End of Days

Gabriele Camera and Filip Veselý[1]

Purdue University

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Abstract

Is autarky the only incentive-feasible trading arrangement if the economy or the record-keeping technology are transient? In the absence of commitment and memory, agents can produce in autarky or specialize and then trade bilaterally exploiting an imperfect record-keeping technology, perishable tokens. Infrequent consumption generates disutility and individual market participation creates an externality. Trading arrangements are endogenously selected. We prove existence of an equilibrium where tokens are exchanged until their demise or, in a finite-horizon economy, some date prior to the last. As the size of the market increases, individual participation confers smaller benefits to others and the equilibrium ceases to exist. JEL E40, C7, D62

1. INTRODUCTION

Economists have long ago embraced the idea that allocations can be improved by specialization and trade, and can be further expanded by innovations in the technology used to carry out transactions. The use of money as a medium of exchange, what Adam Smith compared to "a highway" and David Hume praised as "the oil which renders the motion of the wheels of trade more smooth and easy", has been at the core of the transaction technology of most societies. While several models have addressed the issue of equilibrium valuation of money[2], the notion of fiat money being a *technological innovation* has been formalized only recently (Kocherlakota, 1998). Money can expand the set of allocations available in the economy because it can act as a useful–although

[1] This is a still preliminary version. We thank Ted Temzelides and seminar participants at Purdue University for helpful comments.Corresponding author's e-mail: Gcamera@mgmt.purdue.edu.

[2] A popular approach has been using overlapping generations models based on Samuelson (1958), where fiat money is valued because of physical restrictions on intergenerational exchange. Others have considered decentralized trading environments with explicit exchange frictions (e.g. Townsend, 1980, Freeman, 1989, and Kiyotaki and Wright, 1989).

imperfect–record-keeping device in the absence of other forms of memory (e.g. a public record of past private transactions).

Virtually all models where a record-keeping technology–call it *tokens*–is a technological innovation assume that such a technology is permanent and the economy is everlasting. Cass and Shell (1980) articulate why this must be so in a rational expectations economy with a final date known to everyone. For tokens to be valued in equilibrium *some* individuals must expect that they can use them in future transactions. Thus, public knowledge of absence of future trading rounds is sufficient to deprive the tokens of value. The same argument applies to an infinitely-lived economy where the record-keeping technology is not permanent (e.g. tokens are perishable), with the same result. This suggests that 'durability' is a crucial property not only of the economy but also of the record-keeping technology, if the latter can be a technological innovation. When this property is absent, decentralized trade cannot be facilitated by the use of tokens and the economy will not take (full) advantage of the benefits of specialization.

These issues have considerable implications for theoretical modeling of payment systems, and a natural question arises. Do a temporary record-keeping technology or a finite horizon *necessarily* prevent agents from taking advantage of even some of the possible gains from specialization and trade? We provide an answer, in the negative, by studying a decentralized trading environment where a record-keeping technology, 'barren' tokens, is essential in expanding the allocation set. The individuals' use of the technology and participation in market activities are endogenous and provide an externality. We show that removing the 'durable' attribute from either the tokens, or the economy, is *not* sufficient to prevent tokens from being a technological innovation. Thus, the main message is that participation externalities have important implications for the viability of the technology that supports a decentralized trading arrangement. When strong, these externalities can overcome the drawbacks due to even the most basic imperfection in the technology, its transience.

We articulate these ideas using a conceptual framework abstract in nature, but simple to understand, where record-keeping devices can be a technological innovation, but have a finite, and *publicly* known, horizon. The model is a limited-commitment and limited-communication matching economy with frictions, and a finite number of agents. Trade-intermediary institutions are absent. There is limited-communication in that transactions are unobservable and there is no communication across locations, as in Townsend (1989). Similar to Kocherlakota (1998), there is limited-commitment in that agents can always refuse to implement a proposed exchange and choose autarky. Capacities and location of different sellers are known, buyers select which seller to meet but matching is "with frictions" as in Burdett, Shi and Wright (forthcoming). In equilibrium too many (or too few) buyers may simultaneously visit a seller, in which case some buyers (or sellers) must be rationed. Finally, there is *no memory:* agents cannot recall anything they have observed in prior dates, as in Corbae, Temzelides, and Wright (2000). This gives an explicit role to tokens, yet allowing us to work with a very simple framework.

The argument we develop goes as follows. Agents can improve over autarky or barter by specializing in production and selecting one of several decentralized-trade arrangements. Suppose agents independently choose to make use of tokens until a date sufficiently ahead of the economy's last. There are incentives to produce for someone who has a token on this date, if by so doing the seller fosters market participation in the remaining periods. There are incentives to *require* the transfer of the token if this helps avoiding an undesirable allocation on this and all subsequent dates. Moving backward, tokens will be used in all prior dates to exploit the benefits from specialization and trade. Three features concur to support such an equilibrium. First, by specializing and participating in market exchange an individual confers benefits to others. Second, an incorrect allocation of goods within the period can be a source of disutility for some individuals. Third, there are alternatives to decentralized market exchange and incentives to abandon it, if too inefficient.

We proceed as follows. In section 2 we review previous relevant studies. Section 3 describes the environment and section 4 discusses the trading mechanism and the equilibrium concept adopted. We study valuation of perishable tokens in both an infinite- and finite-life economy, in section 5. We prove that in both instances there exists an equilibrium where the record-keeping technology is used to facilitate market transactions. Section 6 concludes. The appendix contains proofs of lemmas and propositions.

2. FIXED-LIFE ECONOMIES IN PRIOR LITERATURE

A few papers have addressed the issue of existence of fiat money in finite economies.[3] Faust (1989) considers a continuous time economy on a finite interval $[0, \tau]$. Fiat money is valued and held as a *precaution* due to a mismatch between payment and consumption dates. He shows existence

[3] Cuadras-Morato' (1998) studies a search economy where valued *commodity* money is finitely lived.

of an hyperinflationary path as the economy approaches τ, with money valueless in the limit. Two features generate this behavior. Marginal utility is *unbounded* in the absence of consumption. Every agent has *some* prospect that she won't be the last in the trading sequence, due to the continuity of time. Thus, although the probability of future use of money steadily decreases over time, there is an incentive for someone to hold cash irrespective of the closeness of the ending date. Doing so insures against the possibility that, should a consumption need arise, she is unable to make a purchase (however small).

Kovenock and de Vries (2000) propose two models of finite and discrete-time economies. In the first, private transactions and own position in the trading sequence are unobservable (agents are "unaware" of the passage of time). Thus, the final transaction date is uncertain for some agents. These traders might prefer to take a gamble, selling for fiat money. When the gains from trade are sufficiently large, the expected value of the gamble is strictly positive. In the last trading period, however, there is no monetary exchange and those who took the gamble in the prior round are made worse off. In the second model, transactions and own position in the trading sequence are observable, and the last trading date is common knowledge. There are multiple non-monetary equilibria in the stage game, due to the presence of an outside economic opportunity (autarky). In this case, agents may choose to play trigger strategies based on the occurrence of monetary exchange in the period next to the last. Autarky by all is triggered in the last round if, in the prior date, someone refused monetary exchange. If in the stage game there is a substantial payoff differential between autarky and the alternative non-monetary transaction pattern, then monetary exchange will be preferred by all, in the period prior to the last.[4]

Kultti (1995) models a decentralized trading environment similar to Kiyotaki and Wright (1989), with a publicly known terminal date. Agents are *exogenously* allowed to participate in market trade, using barter or monetary exchange, *only if* they have a good or money. Costly production and trade are mutually exclusive. Since own output cannot be consumed and barter is possible, it may be worthwhile to produce in the penultimate period to have access to the market in the last period. In the last period the agent may be lucky and barter her good. She may be unlucky if she can't

[4] We note, however, that here the use of fiat money is not essential to achieve a Pareto superior outcome. The same trigger strategy can be used to support an equilibrium whereby agents exchange gifts up until the penultimate round. Commodities can also serve as a medium of exchange, since they are durable.

barter or if she meets someone with money. Since the production cost is sunk, however, it doesn't hurt to make a *gift* to someone who has money (monetary transfer is inessential in that date). Knowing this, it may be optimal to take a gamble in the penultimate period, by accepting money and avoiding costly production.

These models share one or both of the following features. First, agents are asymmetrically informed on either the last trading date or their position at that date (buyer or seller). Second, money transfers are not necessary at all dates, although they might take place just because it doesn't hurt to do so. In contrast we model a discrete-time environment where (i) the duration of the economy and the record-keeping technology is common knowledge, (ii) agents have no uncertainty over their own position in the last trading date, and (iii) the physical transfer of tokens is necessary in every period in which the record-keeping technology is used to facilitate decentralized trade.

3. ENVIRONMENT

Time is discrete and continues forever. There is a population of $2N > 4$ spatially separated individuals, identified by $i \in \{1, ...2N\}$. There are three kinds of indivisible objects: storable tokens, and two non-storable market goods, a *specialty* and a *common* consumption good. An agent can hold at most one of these objects, and can freely discard it. The specialty commodity can be of $2N$ different varieties, and the common good is homogeneous.[5] At the beginning of $t = 0$, $M = N$ tokens are distributed to every other agent.

Agent i derives utility from consumption of any common good and variety i of the specialty good. No utility is derived from consumption of other varieties, tokens, or own production; future utility is discounted by $\beta \in (0, 1)$. Absence of any consumption for three or more periods generates disutility $c > 0$. Let $q_t^i = 1, 2...$ (q_t^B) denote the units consumed of variety i of the specialty good (of the common good) during t. Let $q_t = q_t^i + q_t^B$, and let $U_i(q_t)$ denote agent i's associated utility:

$$U_i(q_t) = \begin{cases} q_t^i u + q_t^B u^B & \text{if } \; q_t \in \{1, 2...\} & \forall \{q_{t-j}\}_{j=1}^t \\ 0 & \text{if } \; q_t = 0 \;\text{ and }\; q_{t-1} + q_{t-2} \geq 1 & \forall \{q_{t-j}\}_{j=3}^t \\ -c & \text{if } \; q_t = q_{t-1} + q_{t-2} = 0 & \forall \{q_{t-j}\}_{j=3}^t \end{cases} \tag{1}$$

At the end of period t an agent may be in one of three states. She might have consumed q_t during the period, enjoying utility $q_t^i u + q_t^B u^B$. In this case she can forego consumption for two more

[5] Enriching the model by assuming storability of commodities generates competition between tokens and commodities as record-keeping devices (much as in Kiyotaki and Wright, 1989). This issue is not central to our discussion.

periods before suffering any disutility. If consumption occurred in either (or both) of the prior two periods, but not today ($q_t = 0$), she enjoys no utility (and risks suffering $-c$ tomorrow, should she be unable to consume then). Absence of consumption today generates $-c$ if the agent has been unable to consume in the past two periods.

Every individual has the ability to produce a common or a specialty good, but not both, during a period. The type of production must be selected at the end of the prior period. Choosing specialty production allows the agent to realize, during the period, one unit of any variety of the specialty good with disutility $e > 0$. The good must be consumed as it is produced (one can think of this as a custom-made service). Alternatively the agent can select to costlessly produce one transportable common good, realized at the beginning of the following period (i.e. the agent produces outside the market for one period). Unsold goods perish at the end of the period in which they have been realized. The agent can choose at any time to permanently leave the market from the beginning of the next period on. By doing so she loses the ability to engage in market production but acquires the ability to generate autarchic production whose output she can consume. The lifetime value of autarky is normalized to zero. It is assumed that

$$0 < u^B < e < \beta u/2 \tag{2}$$

hence the surplus generated from specialty production is larger than that generated from common production. The first inequality provides a lower bound for the surplus from specialized trade: it is larger than the cost of specialty production, $\beta u - e > e$. The second and third inequalities provide bounds for the gains from bartering the common good.

Trade occurs via a matching process which is endogenous, bilateral, memoryless, and subject to frictions. Specifically, it is assumed that market production and search for a partner are mutually exclusive activities. Those who choose to produce during the period are called *sellers;* everyone else is a *buyer*, who travels to meet sellers. At each date buyers and sellers are randomly assigned to a location in such a manner that each buyer has one closest seller. Figure 1 is one possible representation, where $N = 4$ (buyers are the empty circles and can move along the dotted lines to reach sellers, the filled-in circles). Travel generates disutility $\varepsilon > 0$ unless it involves the closest location, costlessly reachable. Trade is bilateral and matches cannot last more than one period. If multiple buyers simultaneously meet and offer to buy from a seller, the latter sells to only one of them adopting a rationing rule such that each buyer has equal probability of making the purchase.[6] Those buyers who are rationed can resume search and travel to at most $N-1$ new locations during the period; thus there may be multiple stages in the trading process. A complete lack of memory is assumed: actions and outcomes observed in prior periods cannot be recalled. Thus, the matching process cannot depend on histories. The partner's actions and inventory are observable and the initial number of individuals and tokens is common knowledge. Furthermore, there is lack of communication across locations and limited commitment, transactions cannot be observed by third parties and trade agreements cannot be enforced.[7]

Although extreme, these assumptions allow us to work with a simple conceptual framework where tokens can prove to be a technological innovation. They can be used to lessen the limitations to memory, communication, and enforcement. The assumptions made also allow us to capture explicitly the following features. First, there are gains from trading scarce resources and overall efficiency is increased by specialization in economic tasks (seller or buyer) and production (common or specialty). Second, individual market participation is endogenous and generates an externality. Third, there are incentives to resort to autarky instead of market exchange because infrequent consumption generates disutility. In particular, buyers may attempt to consume excessively during a period leaving someone else hungry, leading to an 'undesirable' market allocation. The mode and extent of market participation affects the equilibrium use of the record-keeping technology, the efficiency of the trading arrangement and, in turn, welfare.

4. TRADING MECHANISM AND EQUILIBRIUM

We illustrate market activity and, following Kocherlakota (1998), the limited-commitment trading mechanism, and the equilibrium concept adopted. Market activity has several stages, which may or may not involve a bilateral match. In each stage the agent can take only one action which induces a single outcome. In a *matched stage*, the agent's action depends only on hers and her partner's inventory. We consider a direct trading mechanism; the two agents play a *coordination game*, simultaneously proposing a feasible transfer of goods and tokens. If consistent, the proposals are implemented, otherwise the agents depart. The outcome is observable only in the match. In

[6] Burdett, Shi, and Wright (2000) formalize the matching and rationing procedure in a similar context where one location per period can be visited.

[7] As in Townsend (1989), matches take place in a foggy location where agents make unobserved deals.

an *unmatched stage*, the agent's action may depend on her state, her inventory, the date, and the actions observed during the period (but not prior periods, because of lack of memory). She chooses whether to participate in the market, currently and in the future. If current participation is chosen, as a buyer the agent selects a location to visit, and a seller she stands ready to produce (she remains inactive otherwise). If future participation is chosen, as a seller the agent must select a type of market production, and as a buyer she carries her token into the next period. The alternative is to leave the market forever undertaking autarchic production, from the following period on. The outcome can be privately observed by some agents.

We model limited commitment by requiring that the trading mechanism satisfy sequential individual rationality. In a matched stage the agent can choose to not produce and to retain her initial inventory, for any action taken by her partner, or by her prior to that stage. Clearly, in the unmatched stage the agent is always free to not participate in market activities currently or from tomorrow on. Thus, the actions allowed and the associated outcomes are such that, at any stage of market activity, the agent can always get her reservation value. We study allocations that are compatible with individual incentives. That is, everyone chooses only equilibrium actions, and all actions and their outcomes (both in and out of equilibrium) satisfy sequential individual rationality. We focus on symmetric rational expectations equilibria in pure strategies. For the limited commitment trading mechanism described, a symmetric equilibrium strategy specifies a sequence of actions that is identical for agents in the same state. Agents take market payoffs and strategies of others as given, and actions are based on the correct evaluation of trade opportunities and payoffs. The equilibrium must also specify "off-equilibrium" actions as strictly optimal responses to knowledge of departures from the equilibrium sequence of actions. Equilibrium actions, however, are always selected because no deviation occurs.

In what follows we restrict attention to equilibria where market participation is stationary, and where the transaction pattern is time-invariant, absent breaks in the availability of transaction technologies. We consider a simple time-dependent transaction pattern, otherwise. We check only unilateral one-stage deviations, and do not allow for multiple deviations (say, two agents deviating in some stage at some date). For simplicity, we focus on the case where a buyer can visit at most two adjacent locations (see figure 1) and where ε is arbitrarily small. Thus, there are three relevant stages (and associated actions) of market activity (see figure 2). If the agent is unmatched and has not consumed during the period, she must choose her current market participation. If the agent is matched she must propose a trade to her partner. If the agent is unmatched *and* has consumed during the period she chooses market participation in remaining part of the period and the next.

5. MARKET EQUILIBRIA

5.1 Durable Tokens and Infinite Horizon

To start, we describe decentralized allocations with and without token exchanges, in an infinite horizon economy with a permanent record-keeping technology. Conjecture a stationary equilibrium where, in each period, N agents buy with a token the specialty good produced by the closer of the remaining N agents. Each agent alternates one period of consumption to one of production and never sustains the travel cost ε. A buyer makes exactly one purchase in even periods ($t = 0, 2, 4...$), and the remaining $M = N$ agents consume in odd periods. We refer to this equilibrium pattern of exchange as *indirect exchange* (*IE*). In a stationary equilibrium the value functions of a representative buyer, v_b, and seller, v_s, satisfy

$$v_b = \frac{u - \beta e}{1 - \beta^2}, \quad v_s = \frac{\beta u - e}{1 - \beta^2} \tag{3}$$

Lemma 1. *In an infinite-horizon economy with durable tokens IE is always an equilibrium.*

Suppose the market expects that at every date *every* seller sells a specialty good for a token. If the cost of specialty production is larger than the utility from consumption of a common good no seller would trade her specialty good for a common one. In equilibrium buyers have the largest lifetime utility. Thus, if a seller is willing to forego common consumption today in order to get a token tomorrow, a buyer would surely reject common consumption today in favor of specialty consumption tomorrow. Thus, a seller cannot increase her lifetime utility by producing a common good once, and then reverting back to equilibrium. Assumption (2) assures that the temporary utility from consumption of the common good is low enough ($u^B < e$), and that alternating production to consumption is a feasible pattern of exchange ($e < \beta u$).

Note that *IE* is not the only stationary equilibrium pattern of exchange existing. For example, if tokens are expected to be valueless the following equilibrium also exists. In each period in which the agent has no good (even periods) she produces a common good, and she barters it in the following period. We refer to this pattern of exchange as *barter exchange* (BE) where the lifetime utilities in odd and even periods are

$$v_{odd} = \frac{u^B}{1-\beta^2} > v_{even} = \frac{\beta u^B}{1-\beta^2} \tag{4}$$

Lemma 2. *In an infinite-horizon economy with durable tokens BE is always an equilibrium.*

Both patterns of exchange, *BE* and *IE*, support deterministic consumption and production in alternate periods. Under a welfare criterion which assigns equal weight to each individual, the equilibrium associated with *IE* is Pareto superior to *BE*.

Clearly there can be also other symmetric stationary equilibria with or without market exchange, but they are less interesting for the purpose of our discussion.[8] What we want to emphasize here, is that although other ways to organize market exchange may exists, the pattern of production and exchange *IE* delivers the highest welfare. The proof is straightforward. First, although common production is costless, it is also inefficient relative to specialty production: it generates lower surplus, due to (2), and it impedes market participation for one period. Second, agents cannot do any better than trading once every other period with the closest location, given the technologies available (production, matching, record-keeping and storage).[9]

Finally, we note that this environment allows for "gift giving" equilibria involving (costless) production and exchange common goods. These equilibria are obviously Pareto inferior to *IE*.

[8] For example, there is an equilibrium without market participation. Equilibria with barter of specialty goods cannot exist because their production and trade are mutually exclusive. Equilibria where tokens are exchanged for common goods also do not exist. A token holder would prefer to dispose of it, and produce a common good. In this way she does not decrease her probability of a trade but can increase the frequency of consumption. Equilibria with valued tokens and travel to the farthest seller (or mix on visits to locations) are Pareto inferior to *IE*.

[9] Societal welfare, however, could be increased by a planner whose choice of actions is not restricted to those incentive feasible. For example the planner could divide the economy into three even groups, A, B and C. Each group produces specialty goods for two consecutive periods and consumes two specialty goods per person, in one period. For instance, in t the agents in group A eat 2 goods each, while agents in B and C produce. In $t+1$ agents in B eat while agents in A and C produce. In $t+2$ agents in C eat and agents in A and B produce. Then the cycle starts again. In this way each agent in group A earns lifetime utility $\frac{2u-\varepsilon-e\beta(1+\beta)}{1-\beta^3}$. Welfare will be larger than the one obtained in *IE* is ε is small and individuals sufficiently impatient.

Because of the extreme lack of memory, there cannot be symmetric equilibria which involve the exchange of specialty goods as "gifts". Absent tokens, the agents would have to divide themselves into two groups alternating between the role of producers and consumers. Actions, however, cannot be based on histories and everyone would rather begin as a consumer, due to discounting.

5.2 Fixed-life Tokens and Infinite Horizon

Suppose it is publicly known that the tokens initially issued last only T periods (i.e. they cease to exist at the end of $T-1$). A new supply $M = N$ of durable tokens is known to be randomly distributed at the beginning of T prior to any action being taken by agents. The N new tokens are assigned with probability $1/2$ to those who started the prior period as sellers, independent of their actions. With the complementary probability the new tokens go to those who started the prior period as buyers.

We prove existence of an equilibrium in which tokens are valued, each one representing a claim to one specialty good. We focus on the case where everyone participates in the market in every date buying from the closest agent or selling a specialty good for a token. Thus, half of the agents expect to deterministically alternate consumption of the specialty good to its production until $T-1$ (for the other half the cycle is inverted). In period T this cycle is re-initialized with consumption and production expected to be equally likely.

Due to the similarity with the pattern of indirect exchange previously discussed, we also refer to this equilibrium pattern of exchange as *IE*. Below we define the actions that characterize it, in and out of equilibrium.

Definition of *IE. In an unmatched stage the agent participates in the market, visiting the closest seller as a buyer, or choosing to produce a specialty good as a seller. In a matched stage, a seller proposes to exchange her specialty good for a token and a buyer to exchange her token for a specialty good. Autarky is chosen if it is the end of the period and the agent is an unmatched buyer who has not consumed, or a seller who has not traded.*

Under the conjecture that *IE* is an equilibrium, let $V_s(t)$ and $V_b(t)$ denote the lifetime utility of sellers and buyers at the beginning of t. In a stationary equilibrium

$$V_s(t) = v_s, \quad V_b(t) = v_b \text{ if } t \geq T$$

and for $t < T$ we can define the lifetime utility recursively using (1):

$$V_s(t) = \begin{cases} -e + \beta \max\{V_b(t+1), 0\} & t < T-1 \\ -e + \beta \max\{v, 0\} & t = T-1 \end{cases} \tag{5}$$

$$V_b(t) = \begin{cases} u + \beta \max\{V_s(t+1), 0\} & t < T-1 \\ u + \beta \max\{v, 0\} & t = T-1 \end{cases} \tag{6}$$

where

$$v \equiv \frac{1}{2}(v_b + v_s) = \frac{u-e}{2(1-\beta)}$$

is the expected value of receiving a new token. At every point in time the agent can choose to be inactive and leave the market from the next period on. The value associated to this option is zero.

The proof that *IE* is an equilibrium is developed via a sequence of lemmas each of which focuses on an equilibrium action or a one-time departure taken in $T-1$, and on an "off equilibrium" action taken following an observed deviation. Specifically, under the conjecture that *IE* is the equilibrium, we consider (i) refusing to produce a specialty good in exchange for a token in $T-1$, (ii) choosing to buy twice during $T-1$, and (iii) choosing to not participate in the market in T and subsequent periods. We provide conditions sufficient to insure that the "off-equilibrium" actions specified in definition 1 are strictly optimal responses to knowledge of a departure. In particular, we show that in $T-1$ and T anyone who has not consumed for two consecutive periods strictly prefers autarky. Losing a participant generates uncertainty in the execution of future trades, a negative externality. This creates incentives to produce for those who have a claim to a specialty good in $T-1$. Despite its uselessness for record-keeping, tokens are exchanged in $T-1$ if this helps to dissipate the risk of an 'undesirable' allocation.

We set the stage by defining functions of the parameters of the model (omitting the arguments, when understood) used to identify boundaries of existence regions in the ensuing discussion:

$$c_L(\beta, e, u) = \frac{u-e}{1-\beta}, \quad \beta_L(c, u) = \max\{\beta_1(c, u), \beta_2(c, u)\}$$

$$N_L(\beta, c, u) = \frac{\beta(2+\beta-2\beta^2)}{(1+\beta)^2(1-\beta)} + \sqrt{\frac{2c\beta^3}{u(1-\beta^2)}}, \quad N_H(\beta, c, u, e) = \frac{A_2 + \sqrt{(A_2)^2 - 4A_1A_3}}{2A_1}$$

where $N_L, N_H > 0$ for all $\beta \in (0,1)$; here $\beta_1(c,u)$ and $\beta_2(c,u)$ denote two critical values of β that lay in the unit interval.[10] Conjecture that *IE* is an equilibrium. In the next three lemmas we consider the choice of abandoning the market, as a specialty seller or as a buyer with a token. We first show that an agent prefers to participate in the market, in equilibrium.

Lemma 3. *Suppose IE is an equilibrium and no deviation has been observed. Then it is individually optimal to participate in the market.*

In the next two lemmas we consider the choice of market participation when a deviation has been observed. We start by discussing the case of a buyer who, during $T-1$, is unable to purchase a specialty good from her closest seller. This may be the consequence of one of the following out of equilibrium actions: the seller (i) has chosen to not produce in exchange for a token, (ii) has chosen to not participate in the market in that period, (iii) has produced for another buyer during the period (and rationed everyone else out), or (iv) she has left the market at the beginning of the period. We provide a sufficient condition for a buyer to choose autarky, if she cannot consume in $T-1$.

Lemma 4. *Suppose IE is an equilibrium and in $T-1$ a buyer is unable to purchase a specialty good from her closest seller. Then, if $c > c_L$ it is individually optimal for the buyer to leave the market in T.*

If $c \leq c_L$ the protracted lack of consumption does not generate much disutility; hence, the buyer who cannot consume a specialty good in $T-1$ does not resort to autarky. This, however, may induce sellers to not produce for a token, in $T-1$; hence the requirement $c > c_L$. There is a trade-off between c_L, the degree of patience β, and the cost of specialty production e (see appendix). Specifically, the more patient the buyer, the larger her lifetime utility; thus, c_L must rise in β. A larger cost of production lowers lifetime utilities, hence c_L must fall in e. For this reason in what follows we restrict attention to those e that are *feasible*, that is all those production costs that

[10] These functions are derived in the appendix. Also, $A_1 = 2(1+\beta)^2(1-\beta)$, $A_2 = \beta\left[1-2\beta-2\beta^2+\left(1+\beta+\beta^2\right)\frac{u}{e}\right]$ and $A_3 = -\beta^2\left[1+\frac{1}{e}\left((1+\beta)c-u\right)\right]$

satisfy (2) and $c > c_L$. It follows that e is feasible whenever it lays between max $\left\{u - c(1-\beta), u^B\right\}$ and $\beta u/2$ (a non-empty open set, since $u^B < \beta u/2$). [11]

Now, suppose that a seller in $T-1$ refuses to produce for a token. We show that, despite the deviation, there is *at least* one equilibrium (but there could be more) with an active market in all $t \geq T$. To do so we consider agent i's choice of market participation given that her closest trade partner has left the market in T, right after the distribution of new tokens. Under the conjectured pattern *IE*, agent i is unable to buy or sell, during T. Focus on the case where he who left had been a buyer in $T-1$ (this is not directly observable).[12] We provide conditions sufficient to guarantee that no one chooses autarky if the token supply is unchanged, following the exit. If the token supply has dropped, one more seller chooses autarky from $T+1$ on.

Lemma 5. *Let the condition in lemma 4 be satisfied. Suppose IE is an equilibrium and someone who was a buyer in $T-1$ leaves the market in T, after the distribution of new tokens. Then, there exists an $N_1(\beta, c, u) > 0$ such that:*

i. If the agent left without a token, it is optimal for everyone to remain in the market if $N > N_1$.

ii. If the agent left with a token, it is optimal for her closest seller to choose autarky from $T+1$.

The departure of an agent in T results in an uneven number of sellers and buyers. This generates trading risk and reduces incentives to market participation. None of the remaining agents prefers to abandon the market, if there is a sufficient number of participants and tokens. In case (*i.*) a

[11] As $u^B \to 0$ feasibility of e depends only on c_L. The examples presented have $u = 1,\ u^B = 0.09,\ c = 8,\ \beta = 0.9$ and $k = 10$; hence $u - c(1-\beta) > u^B$. Figure 3 illustrates that the feasible e's lay above the horizontal line labeled $c = c_L$.

[12] We could consider exit by someone who was a seller in $T-1$, or exit in other dates. The case discussed, however, is the most general. The date T is the most stringent case since the loss $-c$ is more likely to arise in T due to the possibility of a break in the cycle of transactions. Second, in equilibrium she who chooses autarky from T on could not have been a seller in $T-1$. Conditional on preferring autarky from T on, it would have been rational for her to exit the market in $T-1$, a period in which she costly produces but earns no assets. Thus, although not directly observable, in equilibrium she who left at T is known to have been a buyer in $T-1$, who has reacted to a deviation as described by Lemma 4.

seller has the certainty to transact, and the probability to consume is not too greatly diminished for a buyer. When the available supply of record keeping devices is reduced (case *ii.*), however, the seller who can't transact in T selects autarky to avoid the future loss $-c$.

What supports the exchange of tokens in $T-1$? In the next two lemmas we focus on two factors. First, a significant participation externality. In that case, a seller strictly prefers to satisfy the buyer's consumption needs in $T-1$, thus deterring her from leaving the market. Second, a buyer must want to consume more than once in $T-1$. Thus, a seller would strictly prefer to settle her $T-1$ transaction with the exchange of a token. In this way she prevents the buyer from attempting to falsely represent her consumption needs to a second seller in that same period, something that would negatively affect future market participation. Thus, although tokens have no record keeping role in $T-1$, token exchange is a fundamental element of the trading mechanism due to the sellers' inability to communicate with each other. It allows them to satisfy the consumption needs of every buyer in a decentralized manner, avoiding a future undesirable allocation. These two elements are analyzed separately, starting from the second. That is, suppose sellers do not request a token in exchange in $T-1$. In equilibrium, if a buyer first attempts to transact at the more distant location she has probability 1/2 to consume. She can then transact with certainty at the closer location. The next lemma provides conditions for this deviation to be individually optimal.

Lemma 6. *Let the conditions in lemmas 4-5 be satisfied. Suppose IE is an equilibrium. There exists a* β_1 *such that* $N_L > N_1$ *if* $\beta > \beta_1$*. Then, if* $N > N_L$ *a buyer who, following a purchase in* $T-1$*, is left with a token prefers to buy and consume once more.*

In equilibrium, consuming twice is individually optimal if the participation externality generated is not too large and if individuals are not too impatient. If buyer i consumes twice in $T-1$, some other buyer is unable to consume at all, and selects autarky from T on. This creates future trading uncertainty for everyone including buyer i. Thus market participation cannot be too limited, $N > N_L$, or the negative externality would be so large that buyer i would not consider the deviation. Discounting cannot be too low either, $\beta > \beta_{1,}$ otherwise those who trade in $T-1$ might prefer future autarky. We now turn to the seller, showing when she strictly prefers to produce for someone who, in $T-1$, has a token.

Lemma 7. *Let the conditions in lemmas 4-6 be satisfied. Suppose IE is an equilibrium. There exists a* β_L *and* $N_H > N_L$ *such that if* $\beta > \beta_L$ and $N_L < N < N_H$, *then in* $T-1$ *it is individually optimal for a seller to produce the specialty good for someone who has a token.*

Recall that in $T-1$ buyers enjoy utility u while sellers suffer $-e$. Thus, to follow the proposed strategy in $T-1$ a seller must have incentives stronger than a buyer. This explains why the requirement on discounting is the most restrictive, $\beta_L > \beta_1$, and why market participation cannot be too large, $N < N_H$. In selecting actions in $T-1$, the seller compares her cost of production to the opportunity cost from not selling. The latter is the expected value of her market activity from T on, given that she cause a buyer to go hungry and then exit. The ensuing trading risk is 'small' when $N \geq N_H$ and $\beta \leq \beta_L$. Hence, the benefits that the buyer's future participation would confer to the seller would also be quite small. This induces the representative seller to refuse production in $T-1$, and in all prior periods, by backward induction. The prior lemmas provide a set of conditions sufficient for an equilibrium where tokens are exchanged, whereby everyone follows the (sequentially rational) actions prescribed by *IE*.

Proposition 1. *Let* $c > c_L$. *If* $\beta > \beta_L$ *and* $N_L < N < N_H$, *then there exists an equilibrium in which perishable tokens have value in every period of their existence for any feasible* e. *Transactions are settled only by the exchange of tokens, even in the final period of their existence.*

We summarize our findings using Figure 3, where the shaded region supports the equilibrium pattern *IE*. Specialty production cannot be too inexpensive, $e > u - c(1-\beta)$. This makes it impracticable to remain in the market for someone whose consumption need was not fulfilled in $T-1$; hence, he resorts to autarky. Less costly production is consistent with an equilibrium when (i) individuals derive less utility from current and future consumption (larger u and β) and (ii) the disutility from infrequent consumption, c, grows toward $\frac{u-u^B}{1-\beta}$. In both of these cases the base of the shaded area moves toward the horizontal axis, until it overlaps with it.[13] Producing for someone

[13] When $c \geq \left(u - u^B\right)/(1-\beta)$ then e is bounded below by u^B, hence specialty production can be arbitrarily inexpensive as long as u^B is arbitrarily small.

with a claim to consumption, in $T-1$, is incentive compatible if the claimant's future market participation provides a substantial externality. This is so if $N < N_H$ since the trade uncertainty existing in a market with unequal numbers of buyers and sellers increases as the number of initial participants falls. There is a trade-off between market participation and gains from trade. The greater the benefits from exchange (or the greater the loss due to inefficient exchange) the less important is the market participation externality. Given e, larger u, β or c increase the incentive to produce in $T-1$, causing a right shift of the curve labeled $N = N_H$. The larger e, the less the incentive to produce; the smaller N the greater the trade risk induced by a market exit. This explains the negative slope of the curve $N = N_H$ and the absence of such a trade-off for sufficiently low N [14] Greater u, lower c, or greater impatience, increase the buyer's desire to misrepresent his claim to consumption in $T-1$. This cause N_L to decrease since greater trading risk is acceptable to a buyer's who has stronger incentives to deviate.[15]

What if buyers could only attempt one purchase per period, as in standard matching models? One can conceive two types of equilibria that need not be supported by *exchange* of tokens in $T-1$. The first is an equilibrium where sellers strictly prefer to produce in $T-1$ and have no incentive to even claim a right to consumption. Tokens are only needed to let agents know what action to play in the absence of memory. In a second possible equilibrium, in $T-1$ a seller could prefer to consume instead of producing. In such a case ownership of a token identifies the individual as having the right to consumption. It does not make the physical transfer of tokens necessary in $T-1$, however. The buyer's possession of it is sufficient to initiate and complete the transaction.

5.3 Fixed-life Tokens and Finite Horizon

In this section we relax the assumption that economy is everlasting and that some record keeping device is always available. The economy's life extends for $2k > 2$ periods beyond $T-1$, and there is only the initial distribution of tokens. We prove existence of an equilibrium with stationary market participation. In it strategies are time-invariant functions in the sub-periods $[0, T-1]$ and

[14] It is a straightforward extension of Lemma 7 to show that if $c \geq \frac{u-u^B}{1-\beta}$ then the *IE* equilibrium exists for all $e \in \left(u^B, \beta u/2\right)$ if $N \in (N_L, N_H^*)$ and $\beta > \beta_L$. $N_H^* \in (N_L, N_H)$ is the value of N_H when $e = \beta u/2$ and $c = \frac{u-u^B}{1-\beta}$.

[15] Limiting the attempted purchases to two in a period provides the mildest support for *IE*. Attempting more than two purchases would strengthen our results. The incentives to deviate would be stronger for a buyer, hence sellers would be even more inclined to request tokens in $T-1$.

$[T, T+2k-1]$, but there is a break in period T. Specifically, tokens are exchanged for specialty goods in $t \leq T-1$, and common goods are bartered in $t \geq T$. Thus, half of the agents expect to deterministically alternate consumption of the specialty good to its production until $T-1$ (the other half does the opposite), and everyone alternates production to consumption of the common good from T on. We refer to this equilibrium pattern of exchange as "mixed" (*ME)* because of the break that occurs in T. The actions used in and out of equilibrium are as follows:

Definition of *ME. The agent follows IE if* $t \leq T-1$, *and BE otherwise. In* $t \geq T$, *in an unmatched stage the agent participates in the market, visiting the closest partner or producing a common good. In a matched stage she proposes barter of common goods. Autarky is chosen if by the end of a period, the agent has not consumed and is unmatched.*

In $t \geq T$ agents produce common goods in even periods, $T, T+2, ..., T+2(k-1)$, and barter it in odd periods, $T+1, T+3..., T+2k-1$. Thus, when *ME* is an equilibrium we denote the lifetime utility of the representative agent in $t \geq T$ as

$$v^B(T+t) = \begin{cases} \frac{1-\beta^{2k-t}}{1-\beta^2}\beta u^B & \text{for } t \in \{0, 2, 4..., 2(k-1)\} \\ \frac{1-\beta^{2k-t+1}}{1-\beta^2} u^B & \text{for } t \in \{1, 3..., 2k-1\} \end{cases}$$

The lifetime utilities in $t \leq T-1$ are given by (5) and (6) with $v = v^B(T)$. Define functions later used to identify boundaries of existence regions:

$$c_L^B(\beta, u) = \frac{\left(1-\beta^{2k}\right)\beta u}{1-\beta^2}, \quad k_L\left(u, u^B\right) = \frac{u}{2u_B}$$

$$N_L^B\left(\beta, k, u, u^B\right) = \frac{1}{2} + \frac{1-\beta^{2k}}{1-\beta^2} \cdot \frac{\beta u^B}{2u}, \quad N_H^B\left(\beta, k, c, e, u^B\right) = \frac{1}{2} + \frac{\beta^2}{2e}\left(c + \frac{1-\beta^{2k}}{1-\beta^2}u^B\right)$$

where $N_L^B, N_H^B > 0$ for all $\beta \in (0,1)$; we also use $\beta_L^B(k, u, u^B) \in (0,1)$ to denote a critical value of β. Conjecture that *ME* is an equilibrium. To prove it we follow the pattern laid out in the previous section, focusing on actions taken in $t \geq T-1$. We start by discussing market participation when no deviation has been observed.

Lemma 8. *Suppose ME is an equilibrium and no deviation has been observed. There exists* β_L^B *and* k_L *such that if* $k > k_L$ *and* $\beta > \beta_L^B$, *then market participation is individually optimal.*

Production in $T-1$ is specialized and costly. Thus, a seller's incentive to stay in the market depends on the present value of all future barter exchanges, so that she must be sufficiently patient, $\beta > \beta_L^B$. The smaller the gain from barter trade, the greater must be β_L^B and the longer must the economy last past $T-1$ (k_L increases as u^B falls). As the value of a barter trade vanishes barter exchange must thus take place over an unbounded sequence of periods ($k_L \to \infty$ as $u^B \to 0$). In the next two lemmas we consider the choice of market participation when some deviation has been observed. We start with the case where in $T-1$ some buyer cannot consume a specialty good.

Lemma 9. *Let the conditions in lemma 8 be satisfied. Suppose ME is an equilibrium, but in $T-1$ a buyer is unable to purchase a specialty good from her closest seller. Then, if $c > c_L^B$ it is individually optimal for the buyer to leave the market in T.*

The intuition is similar to the previous section. Autarky is preferred whenever the disutility generated by protracted absence of consumption is severe. Unlike the previous section, however, there is no trade-off between e and c, since specialty production does not take place after $T-1$ (unlike c_L, c_L^B does not depend on e). For this reason, now we define as *feasible* all those e that satisfy (2), that is $e \in \left(u^B, \beta u/2\right)$. We now prove the equivalent of Lemmas 5-7.

Lemma 10. *Let the conditions in lemmas 8-9 be satisfied. Suppose ME is an equilibrium*

i. Suppose someone has left the market in T. Then her closest partner in $T+1$ optimally chooses to live in autarky from $T+2$.

ii. There exists N_L^B such that if $N > N_L^B$ then a buyer who, following a purchase in $T-1$ is left with a token, prefers to buy and consume once more.

Lemma 11. *Let the conditions listed in lemmas 8-10 be satisfied. Suppose ME is an equilibrium. There exists $N_H^B > N_L^B$ such that if $N < N_H^B$ then in $T-1$ it is individually optimal for a seller to produce the specialty good for someone who has a token.*

The intuition is as in the previous section. If a buyer is able to misrepresent her consumption needs in $T-1$, one agent will select autarky from $T+1$ on. This generates trading uncertainty. The associated externality cannot be too large, $N > N_L^B$, otherwise the buyer would prefer not to deviate in $T-1$. It cannot be too small either, $N < N_H^B$, otherwise in $T-1$ a seller would have little to gain from fostering participation in the future. The absence of a lower bound on β, relative to Lemma 7, reflects the absence of production costs for $t \geq T$.

Proposition 2. *Let $c > c_L^B$. If $\beta > \beta_L^B$, $k > k_L$, and $N_L^B < N < N_H^B$ then there exists an equilibrium in which perishable tokens have value in every period of their existence for any feasible e. Transactions are settled by the exchange of tokens, even in the final period of their existence.*

The main difference from the previous section is that specialty production does not take place after $T-1$, due to the unavailability of a record-keeping technology after that date. The shaded region in Figure 4 provides a summary of our findings. Note that relative to Figure 3, there is no lower bound on e nor N since common goods are costlessly produced. What matters in supporting specialty production up to $T-1$ is only the discounted value of all future barter trades. This explains why $k > k_L$ and $\beta > \beta_L^B$: the greater the number of barter trading rounds and the degree of patience, the greater the incentive to participate in the future barter market. The proposition has an interesting implication. Suppose an economy without memory has an horizon of $J < \infty$ periods, and a record-keeping technology–durable tokens–is available. Then, there exist equilibria where specialty production is exchanged for tokens for at most $J - 2(k_L + 1)$ periods, and barter of common goods is undertaken in all subsequent periods.

6. CONCLUDING REMARKS

We have constructed limited-commitment and limited-communication economies where a temporary record-keeping technology can be used to expand the set of allocations. Agents can improve over autarky or barter by choosing to participate in market activities using a transaction arrangement based on the exchange of 'barren' perishable tokens for a *finite* number of periods. We have shown that participation externalities and incentives to abandon an inefficient market have implications for the viability of the trading technology. If strong, they can overcome the drawbacks due the technology's fundamental imperfection, its finite horizon. This ideas have been articulated by considering a simple environment with complete lack of memory, and in which the record-keeping technology (or the economy's horizons, or both) are *publicly* known to be finite. Introducing incomplete information, as uncertainty on the final transaction date, would more easily support equilibria where barren tokens are adopted to facilitate transactions (see Kovenock and de Vries, 2000).

The analysis leads to the following insights. First, it appears that neither a finite economic horizon, nor a temporary record-keeping technology, are in general *sufficient* to prevent agents from exploiting (some of the) possible gains from specialization and trade. Agents may be willing to sustain a cost today, by producing for someone who has a claim to consumption, in order to reap future gains from trade. As Keynes wrote, in our study "the importance of money essentially flows from its being a link between the present and the future," despite our 'money' being an imperfect record-keeping technology, and even though the future may be known to come to an end. Second, our study suggests that when trading decisions are made in isolation tokens may be viewed as something more than a substitute for societal memory. In our environment tokens serve as communication devices in $T-1$, overcoming the sellers' inability to transfer information within the period. Requiring a token transfer dissipates the risk of an 'undesirable' allocation. Sellers would probably be willing to sustain a cost to coordinate their actions, even if past trading records were available.

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Appendix

Proof of Lemma 1. Suppose *IE* is an equilibrium. Since $v_b = \max\{u + \beta v_s, \beta v_b\}$ and $v_s = \max\{-e + \beta v_b, \beta v_s\}$, then in equilibrium $v_b = \frac{u-\beta e}{1-\beta^2}$ and $v_s = \frac{\beta u - e}{1-\beta^2}$. *IE* is an equilibrium if a seller is willing to produce a specialty good in exchange for a token, i.e. $-e + \beta v_b > \beta v_s \Rightarrow e < \beta u$, which is satisfied by (2). The latter implies $v_b > v_s > 0$, hence tokens are not discarded. A seller must also prefer production of a specialty good to that of a common good. A sufficient condition for this is that she cannot sell a common good. This occurs if buyers do not buy it, and specialty sellers do not agree to swaps for a common good. A buyer does not buy a common good if she strictly prefers to wait one more period to purchase a specialty good. That is, $u^B + \beta v_s < \beta v_b \Rightarrow u^B < \frac{\beta}{1+\beta}(u+e)$. Recall (2); since $e < \beta u$, a lower bound for $\frac{\beta}{1+\beta}(u+e)$ is e. Since $u^B < e$, then $u^B < \frac{\beta}{1+\beta}(u+e)$ is always satisfied. A seller of a specialty good would not exchange hers for a common good if she prefers to sell to a buyer in the future. This occurs if $u^B + \beta v_s - e < \beta v_s$, always satisfied because of (2).■

Proof of Lemma 2. Suppose *BE* is an equilibrium. Production occurs in $t = 0, 2...$, and barter in odd periods, so $v_{odd} = \frac{u^B}{1-\beta^2} > v_{even} = \frac{\beta u^B}{1-\beta^2}$. We provide conditions such that in equilibrium an agent (i) does not leave the market, (ii) does not prefer to produce a specialty good and (iii) always agrees to barter. Since $u^B > 0$ there are gains from trade, hence autarky is not preferred. An agent does not prefer to produce a specialty good since her net payoff from bartering it would be negative, due to (2). Production of the common good is costless, thus everyone has an incentive to produce it in even periods. Furthermore, agents may suffer $-c$ from not consuming in more than two consecutive periods. Hence, everyone strictly prefers to barter her common good.■

Proof of Lemma 3. Conjecture *IE* is an equilibrium and no deviation from it has been observed. From Lemma 1, (2) is sufficient to support *IE* for $t \geq T$. Now consider $t < T$. It is individually optimal to stay in the market if $V_k(t) > 0$ for $k = b, s$. Since $v > 0$ and a buyer prefers to stay in the market in $t \geq T$, then an agent prefers to stay in the market as a buyer in $T-1$. This is evident from (6) since the agents consumes with certainty in $T-1$, as a buyer, and with probability 1/2 in T. Consider a seller. The most stringent case in which she might want to be out of the market is in $T-1$. This is so because, due to (5), production is costly and tokens perish at the end of $T-1$. The strongest incentive to be in autarky in $T-1$ exists for someone who has consumed in $T-2$ and wants to avoid being a seller in $T-1$. Conditional on following the equilibrium strategy in the future, staying in the market as a seller in $T-1$ is individually optimal only if $V_s(T-1) > 0$ $\Rightarrow \ e < \frac{u\beta}{2-\beta}$. Note that $\frac{u\beta}{2-\beta} > \frac{\beta u}{2}$ hence (2) is sufficient for $e < \frac{u\beta}{2-\beta}$. Since $V_b(T-1) > V_s(T-1)$ then buyers and sellers prefer to stay in the market, in equilibrium.■

Proof of Lemma 4. Conjecture *IE* is an equilibrium. Suppose buyer i cannot consume in $T-1$. This may be due to a seller not having wanted to sell to i, or some other buyer having bought more than one specialty commodity. In equilibrium it is not optimal for i to travel to the more distant location to attempt a purchase. She expects no goods are available at that location and it is costly to travel, ε. Let v_{b0} denote buyer $i's$ expected lifetime utility at the beginning of T *prior* to the distribution of the new tokens: $v_{b0} = \frac{1}{2}(-c + \max\{v_s, 0\}) + \frac{1}{2}v_b \equiv \frac{-c}{2} + v$. At the end of $T-1$ buyer i chooses autarky if $-c/2 + v < 0 \ \Rightarrow \ c > c_L(\beta, e, u) = \frac{u-e}{1-\beta}$; since $e \in (u^B, \beta u/2)$ then $c_L < \frac{u-u^B}{1-\beta}$. If $c > c_L$ the expected value of being in the market in T is less than that of exiting *before* the possible loss $-c$ is realized (if in T agent i does not receive a new token she does not consume for three periods in a row). If $c < c_L$, buyer i chooses to remain in the market in T. In this case she does not leave the market in $T+1$ even if she does not receive a token in T; the disutility $-c$ realized in T is a sunk cost by then, while $v_s > 0$ since everyone follows *IE* having observed no deviation (in $T-1$ or T).■

Proof of Lemma 5. Let $c > c_L$ and e be feasible Conjecture *IE* is an equilibrium. Suppose an agent who was a buyer in $T-1$ has left the market at the beginning of T, after the distribution of new tokens. Depending on the distribution there can be N buyers and $N-1$ sellers or vice versa. Denote by $-b$ $(-s)$ the case where he who leaves has (has not) received a new token.

Case I: N sellers and $N-1$ buyers. Each agent who in $T-1$ was a buyer has received a token in T. When one these agents leaves the market in T, N sellers and $N-1$ buyers remain. The seller closest to the agent who exited, call her agent i, observes the buyer's exit in T. Under the conjecture that *IE* is the equilibrium, i expects that if she remains in the market she transacts with probability $(N-1)/N$, as a seller, and with certainty as a buyer. We show that it is individually optimal for agent i to leave the market in $T+1$.

Suppose that, following the observed exit, everyone remains in the market from $T+1$ on. In this case, denote by $v_{s0}(-b)$ the value function of an agent who is a seller in $t > T$ *and* did not consume in the prior period. Let $v_s(-b)$ be the value function of an agent who is a seller in $t > T$ *and* consumed in the prior period (in T no seller has consumed in the prior period). Let $v_b(-b)$ be the lifetime utility of a buyer in $t > T$. Thus

$$v_s(-b) = \frac{N-1}{N} \max\{-e + \beta v_b(-b), 0\} + \frac{1}{N}\beta \max\{v_{s0}(-b), 0\}$$

$$v_{s0}(-b) = \frac{N-1}{N} \max\{-e + \beta v_b(-b), 0\} + \frac{1}{N}\beta \max\{-c + v_{s0}(-b), 0\} = v_s(-b) - \frac{c\beta}{N}$$

$$v_b(-b) = u + \beta \max\{v_s(-b), 0\}.$$

In the conjectured equilibrium agents do not leave the market iff for all $t > T$

$$-c + v_{s0}(-b) > 0 \tag{7}$$

$$-e + \beta v_b(-b) > 0 \tag{8}$$

which jointly imply $v_s(-b) > 0$. Note that $v_b(-b) > 0$ since a buyer consumes with certainty (and then can leave the market, in the worse case scenario). If (7)-(8) hold $v_{s0}(-b) < v_s$ because a seller experiences trade risk (a buyer is missing from the market). Recall also that $c > c_L$ implies $-c/2 + v < 0$; in turn, this implies $-c + v_{s0}(-b) < 0$, since $v = (v_b + v_s)/2 > v_s$. Thus (7) is violated when $c > c_L$. It follows that seller i leaves the market in $T+1$ if she believes that no one else does so in $t > T$. This belief is rational since after she leaves there are $N-1$ buyers and sellers with lifetime utilities v_b and v_s $\forall t > T$. Thus, in this specific case

$$v_{s0}(-b) = \frac{N-1}{N}(-e+\beta v_b) = \frac{N-1}{N} \cdot v_s \tag{9}$$

$$v_b(-b) = u + \beta v_s = v_b \tag{10}$$

Case II: $N-1$ sellers and N buyers. Agents who in $T-1$ were buyers *do not* received a token in T. When one them leaves the market in T, $N-1$ sellers and N buyers remain. Thus her closest buyer in T, call him agent i, cannot consume. Under the conjecture that *IE* is an equilibrium, it is not optimal for i to attempt a purchase at the more distant location (she expects no goods to be available at that location and travel is costs ε). If i stays in the market she expects to transact with certainty as a seller, and with probability $(N-1)/N$ as a buyer. We show that it is individually optimal for agent i to remain in the market. Suppose that everyone remains in the market from $T+1$ on. For $t > T$, denote by $v_s(-s)$ the value function of a seller. Let $v_b(-s)$ refer to the value function of a buyer *given that* he has not consumed in the prior period, but has consumed two periods earlier. Denote by $v_{b0}(-s)$ the lifetime utility of a buyer who hasn't consumed for at least two (or more) periods. Thus

$$v_s(-s) = \max\{-e+\beta v_b(-s), 0\}$$

$$v_b(-s) = \frac{N-1}{N}[u+\beta\max\{v_s(-s),0\}] + \frac{1}{N}\beta\max\{v_{b0}(-s),0\}$$

$$v_{b0}(-s) = \frac{N-1}{N}[u+\beta\max\{v_s(-s),0\}] + \frac{1}{N}[-c+\beta\max\{v_{b0}(-s),0\}] = v_b(-s) - \frac{c}{N}$$

where $v_b(-s) > v_{b0}(-s)$, due to $-c/N$. These expressions assume that no-one disposes of their token (we will provide conditions below).

A seller may choose to not participate in the market in $t > T$ (avoiding production costs) and then leave. A buyer has certainty to consume. This explains why $v_b(-s) > v_s(-s)$. Under the conjecture that *IE* is played, it is optimal to remain in the market if

$$-e+\beta v_b(-s) > 0 \iff v_s(-s) > 0 \tag{11}$$

$$v_{b0}(-s) > 0. \tag{12}$$

Since $v_b(-s) > v_{b0}(-s)$, if (12) holds then no one has an incentive to leave. If a buyer who hasn't consumed in the last two (or more) periods prefers to be in the market, so does everyone else (since they are not suffering $-c$). Suppose that (11) and (12) hold, then

$$v_s(-s) = -e + \beta v_b(-s) \quad \text{and} \quad v_b(-s) = \frac{N-1}{N}\left[u + \beta v_s(-s)\right] + \frac{1}{N}\beta v_{b0}(-s)$$

$$v_b(-s) = \frac{N(N-1)(u-\beta e) - \beta c}{N(1-\beta)\left[N(1+\beta) - \beta\right]} \tag{13}$$

$$v_s(-s) = \frac{\beta\left[N(N-1)u - \beta c\right] - e(N-\beta)N}{N(1-\beta)\left[N(1+\beta) - \beta\right]} \tag{14}$$

$$v_{b0}(-s) = \frac{N(N-1)(u-\beta e) - \beta c}{N(1-\beta)\left[N(1+\beta) - \beta\right]} - \frac{c}{N}. \tag{15}$$

Use (13)-(15) in (11) and (12). First, $v_s(-s) > 0$ iff $\frac{\beta[N(N-1)u-\beta c]-e(N-\beta)N}{N(1-\beta)[N(1+\beta)-\beta]} > 0$. The denominator is always positive, and the numerator is positive if

$$e < \frac{N(N-1)\beta u - \beta^2 c}{N(N-\beta)} \tag{16}$$

The RHS of (16) is always increasing in N, and it is strictly larger than $\beta u/2$ if

$$N > \left(1 + \frac{\beta}{2}\right) + \sqrt{\left(1 + \frac{\beta}{2}\right)^2 + 2\beta\frac{c}{u}}$$

and since $\sqrt{x^2 + y} < x + \sqrt{y}$ then and upper bound for the right hand side of the inequality above is $2 + \beta + \sqrt{\frac{2\beta c}{u}}$

$$N > N_1(\beta, c, u) \equiv 2 + \beta + \sqrt{\frac{2\beta c}{u}} \tag{17}$$

so that (11) is satisfied always whenever $N > N_1$.

Second, $v_{b0}(-s) > 0$ if $c < \frac{N(N-1)(u-\beta e)}{N(1-\beta^2)+\beta^2}$ which can be shown to be satisfied $\forall e$ whenever (16) holds. Hence $N > N_1(\beta, c, u)$ is sufficient to satisfy both (11) and (12) in which case market participation is individually optimal.

Next, we provide conditions such that tokens are not disposed in any $t \geq T$. This may be tempting when there are too many buyers and not enough sellers. If buyer i disposes of his token he becomes a seller in the *following* period, hence the market will have N sellers and $N-1$ buyers. By doing so he suffers $-c$ *with certainty* in $T+1$, and does not increase his chances to trade as a seller (there are now $N-1$ buyers and N sellers). It follows that buyer i does not discard his token at the end of T if he expects to be worse off, i.e. if $v_{b0}(-s) > -c + v_{s0}(-b)$. Since $v_{b0}(-s) > 0$ if (17) holds, and $-c + v_{s0}(-b) < 0$ when $c > c_L$, then no tokens are ever discarded.

We now sum up case I and II. Let $c > c_L$, $N > N_1$ and e be feasible. Conjecture *IE* is an equilibrium. Suppose that in T a buyer (seller) cannot transact because her closest seller (buyer) has left the market. Then it is individually optimal for the buyer to remain in the market, and for a seller to leave in $T + 1$.■

Proof of Lemma 6. Let $c \geq c_L$, $N > N_1$ and e be feasible. Conjecture *IE* is an equilibrium. Suppose that in $T - 1$ a buyer, say agent i, deviates by purchasing the output produced by a seller of a more distant location, and is able to retain her token. Doing so leaves another buyer without consumption for the period, and she chosees autarky from T on (since $N > N_1$). Conditional on this exit, let $v_{bn} = \frac{1}{2}v_b(-b) + \frac{1}{2}v_s(-s)$ be the expected lifetime utility of buyer i in T. With probability 1/2 agent i receives a new token in T, and nets $v_b(-b)$, otherwise she gets $v_s(-s)$.

Suppose agent i is in a matched stage with the seller at the closer location, in $T - 1$, and this seller has yet to produce. Agent i prefers to buy once more if $2u - \varepsilon + \beta v_{bn} > u + \beta v$, i.e. $u - \varepsilon > \beta (v - v_{bn})$. Using the prior definitions and recalling that $v_b(-b) = v_b$

$$v - v_{bn} = \frac{1}{2}[v_s - v_s(-s)] \equiv \frac{1}{2}\left[\frac{\beta u - e}{1 - \beta^2} - \frac{\beta [N(N-1)u - \beta c] - e(N - \beta)N}{N(1-\beta)[N(1+\beta) - \beta]}\right]$$

hence $u - \varepsilon > \beta (v - v_{bn})$ whenever

$$N\beta^3 e > (1+\beta)\beta^3 c - uN\left[2N(1-\beta)(1+\beta)^2 - \beta\left(2+\beta-2\beta^2\right)\right] + \varepsilon N\left(1-\beta^2\right)[N(1+\beta) - \beta]$$

For $\varepsilon > 0$ arbitrarily small, the RHS of the inequality above is negative if

$$\begin{aligned} N &> \frac{\beta\left(2+\beta-2\beta^2\right) + \beta\sqrt{\left(2+\beta-2\beta^2\right)^2 + 8(1+\beta)^3(1-\beta)\beta\frac{c}{u}}}{2(1+\beta)^2(1-\beta)} \\ &> N_2(\beta, c, u) \equiv \frac{\beta\left(2+\beta-2\beta^2\right)}{(1+\beta)^2(1-\beta)} + \frac{\beta\sqrt{2\left(1-\beta^2\right)\beta\frac{c}{u}}}{1-\beta^2} \end{aligned}$$

Observe that $N > \max\{N_1(\beta, c, u), N_2(\beta, c, u)\}$ if

$$N > N_L(\beta, c, u) \equiv \max\left\{2+\beta, \frac{\beta\left(2+\beta-2\beta^2\right)}{(1+\beta)^2(1-\beta)}\right\} + \sqrt{\frac{2\beta c}{u}} \cdot \max\left\{\beta\left(1-\beta^2\right)^{-\frac{1}{2}}, 1\right\}$$

where the arguments β, c and u are omitted when no confusion arises. $N_2 \to \infty$ as $\beta \to 1$ and $N_2 \to 0$ as $\beta \to 0$. Also, $N_1 \to 3 + \sqrt{\frac{2c}{u}}$ as $\beta \to 1$ and $N_2 \to 2$ as $\beta \to 0$. Thus, by the intermediate value theorem there exists a $\beta_1(c,u) \in (0,1)$ such that $N_L = N_2$ for all $\beta > \beta_1$.■

Proof of Lemma 7. Let $c > c_L$, $\beta > \beta_1$, $N > N_L$, and e be feasible. Suppose *IE* is an equilibrium. Suppose in $T-1$ a seller, say agent i, does not sell. One buyer cannot consume and leaves the market in T. Contingent on that, let $v_{sn} = \frac{1}{2}v_{s0}(-b) + \frac{1}{2}v_{b0}(-s)$ be the expected lifetime utility to agent i in the following period, T. With probability 1/2 she receives a new token and nets $v_{s0}(-b)$; else, she gets $v_{b0}(-s)$. Agent i in $T-1$ strictly prefers to sell if $-e + \beta v > \beta v_{sn} \Rightarrow e < \beta\left(v - v_{sn}\right)$, satisfied by some $e > u^B$ (since $v > v_{sn}$). Using prior definitions

$$v - v_{sn} = \frac{1}{2}\left[\frac{v_s}{N} + v_b - \frac{N(N-1)(u-\beta e) - \beta c}{N(1-\beta)\left[N(1+\beta) - \beta\right]}\right]$$

hence $e < \beta\left(v - v_{sn}\right)$ whenever

$$e < \frac{(1+\beta)\,\beta^2 c + \left[N\left(1+\beta+\beta^2\right) - \beta^2\right]\beta u}{2N^2\,(1+\beta)^2\,(1-\beta) - N\beta\left(1 - 2\beta - 2\beta^2\right) - \beta^2} \tag{18}$$

Rearrange (18) as $\Delta(N) < 0$ where $\Delta(N) \equiv N^2 A_1 - N A_2 + A_3$ and

$$A_1 = 2\,(1+\beta)^2\,(1-\beta)\,, \quad A_2 = \beta\left[1 - 2\beta - 2\beta^2 + \left(1+\beta+\beta^2\right)\frac{u}{e}\right]$$

$$A_3 = -\beta^2\left[1 + \frac{1}{e}\left((1+\beta)\,c - u\right)\right]$$

Note that $A_1 > 0$, $A_2 > 0$ (it decreases in e and is positive when $e = \beta u/2$), and $A_3 < 0$ since $c > c_L$. It follows that $\Delta(N) = 0$ has only one positive root $N_H(\beta, c, u, e) = \frac{A_2 + \sqrt{(A_2)^2 - 4A_1A_3}}{2A_1}$, decreasing in e, and increasing in c. $\frac{\partial N_H}{\partial c} = \frac{\beta^2(1+\beta)}{4e} \cdot \left[(A_1)^2 - 4A_1A_3\right]^{-\frac{1}{2}}$ is decreasing in e. When $\beta > \beta_1$ one can show that $\left.\frac{\partial N_H}{\partial c}\right|_{e=\beta u/2} > \frac{\partial N_L}{\partial c} = \sqrt{\frac{\beta^3}{2cu(1-\beta^2)}} > 0\ \forall c > c_L$. Thus, for $\beta > \beta_1$ and feasible e, N_H grows faster than N_L as c rises. We provide a sufficient condition for $N_L < N_H$, when $\beta > \beta_1$. If $c > c_L$ then $N_H > N_L$ as $\beta \to 0$ and $\beta \to 1$. By the intermediate value theorem, there exists a $\beta_2(c,u) \in (0,1)$ such that $\forall \beta > \beta_L(c,u) \equiv \max\{\beta_1(c,u), \beta_2(c,u)\}$ then $N_H > N_L$ $\forall$ feasible e. This concludes the proof.■

Proof of Proposition 1. Let $c > c_L$, $\beta > \beta_L$ *and* $N_L < N < N_H$ and e be feasible. Conjecture *IE* is an equilibrium. Lemma 7 implies that a seller prefers to produce for a buyer, in $T-1$. If the buyer has a token he would agree to the purchase even if he has already consumed, by Lemma 6. Suppose the seller does not request the token in exchange. Given the matching process, there is a positive probability (however small) that the buyer she is matched with may be able to consume twice in $T-1$. This occurrence would lower the expected lifetime utility to the seller since $v > v_{sn}$. Thus, it is individually optimal for the seller to request the token in the transaction, even if she is certain not to use it in a future purchase. Finally it is easy to see, using Lemma 1, that no one has incentive to produce a common good in $t \leq T-1$. A buyer would not accept a common good in exchange for her token, since $u^B < u$. Producing a specialty good in exchange for a common good is also suboptimal, since $u^B < e < \beta u$.■

Proof of Lemma 8. Conjecture *ME* is an equilibrium and no deviation has been observed. For $t \geq T$ Lemma 2 can be extended in an obvious way to prove existence of the equilibrium *BE*. In $t \leq T-1$ it is individually optimal to stay in the market if $V_j(t) > 0$ for $j = s, b$. The most stringent case in which an individual might want to leave the market occurs in $T-1$, as a seller. Conditional on *ME* being an equilibrium, it is individually optimal for the seller to participate in the market in $T-1$ if $V_s(T-1) = -e + \beta v^B(T) > 0$. Since $e < \beta u/2$, then $e < \beta v^B(T)$ whenever

$$\frac{u}{2u^B} < \frac{\beta\left(1-\beta^{2k}\right)}{1-\beta^2}. \tag{19}$$

Define $k_L\left(u, u^B\right) = \frac{u}{2u^B}$, and note that (2) implies $k_L > 1$. Let $k > k_L$. The RHS of (19) is increasing in β, and converges to k as $\beta \to 1$. By the intermediate value theorem, it follows that there exists a unique $\beta_L^B(k, u, u^B) \in (0,1)$, decreasing in u^B, such that if $\beta > \beta_L^B$ then (19) is satisfied. Since $V_b(T-1) > V_s(T-1)$ then $k > k_L$ and $\beta > \beta_L^B$ guarantee that everyone chooses to participate in the market.■

Proof of Lemma 9. Let $k > k_L$ and $\beta > \beta_L^B$. Conjecture *ME* is an equilibrium. Suppose agent i was unable to buy during $T-1$, as a buyer. It is not optimal for her to travel to a more distant location to attempt a purchase, since she expects no goods to be available at that location and it is costly to travel. Let $v_0^B(T) = \max\left\{-c + v^B(T), 0\right\}$ denote agent i's expected lifetime utility at the beginning of T, where $v^B(T) = \frac{\beta\left(1-\beta^{2k}\right)}{1-\beta^2}u$. Agent i leaves the market if $-c + v^B(T) < 0$, i.e. if $c > c_L^B(\beta, u) = \frac{\beta\left(1-\beta^{2k}\right)u}{1-\beta^2}$.■

Proof of Lemma 10. Let $k > k_L$, $\beta > \beta_L^B$ and $c > c_L^B$. Recall that $k > 1$. Conjecture ME is an equilibrium.

Suppose an agent leaves the market at the beginning of T. Thus, in T there are $2N-1$ individuals who can engage in barter from $T+1$ on. One agent cannot barter in $T+1$ and will be able to barter no sooner than $T+3$ (recall that in each period agents are randomly distributed across locations). If she remains in the market in $T+2$, her expected lifetime utility is $-\beta c + \beta^2 \tilde{v}^B(T+3)$, where $\tilde{v}^B(T+3) < v^B(T+3)$ since barter is uncertain in $T+3$. Thus, a sufficient condition for exit from the market in $T+2$ is $-\beta c + \beta^2 v^B(T+3) < 0 \;\Rightarrow c > \beta v^B(T+3)$. Since common good production is costless but takes a full period, then $\beta v^B(T+3) = v^B(T+2) < v^B(T)$. Since $c > v^B(T)$, it follows that $c > \beta v^B(T+3)$.

Suppose that in $T-1$ a buyer, call him agent i, consumes twice then some other buyer does not consume and exits from the market in T. Agent i has expected lifetime utility

$$V_{-1}^B(T) = \frac{2(N-1)\beta v^B(T+1)}{2N-1} = \frac{2(N-1)v^B(T)}{2N-1}$$

since i takes into account that in $T+1$ she will unable to barter with probability $1/(2N-1)$; in that case she leaves the market (as seen above). Someone who in $T-1$ was a seller, at the beginning of T has expected lifetime utility $V_{-1}^B(T) - \frac{\beta c}{2N-1}$, since if she is unable to barter (with probability $1/(2N-1)$) she suffers $-c$ and then leaves the market. Thus, agent i would prefer to consume twice in $T-1$, if

$$u + \beta V_{-1}^B(T) > \beta v^B(T) \tag{20}$$

If $k = 1$ then (20) is satisfied since $u > \beta^2 u^B$ (it's always better to consume a specialty good today, than a common good in the future). Since $k > k_L > 1$, however, (20) is satisfied whenever

$$u^B < \frac{(2N-1)\left(1-\beta^2\right)}{\beta\left(1-\beta^{2k}\right)}u \quad \Rightarrow \quad N > N_L^B\left(\beta, k, u, u^B\right) \equiv \frac{1}{2} + \frac{\beta\left(1-\beta^{2k}\right)u^B}{2\left(1-\beta^2\right)u}. \blacksquare$$

Proof of Lemma 11. Let $k > k_L$, $\beta > \beta_L^B$, $c > c_L^B$ and $N > N_L^B$. Conjecture ME is an equilibrium. By lemma 9, if in $T-1$ a seller does not produce, a buyer cannot consume and leaves the market in T. By implementing this deviation and remaining in the market the seller has expected lifetime utility $V_{-1}^{B}(T)-\frac{\beta c}{2N-1}$, next period. Thus, it is individually optimal for her to sell in $T-1$ if

$$-e+\beta v^{B}(T)>\beta V_{-1}^{B}(T)-\frac{\beta^{2}c}{2N-1}$$

Using the expressions defined in previous lemmas it is shown that the above holds if

$$N<N_{H}^{B}(\beta,k,c,e,u^{B})\equiv\frac{\beta^{2}c+\beta v^{B}(T)}{2e}+\frac{1}{2}=\frac{1}{2}+\frac{\beta^{2}}{2e}\left(c+\frac{1-\beta^{2k}}{1-\beta^{2}}\cdot u^{B}\right)$$

We conclude that if e is feasible it is individually optimal for a seller to sell in $T-1$ if $N<N_{H}^{B}$. Note that $N_{L}^{B}<N_{H}^{B}$ always. Finally we notice that N_{H}^{B} is decreasing in e, and both N_{L}^{B} and N_{H}^{B} increase in k.■

Proof of Proposition 2. It follows from Lemmas 8-11, as the proof of Proposition 1.■

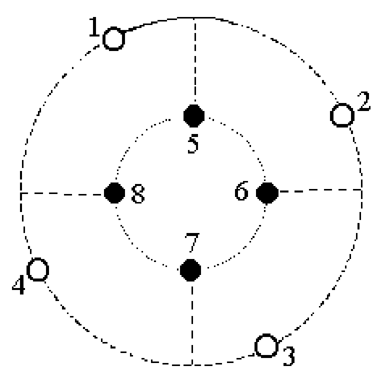
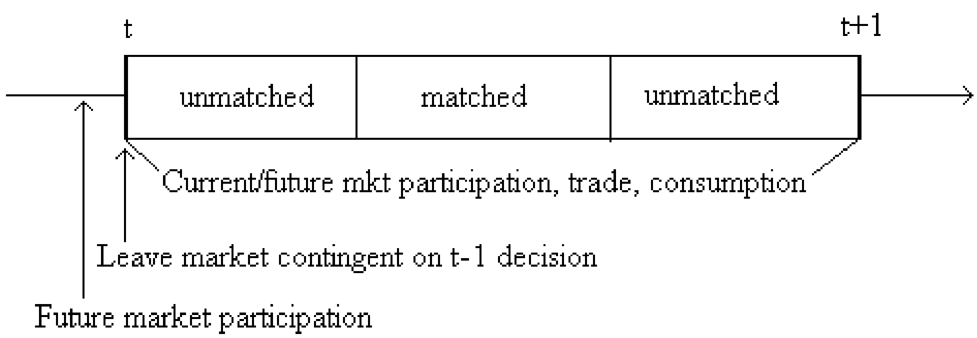


Figure 1



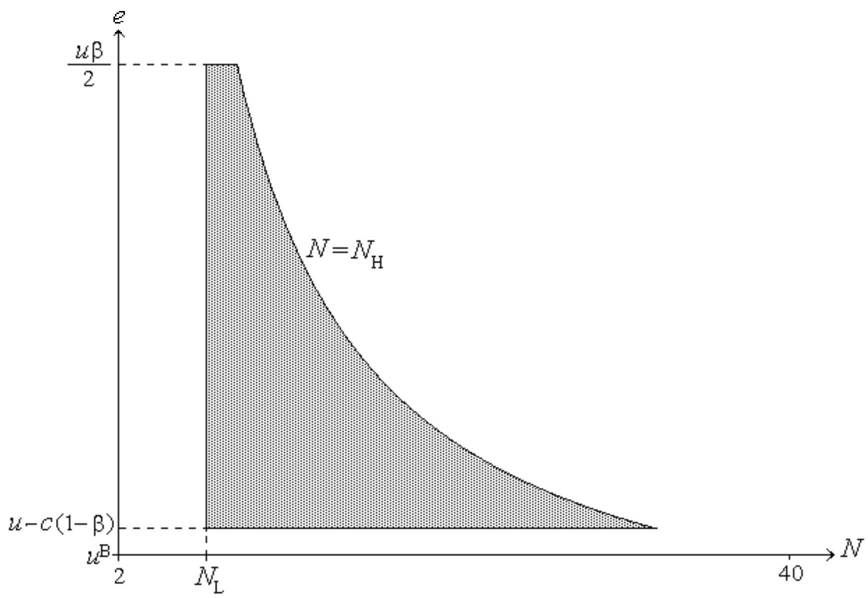


Figure 3

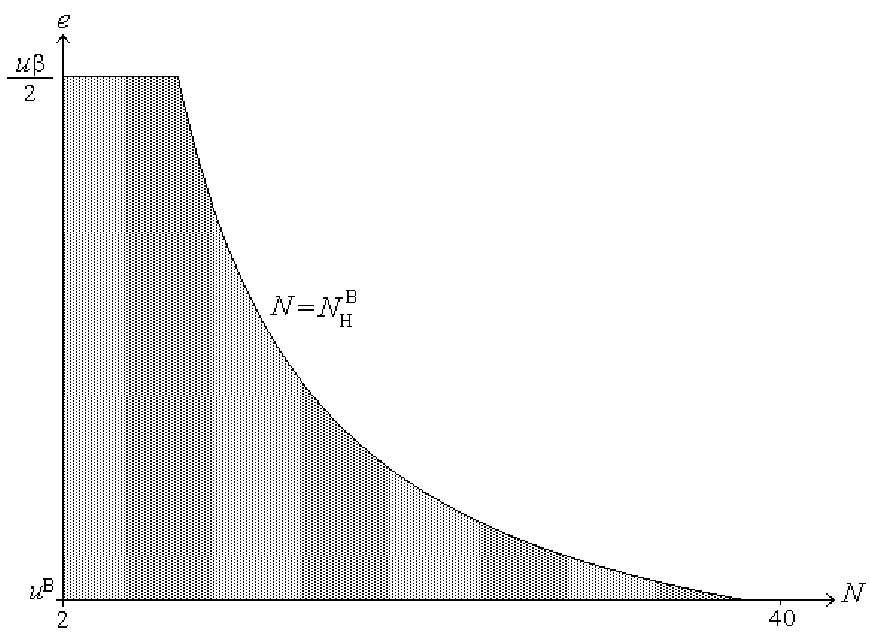


Figure 4