, $h'(\cdot) - c'(\cdot) \leq 0$. To satisfy $U(z_H) \geq 0$, I need another assumption summarized in part 2 of Assumption 5.

Proof of Proposition 3 under Part 2 of Assumption 5

Proof. Now Suppose part 2 of Assumption 5 holds. The proof is similar to the previous part. Because $h'(z) - c'(z)$ is negative, according to Equation 38, the first constraint in Problem 7 is satisfied. We just need to show that $U(z_H)$ is positive. Again from equation 33, one can write

$$U(z_H) = \int \left[m(\theta^{FB}(z))[h(z) - c(z)] - k\theta^{FB}(z) - m(\theta^{FB}(z))c'(z) \frac{F(z)}{F'(z)} \right] F'(z)dz.$$

A sufficient condition for the integral to be positive is that the sum of the terms in the brackets is always positive. That is, for all z : $m(\theta^{FB}(z))[h(z) - c(z)] - k\theta^{FB}(z) - m(\theta^{FB}(z))c'(z) \frac{F(z)}{F'(z)} \geq 0$. But at the solution $m'(\theta^{FB}(z))[h(z) - c(z)] = k$, therefore

$$\begin{aligned} & \frac{m(\theta^{FB}(z))[h(z) - c(z)] - k\theta^{FB}(z)}{m(\theta^{FB}(z))} \\ = & \frac{m(\theta^{FB}(z))[h(z) - c(z)] - m'(\theta^{FB}(z))[h(z) - c(z)]\theta^{FB}(z)}{m(\theta^{FB}(z))} = - \frac{\theta^{FB}(z)q'(\theta^{FB}(z))}{q(\theta^{FB}(z))}(h(z) - c(z)). \end{aligned}$$

Hence, for $U(z_H)$ to be positive, it is sufficient to have:

$$\eta(\theta^{FB}(z)) \frac{h(z) - c(z)}{c'(z)} \geq \frac{F(z)}{F'(z)} \text{ for all } z.$$

³⁵The proposition and its proof can be written in the exactly same fashion if instead $c(\cdot)$ is strictly decreasing and $h(\cdot) - c(\cdot)$ is increasing. The result are not reported to save space.

From $m'(\theta^{FB}(z))[h(z) - c(z)] = k$, I can write $\theta^{FB}(z) = m'^{-1}(\frac{k}{h(z)-c(z)})$. Replacing $\theta^{FB}(\cdot)$ in the sufficient condition yields $\psi(\frac{k}{h(z)-c(z)})\frac{h(z)-c(z)}{c'(z)} \geq \frac{F(z)}{F'(z)}$ which is the same as the left hand side of part 2 of Assumption 5. This concludes the proof³⁶. \square

$\tilde{p}(z)$ is increasing in z

I take a derivative of Equation 32 with respect to z to get the following:

$$\frac{d\tilde{p}(z)}{dz} = -\frac{m'(\tilde{\theta}(z))}{m(\tilde{\theta}(z))} \frac{d\tilde{\theta}(z)}{dz} (\tilde{p}(z) - c(z)) \geq 0 \quad (39)$$

The inequality holds because $\tilde{\theta}(z)$ is decreasing in z following the fact that the allocation satisfies IC. Moreover, $\tilde{p}(z) - c(z)$ is positive following the fact that the allocation satisfies the participation constraint.

Constructing an implementable allocation from the direct mechanism

So far, I have constructed the direct mechanism for Proposition 3. I construct the associated implementable allocation $\{P, G, \theta, \mu, t, T\}$ as follows.

$$P \equiv [p_L, p_H] \subseteq P = R_+ \text{ where } p_L \equiv \tilde{p}^{CE}(z_L) \text{ and } p_H \equiv \tilde{p}^{CE}(z_H)$$

and \tilde{p}^{CE} is given by³⁷ Equation 36. The market tightness for this allocation is given by

$$\begin{cases} \theta(p) = \infty & \text{for } p \leq c(z_L) \\ m(\theta(p)) = \min\{1, \frac{U(z_L)}{p-c(z_L)}\} & \text{for } p \in (c(z_L), p_L) \\ \theta(p) = \theta^{FB}(\tilde{p}^{CE-1}(p)) & \text{for } p \in [p_L, p_H] \\ m(\theta(p)) = \min\{1, \frac{U(z_H)}{p-c(z_H)}\} & \text{for } p \in (p_H, \infty) \end{cases}$$

The rest of elements are given as follows:

$$G(p) = \begin{cases} 0 & \text{for } p < p_L \\ \int_{p_L}^p \theta(p) F'(\tilde{p}^{CE-1}(p)) dp & \text{for } p \in [p_L, p_H] \\ 1 & \text{for } p > p_H \end{cases}$$

$$t(p) = \begin{cases} h(z_L) - p & \text{for all } p < p_L \\ h(\tilde{p}^{CE-1}(p)) - p - \frac{k}{q(\theta(p))} & \text{and } p \in [p_L, p_H] \\ h(z_H) - p & \text{for } p > p_H \end{cases}$$

³⁶Note that when $c'(\cdot) < 0$ and $h'(\cdot) - c'(\cdot) \geq 0$, a similar result can be obtained.

³⁷I showed above that $\tilde{p}(z)$ is strictly increasing in z . Also $\tilde{p}(z)$ is continuous, therefore the set of prices in the constructed implementable mechanism is $P \equiv [p_L, p_H]$.

$$\int \mu(z|p)dz = 1 \text{ for all } p \text{ and } \mu(z|p) = \begin{cases} 0 & \text{for } p < p_L \text{ and } z \neq z_L \\ 0 & \text{for } p \neq \tilde{p}(z) \text{ and } p \in [p_L, p_H] \\ 0 & \text{for } p > p_H \text{ and } z \neq z_H \end{cases}$$

$$T = 0$$

The construction is straightforward. We allocate all types the same market tightness and transfer that they were given in the direct mechanism. For construction of off-the-equilibrium-path beliefs, if $p < p_L$, then the only type attracted to this post is z_L . Therefore, $\mu(z|p) = 0$ for all $z \neq z_L$, and $\mu(z|p)$ has a mass point at $z = z_L$. Similarly if $p > p_H$, then the only type attracted to this post is z_H . Therefore, $\mu(z|p) = 0$ for all $z \neq z_H$. Given the above beliefs, we construct the tax amount for all p such that buyers get a net profit of exactly 0 for $p \in P$ and $-k$ for $p \notin P$. Note that choice of t is not unique for $p \notin P$. We could construct t differently such that buyers get any non-positive amount of profit for $p \notin P$. $G(p)$ is easily constructed given the construction of $\theta(\cdot)$.

Now I check the conditions of implementability. The buyers maximization and free entry condition is satisfied due to the construction of t (easy to check). Feasibility or market clearing is also trivially satisfied due to the construction of G . The budget-balance condition is satisfied due to the choice of $U(z_H)$.

Regarding the sellers' optimal search condition, first note that the restriction on off-the-equilibrium-path beliefs is equivalent to³⁸:

$$m(\theta(p)) = \min\left\{1, \inf_{z \in \{z|c(z) < p\}} \frac{U(z)}{p - c(z)}\right\},$$

if $\{z|c(z) < p\}$ is non-empty. Otherwise, set $\theta(p) = \infty$. Now it is easy to see that sellers' optimal search is also satisfied due to the construction of $\theta(p)$. The only thing worth explaining here is why only z_L is attracted to any price less than p_L (and similarly why only z_H is attracted to any price greater than p_H). To see why, I begin by writing the incentive compatibility condition for any feasible mechanism:

$$m(\tilde{\theta}(z_L))(\tilde{p}(z_L) - c(z)) \leq U(z) \text{ for all } z.$$

After using the fact that $U(z_L) = m(\tilde{\theta}(z_L))(\tilde{p}(z_L) - c(z_L))$, we can write:

$$U(z_L) - U(z) \leq m(\tilde{\theta}(z_L))(c(z) - c(z_L)) \text{ for all } z.$$

³⁸See [Chang \(2012\)](#) for a more detailed discussion.

Therefore,

$$\begin{aligned} U(z_L) - U(z) &\leq m(\tilde{\theta}(z_L))(c(z) - c(z_L)) = \frac{U(z_L)}{\tilde{p}(z_L) - c(z_L)}(c(z) - c(z_L)) \\ &\leq \frac{U(z_L)}{p - c(z_L)}(c(z) - c(z_L)) \text{ for all } z \text{ and for } p \in (c(z_L), p(z_L)), \end{aligned}$$

or equivalently,

$$\frac{U(z_L)}{p - c(z_L)} \leq \frac{U(z)}{p - c(z)} \text{ for all } z \text{ and for } p \in (c(z_L), p(z_L)).$$

Therefore by setting $m(\theta(p))$ to be equal to $\frac{U(z_L)}{p - c(z_L)}$, the restriction on off-the-equilibrium-path beliefs is satisfied.

Understanding Part 2 of Assumption 5 better

It is easy to show that if $1/q(\theta)$ is convex, $\eta(\theta)$ is increasing in θ . Also, m' is decreasing in θ by assumption. Hence, $\psi(\cdot)$ is a decreasing function. The second part of Assumption 5 states that for a given distribution, a given z and a given value for $c'(z)$, $h(z) - c(z)$ should be sufficiently high or k should be sufficiently low. The intuition is that the surplus generated by type z should be sufficiently high (or the entry cost sufficiently low) to provide enough resources for the planner to implement the first best allocation. This assumption is exactly the counterpart of part 3 of Assumption 2 in the discrete type case.

8.6.2 Even if $c'(\cdot) \geq 0$ and $h'(\cdot) \geq 0$, the optimal tax schedule may not be monotone

Assume that part 2 of Assumption 5 holds, so FB is achievable and $\tilde{\theta}(z) = \theta^{FB}(z)$. I suppress the superscript CE in this section to reduce the notation. I calculate $m(\tilde{\theta}(z))t(z)$, take its derivative with respect to z and then show that if $c'(\cdot) \geq 0$, $h'(\cdot) \geq 0$ with strict inequality for a positive measure of z and $h'(z_L) = 0$, then $\frac{dt(p)}{dp} \big|_{p=p_L} < 0$.

$$\begin{aligned} m(\tilde{\theta}(z))\tilde{t}(z) &= m(\theta^{FB}(z))(h(z) - c(z)) - k\theta^{FB}(z) - U(z_H) - \int_z^{z_H} m(\theta^{FB}(z_0))c'(z_0)dz_0 \\ &= U^{FB}(z) - U(z_H) - \int_z^{z_H} m(\theta^{FB}(z_0))c'(z_0)dz_0 \end{aligned}$$

The derivative of $m(\theta^{FB}(z))t(z)$ with respect to z is given by:

$$\begin{aligned} \frac{\partial}{\partial z}[m(\tilde{\theta}(z))t(z)] &= \frac{\partial}{\partial z}U^{FB}(z) + m(\theta^{FB}(z))c'(z) \\ &= m(\theta^{FB}(z))(h'(z) - c'(z)) + m(\theta^{FB}(z))c'(z) = m(\theta^{FB}(z))h'(z) \geq 0 \end{aligned} \quad (40)$$

The second equality is derived by applying Envelop theorem to the following maximization problem: $U^{FB}(z) = \max_{\theta} \{m(\theta)(h(z) - c(z)) - k\theta\}$. The inequality holds by assumption.

Similar to the previous proof, it is easier to work with the direct mechanism. Because $\tilde{\theta}(\cdot)$ is strictly decreasing, then the associated implementable allocation must be separating. I show below that $\tilde{t}(z)$ is decreasing in z at $z = z_L$. Then it is readily concluded that tax function must be also decreasing in the price at $p = p_L$, because the allocation is separating and price in each submarket is strictly increasing in the type applying to that submarket.

According to Equation 40, $\frac{d[m(\tilde{\theta}(z))\tilde{t}(z)]}{dz} = m(\tilde{\theta}(z))h'(z)$, therefore

$$\tilde{t}'(z) = h'(z) - \frac{m'(\tilde{\theta}(z))}{m(\tilde{\theta}(z))} \frac{d\tilde{\theta}(z)}{d(z)} \tilde{t}(z)$$

Consider this equality for $z = z_L$. Given the assumption that $h'(z_L) = 0$ and given the fact that $\tilde{\theta}'(z) < 0$, it is sufficient to show that $\tilde{t}(z_L) < 0$. Then it follows that $\tilde{t}'(z_L) < 0$. To calculate $\tilde{t}(z)$, I use the planner's budget-balance condition to write:

$$\int m(\tilde{\theta}(z))\tilde{t}(z)dF(z) = 0.$$

Let $\chi(\cdot) \equiv m(\tilde{\theta}(z))\tilde{t}(z)$. Then,

$$0 = \int \chi(z)dF(z) = -\chi(z)(1 - F(z)) \Big|_{z_L}^{z_H} + \int \chi'(z)(1 - F(z))dz$$

by using integration by parts. Therefore

$$\chi(z_L) = - \int \chi'(z)(1 - F(z))dz < 0.$$

The inequality holds because $\chi'(z) = m(\tilde{\theta}(z))h'(z)$ from Equation 40 and the fact that $h'(z) \geq 0$. Also, the inequality is strict because $h'(z) > 0$ for some z . But $\tilde{t}(z_L) = \frac{\chi(z_L)}{m(\tilde{\theta}(z_L))} < 0$ by definition of $\chi(\cdot)$. The proof is complete.

8.6.3 What if complete information allocation is not achievable

I keep the assumption that $c'(z) > 0$ and $h'(z) - c'(z) \leq 0$, but now assume that the distribution of types is such that the planner cannot achieve the first best. I show in the next proposition that the probability of matching for almost all types must be distorted (relative to the first best) so that IC and budget constraint are both satisfied.

Proposition 3 requires $m(\theta^{FB}(z))[h(z) - c(z) - c'(z)\frac{F(z)}{F'(z)}] - k\theta^{FB}(z)$ to be positive (if $h'(z) \leq 0$ is not satisfied for some z). However, if this expression is negative for some types, then the following result (Proposition 6) requires at least $h(z) - c(z) - c'(z)\frac{F(z)}{F'(z)}$ to be positive

for all z . Also this proposition requires $h(z) - c(z) - c'(z) \frac{F(z)}{F'(z)}$ to be decreasing in z to ensure that the monotonicity constraint is satisfied ($\frac{d\hat{\theta}(z)}{dz} \leq 0$).

Note that generally $\tilde{t}^{CE}(z) \neq 0$ for almost all types. This implies that although the first best is not achievable under the premises of this proposition, the planner can use transfers effectively to achieve higher welfare than the equilibrium. The intuition is the same as in the simple two-type example.

Assumption 7. For all z , $h(z) - c(z) - c'(z) \frac{F(z)}{F'(z)} > 0$ and $\frac{d}{dz} [h(z) - c(z) - c'(z) \frac{F(z)}{F'(z)}] \leq 0$.

Proposition 6. Assume $c'(z) > 0$, $h'(z) - c'(z) \leq 0$ and $U^{FB}(z) > 0$ for all z . Also, suppose Assumption 7 holds. If the first best is not achievable, then there exists a $\nu > 0$ such that the market tightness $\tilde{\theta}^{CE}(z)$ solves the following equations:

$$m'(\theta(z)) \left[h(z) - c(z) - \frac{\nu}{1 + \nu} c'(z) \frac{F(z)}{F'(z)} \right] = k, \quad (41)$$

$$\int \left[m(\theta(z)) \left[h(z) - c(z) - c'(z) \frac{F(z)}{F'(z)} \right] - k\theta(z) \right] F'(z) dz = 0. \quad (42)$$

Moreover, $\tilde{s}^{CE}(z) = 0$ without loss of generality and $\tilde{p}^{CE}(z)$ is obtained similarly as before:

$$\tilde{p}^{CE}(z) = c(z) + \frac{U(z_H) + \int_z^{z_H} m(\tilde{\theta}^{CE}(z_0)) c'(z_0) dz_0}{m(\tilde{\theta}^{CE}(z))},$$

where $U(z_H) = 0$.

Proof. In this case by assumption, the complete information allocation is not achievable. Therefore, the guess-and-verify approach does not work. To solve for planner's problem, consider Problem 7. I first ignore monotonicity constraint. I form the Lagrangian and derive first order condition (FOC). Then I verify that the monotonicity constraint (and consequently IC) is also satisfied. Denote the Lagrangian by \mathcal{L} and the Lagrangian multiplier by ν :

$$\begin{aligned} \mathcal{L} = & \int [m(\theta(z))(h(z) - c(z)) - k\theta(z) \\ & + \nu[m(\theta(z))(h(z) - c(z)) - k\theta(z) - m(\theta(z))c'(z) \frac{F(z)}{F'(z)} - U(z_H)]] F'(z) dz. \end{aligned} \quad (43)$$

The FOC with respect to $\theta(z)$ for all z is given by:

$$m'(\theta(z))(h(z) - c(z)) - k + \nu(m'(\theta(z))(h(z) - c(z)) - k) - \nu m'(\theta(z))c'(z) \frac{F(z)}{F'(z)} = 0. \quad (44)$$

It can be simplified to conform to Equation 41 exactly.

According to the assumptions of the proposition, $h - c$ and $h - c - c' \frac{F(z)}{F'(z)}$ are decreasing in z . Also, ν is non-negative, so $\frac{1}{1+\nu}(h(z) - c(z)) + \frac{\nu}{1+\nu}(h(z) - c(z) - c'(z) \frac{F(z)}{F'(z)}) = h(z) - c(z) - \frac{\nu}{1+\nu} c'(z) \frac{F(z)}{F'(z)}$ is also decreasing in z . Therefore, FOC implies that $\theta(z)$ is decreasing in z as well. As a result, the monotonicity constraint ($c'(z) \frac{d\theta(z)}{dz} \leq 0$) is satisfied. The first best is not achievable, so if the planner allocates all types the market tightness $\theta^{FB}(z)$ (and corresponding transfers from equation 32), then the derived value for $U(z_H)$ becomes negative. (Otherwise first best would be achievable)³⁹.

Assume there exists $\nu > 0$ such that the FOC and BB both hold. Since the objective function is strictly concave in $\theta(\cdot)$, and the objective function is just the sum of concave functions, the objective function is also concave in $\theta(\cdot)$ ⁴⁰. Because of concavity of the objective function, the FOC is sufficient for the solution. Hence, it only remains to show that such $\nu > 0$ exists.

Note that $\theta(\cdot)$ obtained from Equation 41 is continuous in ν . Accordingly, the LHS of Equation 42 is continuous in ν as well. I need to show that the LHS of Equation 42 is negative at $\nu = 0$ and is positive when $\nu \rightarrow \infty$.

If $\nu = 0$, then $\theta(z) = \theta^{FB}(z)$ is the solution to Equation 41. The first best is not

³⁹Among all allocations in which all types are active, the proposed allocation is globally optimum, because the objective function is concave and the constraint set is convex. However, since FB is not achievable, we need to compare the level of welfare from this case (where all types are active) with another case where some types are inactive. Notice that since $U(z)$ is decreasing in z , if some types are not active, then all types above them will not be active either. Therefore, there is a threshold under which all types are active and above which all types are not active. Then, we can apply the same method to the set of types below the threshold in order to calculate welfare. To show that exclusion of some types does not increase welfare, note that we can write the value of the planner's objective function as

$$\int [m(\theta(z))(h(z) - c(z)) - k\theta(z)] F'(z) dz = \int m(\theta(z)) c'(z) F(z) dz.$$

The equality follows from Equation 42 (which is implied by the fact that $U(z_H) = 0$). Since $c'(z) > 0$ for all z , if some types are excluded, then the value of the objective function will be strictly lower. Therefore, it is never optimal to exclude some types.

⁴⁰To show this point, consider a simpler version where the objective is a function of two variables, that is, $g(x_1, x_2) = f(x_1) + h(x_2)$. Also assume $f(\cdot)$ and $h(\cdot)$ are concave in x_1 and x_2 respectively. I want to show that g is concave in (x_1, x_2) . To show that, I form the Hessian as follows:

$$\begin{bmatrix} f'' & 0 \\ 0 & h'' \end{bmatrix}$$

Since f'' and h'' are both negative, the determinant of Hessian is negative. Therefore g is concave.

achievable, so the other constraint in Problem 7 must be violated:

$$\int \left[m(\theta^{FB}(z)) \left[h(z) - c(z) - c'(z) \frac{F(z)}{F'(z)} \right] - k\theta(z) \right] F'(z) dz < 0.$$

If $\nu \rightarrow \infty$,

$$\begin{aligned} & \int \left[m(\theta(z)) \left[h(z) - c(z) - c'(z) \frac{F(z)}{F'(z)} \right] - k\theta(z) \right] F'(z) dz \\ &= \lim_{\nu \rightarrow \infty} \int \left[m(\theta(z)) \left[h(z) - c(z) - \frac{\nu}{1+\nu} c'(z) \frac{F(z)}{F'(z)} \right] - k\theta(z) \right] F'(z) dz \\ &= \int \left[\frac{km(\theta(z))}{m'(\theta(z))} - k\theta(z) \right] F'(z) dz = k \int \frac{m(\theta(z)) - \theta(z)m'(\theta(z))}{m'(\theta(z))} F'(z) dz \\ &= -k \int \frac{\theta(z)^2 q'(\theta(z))}{m'(\theta(z))} F'(z) dz > 0, \end{aligned}$$

where the second equality is derived from the FOC for any ν , that is, $m'(\theta(z))[h(z) - c(z) - c'(z) \frac{F(z)}{F'(z)}] = k$, and the last inequality follows from $q' < 0$ and $m' > 0$. According to intermediate value theorem, there exists a strictly positive ν which satisfies 42. \square

8.7 Proof of Proposition 5

First I define the implementable allocation with two types of taxes similar to the definition of implementable allocation in Definition 6. One difference is that here there is also another tax that buyers should pay when they want to enter each submarket, $t_e(p)$. This tax is collected before agents find a match. Due to this difference, we have to take into account the fact that the amount of entry tax for the open submarkets, $p \in P$, cannot exceed $-k$, otherwise buyers don't have incentive to stay in the matching stage, i.e., they will leave after the entry tax (subsidy, in fact) is paid. This is reflected in condition (i) of the following definition.

Another difference is that here the set of admissible prices is assumed to be $(c(z_L), \infty)$. This is because for any $p \leq c(z_L)$, no seller would have incentive to apply to that submarket, because the seller would get a negative payoff. Therefore, we just assume that such a price cannot be posted. This assumption is made to avoid some technical difficulties.

Definition 10. *An implementable allocation, $\{P, G, \theta, \mu, t, t_e, T\}$, is a measure G on the set of all possible prices, $\mathbb{P} \equiv (c(z_L), \infty)$, with support P , a function $\theta : \mathbb{P} \rightarrow [0, \infty]$, a conditional density function of buyers' beliefs regarding sellers' types who would apply to p , $\mu : \mathbb{P} \times \mathbb{Z} \rightarrow [0, 1]$, a tax function denoting the amount of tax to be imposed on buyers at each submarket conditional on trade, $t : \mathbb{P} \rightarrow \mathbb{R}$, another tax function denoting the amount of tax*

to be imposed on buyers at each submarket conditional on entry, $t_e : \mathbb{P} \rightarrow \mathbb{R}$, and finally the amount of the numeraire good to be transferred to sellers in a lump sum way, $T \in \mathbb{R}_+$, which satisfies the following conditions:

(i) **Buyers' profit maximization, free entry and no commitment**

For any $p \in \mathbb{P}$,

$$q(\theta(p)) \left[\int h(z) \mu(z|p) dz - t(p) \right] \leq k + t_e(p),$$

with equality if $p \in P$. Also,

$$0 \leq k + t_e(p)$$

for any $p \in P$.

(ii) **Sellers' optimal search**

Let $U(z) = \max \left\{ 0, \max_{p' \in P} \{ m(\theta(p'))(p' - c(z)) \} \right\} + T$ and $U(z) = T$ if $P = \emptyset$. Then for any $p \in \mathbb{P}$ and z , $U(z) \geq m(\theta(p))(p - c(z)) + T$ with equality if $\theta(p) < \infty$ and $\mu(z|p) > 0$. Moreover, if $p - c(z) < 0$, either $\theta(p) = \infty$ or $\mu(z|p) > 0$.

(iii) **Feasibility or market clearing**

For all z , $\int_P \frac{\mu(z|p)}{\theta(p)} dG(p) \leq F'(z)$, with equality if $U(z) > T$.

(iv) **Planner's budget constraint**

$$\int_P [q(\theta(p))t(p) + t_e(p)] dG(p) \geq T.$$

Let $\{(\tilde{p}(\cdot), \tilde{s}(\cdot), \tilde{\theta}(\cdot))\}$ denote the direct mechanism with the properties given in Proposition 5. Since all types get a strictly positive payoff and also that the market tightness allocated to different types is all different, if $\tilde{s}(z) \neq 0$ for some z , we can substitute $\tilde{p}(z)$ by $\tilde{p}(z) + \frac{\tilde{s}(z)}{m(\tilde{\theta}(z))}$ for almost all types (because $\tilde{\theta}(z) \neq 0$ for almost all types). Therefore, we can assume without loss of generality that $\tilde{s}(z) = 0$ for almost all z . Furthermore, to avoid technical difficulties, assume that the $\tilde{p}(z)$ and $\tilde{\theta}(z)$ are both differentiable in z .

As shown in part 1 of Lemma 3, $c'(z) \frac{d\tilde{\theta}(z)}{dz} \leq 0$ for all z is a necessary condition for any mechanism which satisfies IC. Since $\tilde{\theta}(z)$ is different for different types by assumption, $\tilde{\theta}(z)$ must be a strictly decreasing **function** in z . Also in the proof of Proposition 3, it was shown that $\tilde{p}(z)$ is given by the following equation in any mechanism that satisfies IC:

$$\tilde{p}(z) = c(z) + \frac{U(z_H) + \int_z^{z_H} m(\tilde{\theta}(z_0))c'(z_0)dz_0}{m(\tilde{\theta}(z))}, \quad (45)$$

where $U(z_H) = \int [m(\tilde{\theta}(z))(h(z) - c(z)) - k\tilde{\theta}(z) - m(\tilde{\theta}(z))c'(z) \frac{F(z)}{F'(z)}] dF(z)$. According to Equation 39, we have $\tilde{p}'(z) = \frac{m'(\tilde{\theta}(z))}{m(\tilde{\theta}(z))} \frac{d\tilde{\theta}(z)}{dz} (\tilde{p}(z) - c(z))$ which implies that $\tilde{p}(z)$ is strictly increasing in z with the assumption of differentiability of $\tilde{\theta}(\cdot)$. This is because $\tilde{p}(z) - c(z)$

is strictly positive, (otherwise that type will get a negative payoff which contradicts the assumption that all types get a strictly positive payoff), and also because $\tilde{\theta}(z)$ is strictly decreasing. Moreover, $\tilde{p}(z)$ is continuous, therefore the set of prices in the constructed implementable mechanism is $P \equiv [p_L, p_H]$ where $p_L \equiv p(z_L)$ and $p_H \equiv p(z_H)$.

I construct the allocation $\{P, G, \theta, \mu, t, t_e, T\}$ as follows and show that if M and M' are chosen sufficiently large, this allocation is implementable and $t_e(p)$ is strictly decreasing and $t(p)$ is strictly increasing in p . The market tightness for this allocation is given by⁴¹:

$$\begin{cases} m(\theta(p)) = \min\{1, \frac{U(z_L)}{p-c(z_L)}\} & \text{for } p \in (c(z_L), p_L) \\ \theta(p) = \tilde{\theta}(\tilde{p}^{-1}(p)) & \text{for } p \in [p_L, p_H] \\ m(\theta(p)) = \min\{1, \frac{U(z_H)}{p-c(z_H)}\} & \text{for } p \in (p_H, \infty) \end{cases}$$

The rest of elements are given as follows:

$$G(p) = \begin{cases} 0 & \text{for } p \in (c(z_L), p_L) \\ \int_{p_L}^p \theta(p) F'(\tilde{p}^{-1}(p)) dp & \text{for } p \in [p_L, p_H] \\ 1 & \text{for } p > p_H \end{cases}$$

$$t_e(p) = \begin{cases} -k + M(p_H - p) & \text{for } p \in (c(z_L), p_H] \\ -k & \text{for } p \in (p_H, \infty) \end{cases}$$

$$t(p) = \begin{cases} h(z_L) - p - \frac{k+t_e(p)}{q(\theta(p))} & \text{for all } p \in (c(z_L), p_L) \\ h(\tilde{p}^{-1}(p)) - p - \frac{k+t_e(p)}{q(\theta(p))} & \text{and } p \in [p_L, p_H] \\ t(p_H) + M'(p - p_H) & \text{for } p > p_H \end{cases}$$

$$\int \mu(z|p) dz = 1 \text{ for all } p \text{ and } \mu(z|p) = \begin{cases} 0 & \text{for } p < p_L \text{ and } z \neq z_L \\ 0 & \text{for } p \neq \tilde{p}(z) \text{ and } p \in [p_L, p_H] \\ 0 & \text{for } p > p_H \text{ and } z \neq z_H \end{cases}$$

$$T = 0$$

Now I check the conditions of implementability. The buyers profit maximization and free entry condition is satisfied due to the construction of t and t_e (easy to check). Feasibility (or market clearing) is also trivially satisfied due to the construction of G . The budget-balance

⁴¹If there are more than one $\theta(p)$ consistent with the above equation, then choose the largest one. This does not happen here though, because we have assumed in this section that m is strictly increasing.

is satisfied due to the choice of $U(z_H)$. Sellers' optimal search condition is satisfied and the argument is exactly similar to one in page 78, so I skip it.

Regarding monotonicity of taxes, it is obvious that $t_e(p)$ is decreasing in p for any $p \in [p_L, p_H]$ for any $M > 0$. It is just left to show that $t(p)$ is increasing in p . I take a derivative of $t(p)$ with respect to p :

$$t'(p) = h'(\tilde{p}^{-1}(p)) \frac{d(\tilde{p}^{-1}(p))}{dp} - 1 + M \frac{q(\theta(p)) + q'(\theta(p))\theta'(p)(p_H - p)}{q(\theta(p))^2}.$$

Now, define

$$M_1 \equiv \min \left\{ 4, \sup_{p \in [p_L, p_H]} \frac{(1 - h'(\tilde{p}^{-1}(p)) \frac{d(\tilde{p}^{-1}(p))}{dp}) q(\theta(p))^2}{q(\theta(p)) + q'(\theta(p))\theta'(p)(p_H - p)}, \sup_{p \in [0, p_L]} \frac{q(\theta(p))^2}{q(\theta(p)) + q'(\theta(p))\theta'(p)(p_H - p)} \right\}.$$

M_1 is a lower bound for M . Note that 4 is just an arbitrary positive number. Also, the third expression in the min has been derived similarly to the second expression but for the case with $p \in (c_L, p_L]$. I want to show that $M_1 < \infty$, so I need to show that the second and third expressions in the min are less than ∞ . Note that if $q(\theta(p)) \rightarrow 0$, then the expression goes to 0, therefore I just need to show that $\frac{d(\tilde{p}^{-1}(p))}{dp} > -\infty$. I have already calculated $\frac{d\tilde{p}}{dz}$ in Equation 39 and have shown that $\frac{d\tilde{p}}{dz} > 0$ for all z . Therefore, $\frac{d(\tilde{p}^{-1}(p))}{dp}$ which is just the inverse of $\frac{d\tilde{p}}{dz}$ is always positive too. Since z lies in a compact interval, $1 - h'(\cdot) \frac{d\tilde{p}}{dz}$ is not greater than 1 and the proof in this part is complete.

For $p \in (c(z_L), p_L)$, we can similarly find $M_2 > 0$ such that if $M > M_2$, then $t(p)$ is strictly increasing. For $p > p_H$, again we can similarly find $M_3 > 0$ such that if $M' > M_3$, then $t(p)$ is strictly increasing. Since $t(p)$ is continuous by construction, therefore the fact that it is increasing in different intervals implies that it is increasing in the entire domain.

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