(In)Stability and Informational Efficiency of Prices

Gabriel Desgranges*and Stéphane Gauthier†

Abstract

This paper analyzes how agents coordinate their forecasts on a Rational Expectations Equilibrium under asymmetric information about fundamentals. We consider the class of linear one-dimensional models where the price is determined by price expectations. We find that REE stability is favored by a small sensitivity of the economy to forecasts, and, more surprinsingly, by a *small* proportion of informed agents. Still, price informational efficiency is favored by a small sensitivity of the economy to forecasts but also by a *large* proportion of informed agents, suggesting a conflict between the two issues of stabilizing fluctuations and transmitting information to the market.

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JEL classification: C62, D82, D84

1 Introduction

We address a problem of expectations coordination in a one step forward looking model with asymmetric information about the fundamentals. In this context, the usual solution concept is the Rational Expectations Equilibrium (REE hereafter). We study stability of this REE under "eductive" learning

*THEMA (university of Cergy-Pontoise, France). 33, blvd du port 95011 Cergy-Pontoise CEDEX, France. gabriel.desgranges@eco.u-cergy.fr

†CREST-LMA (INSEE, France) and GEMMA (university of Caen, France). CREST, Laboratoire de macroéconomie, 15, boulevard Gabriel Péri 92245 Malakoff CEDEX France. gauthier@ensae.fr

à la Guesnerie (1992).[1] The REE concept assumes full coordination of expectations of agents, i.e. REE assumes that all the agents have succeeded to form correct (self-fulfilling) beliefs. Still, forming a rational expectation is not *a priori* correct, it is the right expectation of the economic outcome only when all the agents have rational expectations as well. In game theoretical words, adopting rational expectations is not a dominant strategy, but rather a Nash strategy. Thus, assuming REE behavior amounts to assume that every agent *a priori* knows that others adopt REE behavior. Why should it be the case? The "eductive" story offers a possible theoretical answer to this question. The "eductive" story does not *a priori* assume that agents know each other expectations and instead examines when this can be justified. It goes as follows. Relax the assumption that every agent expects the REE prices into a weaker assumption that every agent expects the prices to be in a neighborhood of the REE prices. Still, assume that the rationality of all the agents is Common Knowledge (CK hereafter) and the structure of the economy is CK as well. Then, consider that every agent enters into an individual strategic reasoning triggered by the CK assumptions. This process of forecasting others' forecast leads each agent to predict a set of possible outcomes. It can be completely successful, i.e. it can lead each agent to predict that the REE is the only possible outcome. In this case, we conclude that the REE is robust to relaxing the assumption that every agent *a priori* expects the REE, and we say that the REE is stable under "eductive" learning. Alternatively, the eductive process can fail to predict a unique outcome. In this case, a set of prices is possible: a non REE price can be observed. The eductive method does not aim at making a precise prediction of what should happen when the REE is unstable. The only conclusion that is drawn from this instability result is that out-of-equilibrium fluctuations are likely to happen. In game theoretical words, the "eductive" method amounts to say that the REE-Nash equilibrium of a certain market game is stable whenever it is the only rationalizable solution of this game.

In this paper, we apply the eductive method to a reduced form linear economy where the fundamentals are uncertain. We first check that, as long as information of agents is symmetric, the initial story of Guesnerie (1992) is straightforwardly transposed. This corresponds to a first result: the stability of the REE obtains when the economy is not very sensitive to expectations, *independently of the volatility of the fundamentals.*

[1] Guesnerie (2002) gives a synthetical assessment of the eductive methodology.

Then, we introduce an asymmetry of information among agents: some agents only observes the state of nature, while the others have no private information. The introduction of informational asymmetries about fundamentals affects the above story in two ways.

Firstly, a striking result is that *the conditions of stability are negatively affected by the proportion of informed agents*. This is due to the fact that an agent's decisions (and then the actual price) are more sensitive to his expectations when he is informed. A noticeable corollary of this result is that, in the limit case where the degree of sensitivity of the economy to price forecasts is CK (the real parameter ϕ in the model is constant), the conditions of stability are not affected by informational asymmetries, as represented by the proportion of informed agents.

Secondly, informational asymmetries raise the question of informational efficiency of the price: is it the case that the price will be fully revealing? i.e. is an agent able to infer the state of nature from observing the price only, without having *a priori* private information about this state of nature? This question is different from the preceding one. Indeed, it can be the case that the eductive process restricts the sets of admissible prices in a given state of nature although it does not predict a unique price in a given state of nature. If the restrictions imposed by the eductive process on these sets of admissible prices are stringent enough, then it can be the case that no price is compatible with two (or more) states of nature. In such a case, the price reveal the underlying state of nature with certainty, although this price may not be a REE price. We find that *informational efficiency is favored by a large proportion of informed agents*. Lastly, we investigate this question in the case where the prior beliefs about the fundamentals of non informed agents are not CK, but private information. We find that the latter result is robust to this higher order uncertainty about fundamentals.

Summing up, *the proportion of informed agents plays an ambiguous role: on the one hand, increasing the proportion of informed agents makes more difficult the coordination of expectations on the REE, and on the other hand, it favors informational efficiency* (i.e. actual prices reveal faster the underlying state).

The related literature includes two trends: the first one describes macroeconomic dynamics when agents make forecasts errors and revise their beliefs (this is macroeconomics dynamics with learning à la Evans and Honkapohja (2001) and Grandmont (1998)) and the second trend examines the justifications of REE (à la Guesnerie (2002)). Precisely, the first trend gives an account of macroeconomic volatility through learning considerations. We aim at contributing to these issues by introducing informational asymmetries (as recently done in the literature about REE justifications). Our results extend existing results concerning eductive stability and confirm the intuitions sustained in those papers.

- Informational asymmetries affect the stability of the learning dynamics.

- The results of Guesnerie (1992) concerning elasticities are robust to the kind of asymmetric information considered in this paper.

- This completes Desgranges and Guesnerie (2000) and Desgranges, Geoffard and Guesnerie (2003) concerning the role played by asymmetric information. The main difference with those papers is that, in the present paper, the uninformed agents cannot extract information from the price, simply because they make their decisions before they observe the price (still, the price conveys *ex post* information about the fundamentals, this is why we are in a position to consider informational efficiency of the price). Our main result of a destabilizing effect of a large proportion of informed agents is reminiscent from results in these papers. Still, the intuition sustaining the results in these papers is totally different. In our model, an informed agent can make a forecast error, which is not the case in the others papers, and this is central to our stability results (as explained above). Namely, in our model, increasing the proportion of informed agents makes the price *less* predictible, which favors REE instability, whereas, in the other papers, increasing the proportion of informed agents makes the price *more* revealing. This triggers a high sensitivity of uninformed agents' beliefs on the informational content of the price, which in turn triggers REE instability.

- Heinemann (2002) has some basic results in a simple framework similar to the present one.

The paper is organized as follows. Section 2 presents the model and the "eductive" story when there is no informational asymmetries. Section 3 considers eductive stability of the REE under asymmetric information. Section 4 considers price informational efficiency. Section 5 investigates robustness issues (individual heterogeneity, non linear framework, higher order uncertainty on the fundamentals). Section 6 concludes.

2 Learning under symmetric information

2.1 The general framework

We consider the class of self-referential linear univariate models where the state of the economic system in any given period is described by a real number p. The value of p in the state of the world ω ($\omega = 1, \ldots, \Omega$) is denoted $p(\omega)$. It is determined by the temporary equilibrium relation

$$p(\omega) = \phi(\omega) \int_0^1 p_i^e di + \eta(\omega). \tag{1}$$

This mapping should be thought of as a situation where there is a continuum of infinitesimal traders $i \in [0,1]$ with individual forecasts p_i^e about the actual state of the economy. These forecasts are formed in the previous period and they influence the system through the aggregate forecast

$$P^e \equiv \int_0^1 p_i^e di. \tag{2}$$

In (1), the underlying fundamentals are summarized by a pair $(\phi(\omega), \eta(\omega))$ in the state of the world ω. The parameter $\phi(\omega)$ represents the sensitivy of the economy to traders' forecasts, while the parameter $\eta(\omega)$ is a mere scale factor. From now onwards, we restrict attention to the case where all the forecast weights $\phi(\omega)$ have the same sign, whatever ω is. Below are two examples where this hypothesis is satisfied.

Example 1. *The cobweb model with aggregate demand uncertainty.*
A continuum of infinitesimal and homogenous firms $i \in [0,1]$ make their individual supply decisions q_i one period before their product is sold. Each firm i has a cost function $C(q_i) = q_i^2/2c$. Price taking behavior leads firm i to maximize profit $p_i^e q_i - q_i^2/2c$ when its price forecast is p_i^e. Thus, $q_i = cp_i^e$ for $i \in [0,1]$. The demand side of the economy is described by an aggregate demand function $\delta(\omega) - \mu(\omega)p$, where p represents the actual price, and both $\delta(\omega)$ and $\mu(\omega)$ are positive parameters. A temporary equilibrium is defined by a price $p(\omega)$ such that aggregate supply equals aggregate demand, that is

$$\int_0^1 cp_i^e di = \delta(\omega) - \mu(\omega)p(\omega). \tag{3}$$

Let $\phi(\omega) \equiv -c/\mu(\omega)$ and $\eta(\omega) \equiv \delta(\omega)/\mu(\omega)$. Then (3) rewrites as (1).

Example 2. *The Lucas aggregate supply model.*
Consider a continuum of infinitesimal and homogenous firms $i \in [0,1]$. Each firm is assumed to increase individual supply $q_i = \beta(p_i - p_i^e)$ only in response to perceived relative price $(p_i - p_i^e)$, where p_i stands for the price of its own product and p_i^e represents its forecast of the aggregate price level. A temporary equilibrium in state ω is an aggregate price level

$$p(\omega) \equiv \int_0^1 p_i(\omega)di,$$

such that aggregate supply equals aggregate demand, $\delta(\omega) - \mu(\omega)p(\omega)$. Thus,

$$\int_0^1 \beta(p_i(\omega) - p_i^e(\omega))di = \delta(\omega) - \mu(\omega)p(\omega). \tag{4}$$

Let $\phi(\omega) \equiv \beta/(\beta + \mu(\omega))$ and $\eta(\omega) \equiv \delta(\omega)/(\beta + \mu(\omega))$. Then (4) rewrites as (1).

In the temporary equilibrium relation (1), individual price forecasts rely on information about the actual state of the world that is available to traders. If, for instance, some trader i already knows the underlying fundamentals of the economy when submitting his price forecast, then p_i^e should be made conditionnally to ω (which has been done in the above examples). In this section, we focus on two polar configurations with symmetric information. Firslty, all the traders are perfectly informed of ω when they form their price forecasts. Secondly, traders are no longer aware of ω at that moment.

2.2 The perfect information case

Let the state of the world ω and the values $(\phi(\omega), \eta(\omega))$ be common knowledge (CK hereafter). The rational expectations equilibrium (REE hereafter) is (as usual) defined as a price vector $(p^*(1), ..., p^*(\Omega))$ where $p_i^e(\omega) = p(\omega)$ for any $i \in [0,1]$ in (1). The price $p^*(\omega)$ is accordingly such that

$$p^*(\omega) = \phi(\omega)\, p^*(\omega) + \eta(\omega). \tag{5}$$

If $\phi(\omega) \neq 1$, there exists a unique price $p^*(\omega)$ solution to (5), and there is then a widespread agreement in the literature to assume that every trader expects such a price to arise. In order to assess the relevance of this hypothesis, we interpret the REE $p^*(\omega)$ as a Nash equilibrium of a strategic game played by traders $i \in [0,1]$. In this game, pure strategies are price forecasts $p_i^e(\omega)$, and the payoff of any player i is $-(p(\omega) - p_i^e(\omega))^2$, where $p(\omega)$ is determined by (1): it is inversely related to its own forecast error. The structure of this game is CK.

In Examples 1 and 2 above, firms maximize profit, they do not minimize forecast errors. However, it is quite intuitive that both goals are equivalent. To show this point formally, consider for instance the cobweb model.

Example 1 (continued). *Strategy and payoff in the cobweb model.*
Recall that a firm i produces $q_i = cp_i^e(\omega)$ whenever it expects $p_i^e(\omega)$ in state ω. Once the actual price $p(\omega)$ is realized, its *ex post* profit is $p(\omega)cp_i^e(\omega) - \left[cp_i^e(\omega)\right]^2/2c$. Notice that the value $p_i^e(\omega)$ minimizing the forecast error $\left(p(\omega) - p_i^e(\omega)\right)^2$ is exactly the value maximizing the quantity $p(\omega)cp_i^e(\omega) - \left[cp_i^e(\omega)\right]^2/2c$ (this value is simply $p(\omega)$). Hence, choosing a correct forecast is equivalent to maximizing profit for a rational trader, i.e. for a trader whose production and forecast satisfy $q_i = cp_i^e(\omega)$.

More generally, in the game associated with the temporary equilibrium relation (1), the REE defined above coincides with the Nash equilbrium of the game. Indeed, in state ω, the optimal forecast $E_i\left[p(\omega)\right]$ of a trader i is

$$E_i\left[p(\omega)\right] = \phi(\omega)\, E_i\left[\int_0^1 p_n^e(\omega)\, dn\right] + \eta(\omega)\,, \tag{6}$$

where $E_i\left[\int_0^1 p_n^e(\omega)\, dn\right]$ is the aggregate forecast, as expected by i. Hence, it is straightforward that a Nash equilibrium is characterized by Equation (5) defining the REE. Indeed, in state ω, the best reply of trader i to every other trader expecting $p^*(\omega)$ is exactly $p^*(\omega)$ from Equation (6).

This interpretation highlights the coordination issue faced by the traders: the optimality of a forecast depends on others' forecasts. In particular, only those traders expecting the aggregate forecast $P^e(\omega)$ to be $p^*(\omega)$ do expect the REE price $p^*(\omega)$. Introducing higher order beliefs leads to the property that expecting $p^*(\omega)$ follows from expecting that all the traders[2] expect the aggregate price forecast to be $p^*(\omega)$. This kind of mental justification of the forecast $p^*(\omega)$ can be iterated *ad infinitum*. We then have the following characterization of a REE: in state ω, the REE price $p^*(\omega)$ is the only price p that is compatible with CK of every trader expecting p.

[2] but a set of zero measure.

A possible way to appreciate the robustness of the price $p^*(\omega)$ (as the "fair" price) consists to relax the above CK assumption supporting the REE. That is, we relax the assumption of CK that every trader expects exactly the REE price into the weaker assumption of that every individual forecast of the price in state ω belongs to a given interval $P^0(\omega)$. $P^0(\omega)$ contains the REE price $p^*(\omega)$, but it does not necessarily reduce to this price. Under this weaker assumption, we address the issue of expectations coordination: are agents still able to correctly guess the actual price? Or, equivalently, are they able to coordinate their forecasts on the price $p^*(\omega)$? If yes, agents have perfect foresight and coordination is successful. Otherwise, some traders do make forecast errors, and coordination fails.

The process through which agents coordinate their forecasts is based on some concept of iterated elimination of dominated strategies, as in Bernheim (1984), Pearce (1984) or Guesnerie (1992). It starts with an anchorage hypothesis ensuring CK that the aggregate forecast $P^e(\omega)$ in state ω belongs to some interval $P^0(\omega)$. We denote $P^0(\omega) = \left[P^0_{\inf}(\omega), P^0_{\sup}(\omega)\right]$ and we assume $p^*(\omega) \in P^0(\omega)$. Formally, in the game under consideration, this process amounts to define the set of rationalizable strategies under the restriction that the "action" $p^e_i(\omega)$ of agent i in state ω lies in $P^0(\omega)$.

We set $\phi(\omega) > 0$ for every ω.[3] Under this hypothesis, the anchorage assumption allows traders to start an iterative learning process based on CK of individual rationality and CK of the model. Indeed, the anchorage Assumption implies that $P^e(\omega) \in P^0(\omega)$. Then, by (6), individual rationality implies that any trader i forms a price forecast in a new interval $P^1(\omega) = \left[P^1_{\inf}(\omega), P^1_{\sup}(\omega)\right]$, where

$$\begin{aligned} P^1_{\inf}(\omega) &= \phi(\omega)P^0_{\inf}(\omega) + \eta(\omega), & (7)\\ \text{and } P^1_{\sup}(\omega) &= \phi(\omega)P^0_{\sup}(\omega) + \eta(\omega). & (8) \end{aligned}$$

Since individual rationality is CK, it is CK that $p^e_i(\omega) \in P^1(\omega)$ for every $i \in [0,1]$. Thus, it is CK that the aggregate price forecast $P^e(\omega)$ also belongs to $P^1(\omega)$. Iterating this argument implies that, if it is CK that $P^e(\omega) \in P^\tau(\omega)$ at the outset of some step $\tau \geq 0$, then it is CK that $P^e(\omega) \in P^{\tau+1}(\omega)$ at the outset of the next step, where the sequence of intervals $P^\tau(\omega)$ is defined by the recurrence equation:

$$P^{\tau+1}(\omega) = \phi(\omega)P^\tau(\omega) + \eta(\omega).$$

[3] All the results go through if we assume instead that $\phi(\omega) < 0$ for every ω. This latter assumtpion corresponds for example to the Cobweb.

We say that expectations coordination is successful if and only if the sequence $(P^{\tau}(\omega), \tau \geq 0)$ converges toward a single price in every state ω. It is obvious that this limit is necessarily the REE price $p^*(\omega)$ (this is the only fixed point of the learning process). As Guesnerie (1992) shows, coordination is successful if and only if $\phi(\omega) < 1$ for every ω, i.e. the economic system is not too sensitive to forecasts in the actual state of the world, or equivalently traders' forecasts are not too sensitive to others' forecasts in (6).

2.3 The hidden state case

Suppose now that uncertainty about the underlying economic fundamentals is not resolved when agents try to coordinate their price forecasts. There are then two different types of uncertainty; a first uncertainty bears on the aggregate price forecast (no trader observes the forecasts of other traders), and a second uncertainty is due to the imperfect knowledge of the state of the world. We assume that all the agents assign the probability $\pi(\omega)$ to state ω, and that this fact is CK. Hence, if a trader i expects the price $p_i^e(\omega)$ to arise in state ω, his price forecast writes

$$\bar{p}_i^e = \sum_{w=1}^{\Omega} \pi(w) p_i^e(w). \tag{9}$$

Under the rational expectations hypothesis, all the agents *a priori* believe that a price $p^*(\omega)$ arises in state ω, and this belief is self-fulfilling:[4]

$$p^*(\omega) = \phi(\omega) \sum_{w=1}^{\Omega} \pi(w) p^*(w) + \eta(\omega) \tag{10}$$

whatever $\omega = 1, \ldots, \Omega$ is. That is, a REE is a Ω-dimensional vector $(p^*(1), \ldots, p^*(\Omega))$ such that (10) holds for any ω.[5]

As in the perfect information case, this equilibrium is the Nash equilibrium of a strategic game where every trader minimzes the expected forecast

[4] For the consistence of notation, we still denote $p^*(\omega)$ the REE price in state ω although it is not the same price as the REE price $p^*(\omega)$ in the preceding section.

[5] Note that individual price forecasts can not be made conditionnally to the actual price. Indeed traders can not extract information about the actual state of the world from the actual price when they form their forecasts: this price is determined by (1) one period after they form their forecasts.

error $\sum_{w=1}^{\Omega}\pi(w)\left(p(w)-p_i^e(w)\right)^2$. Indeed, in this game, whenever the aggregate forecasts "played" by traders is some $\bar{P}^e=\sum_{w=1}^{\Omega}\pi(w)P^e(w)$, the best response of any player i is the price forecast

$$\bar{p}_i^e=\sum_{w=1}^{\Omega}\pi(w)\left[\phi(w)\bar{P}^e+\eta(w)\right]\equiv\bar{\phi}\bar{P}^e+\bar{\eta}, \tag{11}$$

where $\bar{\phi}=\sum_{w=1}^{\Omega}\pi(w)\phi(w)$ and $\bar{\eta}=\sum_{w=1}^{\Omega}\pi(w)\eta(w)$. A REE can consequently be thought of as a situation where it is CK that, according to (9), all the traders expect the average REE price:

$$\bar{p}^*=\sum_{w=1}^{\Omega}\pi(w)p^*(w). \tag{12}$$

We now consider the same coordination process as the one defined in the complete information case. First, we maintain the Anchorage Assumption. Notice however that a valid (slightly different) anchorage Assumption is that it is CK that the aggregate forecast $\bar{P}^e$ belongs to some interval $P^0=\left[P^0_{\inf},P^0_{\sup}\right]$ (with $\bar{p}^*\in P$). Individual rationality then implies that $\bar{p}_i^e\in P^1=\left[P^1_{\inf},P^1_{\sup}\right]$ for any $i\in[0,1]$, where

$$\begin{aligned} P^1_{\inf} &= \bar{\phi}P^0_{\inf}+\bar{\eta}, && (13)\\ \text{and}\quad P^1_{\sup} &= \bar{\phi}P^0_{\sup}+\bar{\eta}. && (14)\end{aligned}$$

If all the traders know that all the traders are rational, then all the traders know that $P^e\in P^1$. More generally, CK of individual rationality and model implies that it is CK that $P^e\in P^\tau$ for any $\tau\geq 0$, where the sequence $(P^\tau,\tau\geq 0)$ is defined recursively as in (13) and (14). The rest point of this dynamical system is the REE price (12). It is asymptotically stable if and only if $\bar{\phi}<1$. Then, agents succeed to coordinate their beliefs on the REE iff $\bar{\phi}<1$. Again, this stability condition $\bar{\phi}<1$ can be (loosely) interpreted as follows: correctly forecasting the actual price does not require to precisely figure out what the others believe.

3 Learning under asymmetric information

We now assume that there are α $(0<\alpha<1)$ traders who observe the actual state of the world ω before they form their forecasts; the $(1-\alpha)$ remaining agents have no knowledge of ω at the time of making their decision (they will observe ω later only). Hence, informed agents only can choose their forecast conditionnally to ω. The uninformed agents "play" an average price forecast, as described below.

We extend the analysis of expectations coordination to this setting with asymmetric information. We shall show that a low proportion of informed traders is required for this process of mutual introspections to allow agents to coordinate their beliefs on rational expectations. Namely, stability obtains only if the actual state of the world is concealed from many traders.

In each period, the timing of the events is as follows:

1. The informed agents $i \in [0, \alpha]$ observe the state of the world ω.

2. All the agents form simultaneously their price forecasts conditionnally to their information about the state of the world. Let $p_i^e(\omega)$ be the price expected by agent i to arise in state ω and

$$\bar{p}_i^e = \sum_{w=1}^{\Omega} \pi(w) p_i^e(w). \tag{15}$$

   Thus, in state ω, the informed agent i expects $p_i^e(\omega)$, while the uninformed agent i's forecast is the average price $\bar{p}_i^e$. It follows that the aggregate forecast in state ω is

$$P^e(\omega) = \int_0^\alpha p_i^e(\omega)\, di + \int_\alpha^1 \bar{p}_i^e di. \tag{16}$$

3. The actual price is determined by (1).

Example 1 (continued). *The Muth Model with demand uncertainty.*
An informed firm $i \in [0, \alpha]$ produces $q_i = cp_i^e(\omega)$ in state ω. An uninformed firm $i \in [\alpha, 1]$ produces $q_i = c\bar{p}_i^e$ that solves the problem

$$\max_q \sum_{w=1}^{\Omega} \pi(w) \left[p_i^e(w) q - \frac{1}{2c} q^2 \right],$$

and the aggregate supply in state ω is consequently equal to $cP^e(\omega)$. The temporary equilibrium price in state ω is such that aggregate supply equals aggregate demand. It is then determined by (16) and (1).

3.1 The rational expectations equilibrium

A REE is now a Ω-dimensional vector $\mathbf{p}^*$ whose ω-th component is such that

$$p^*(\omega) = \phi(\omega)\left[\alpha p^*(\omega) + (1-\alpha)\bar{p}^*\right] + \eta(\omega), \tag{17}$$

where $\bar{p}^* = \sum_{w=1}^{\Omega} \pi(w) p^*(w)$. It is straightforward that, generically, there is a unique REE.

Furthermore, this unique REE coincides with the Nash equilibrium of a Bayesian game that is analogous to the ones described in the preceding section. Precisely, in this game, the strategy of an informed agent consists in a vector of price forecast $(p_i^e(1), ..., p_i^e(\Omega))$, and the strategy of an uninformed agent is an average forecast $\bar{p}_i^e$. Every agent minimizes the expected forecast error. Therefore, when the aggregate forecast in state ω is $P^e(\omega)$, the best reply of an informed agent $i \in [0, \alpha]$ is to play

$$p_i^e(\omega) = \phi(\omega) P^e(\omega) + \eta(\omega) \equiv R_\omega\left(P^e(\omega)\right), \tag{18}$$

in every state ω, whereas the best reply of an uninformed agent is

$$\bar{p}_i^e = \sum_{w=1}^{\Omega} \pi(w) R_w\left(P^e(w)\right). \tag{19}$$

It is then straightforward that the Nash equilibrium of this game consists in every informed "playing" $\mathbf{p}^*$ and every uninformed "playing" $\bar{p}^*$.

3.2 The learning dynamics

Rational expectations can be interpreted as a situation where, provided that everyone is expected to play Nash, there is no individual incentives to deviate from Nash behavior, i.e. from forming forecasts according to the rational expectations hypothesis. Thus, in such an equilibrium, it is CK that the aggregate price forecast is equal to

$$\alpha p^*(\omega) + (1-\alpha)\sum_{w=1}^{\Omega} \pi(w) p^*(w) \tag{20}$$

if the actual state of the world is ω, whatever $\omega = 1, \ldots, \Omega$ is.

Let us now relax this assumption. Assume instead that it is CK that the aggregate price forecast $P^e(\omega)$ in state ω $(\omega = 1, \ldots, \Omega)$ belongs to some interval $P^0(\omega) = [P^0_{\inf}(\omega), P^0_{\sup}(\omega)]$ which includes the aggregate forecast (20), but does not necessarily reduces to it. Since $E_i P^e(\omega) \in P^\tau(\omega)$ for any player $i \in [0,1]$, a rational informed trader $i \in [0,\alpha]$ plays a price forecast $p^e_i(\omega)$ that is a best response to $E_i P^e(\omega)$ through the reaction function $R_\omega(\cdot)$, and so

$$p^e_i(\omega) \in \left[R_\omega(P^0_{\inf}(\omega)), R_\omega(P^0_{\sup}(\omega))\right]. \tag{21}$$

On the other hand, a rational uninformed trader $i \in [\alpha, 1]$ plays a price forecast

$$p^e_i \in \left[\sum_{w=1}^{\Omega} \pi(w) R_w(P^0_{\inf}(w)), \sum_{w=1}^{\Omega} \pi(w) R_w(P^0_{\sup}(w))\right]. \tag{22}$$

It follows that the aggregate price forecast $P^e(\omega)$ belongs to a new interval $P^1(\omega) = \left[P^1_{\inf}(\omega), P^1_{\sup}(\omega)\right]$ whose bounds are such that

$$\begin{aligned} P^1_{\inf}(\omega) &= \alpha R_\omega(P^0_{\inf}(\omega)) + (1-\alpha) \sum_{w=1}^{\Omega} \pi(w) R_w(P^0_{\inf}(w)), && (23) \\ \text{and} \quad P^1_{\sup}(\omega) &= \alpha R_\omega(P^0_{\sup}(\omega)) + (1-\alpha) \sum_{w=1}^{\Omega} \pi(w) R_w(P^0_{\sup}(w)). && (24) \end{aligned}$$

Now, if all the agents know that all the agents are rational, then they all know that $P^e(\omega)$ belongs to $P^1(\omega)$ in state ω ($\omega = 1, \ldots, \Omega$). More generally, CK of rationality implies that all the agents know that $P^e(\omega)$ belongs to $P^\tau(\omega)$ in state ω ($\omega = 1, \ldots, \Omega$) at outset of step τ ($\tau \geq 0$).

The only fixed point of the infinite sequence of intervals $(P^\tau(\omega), \tau \geq 0)$ defined recursively as in (23) and (24) is such that boundary prices $P^\tau_{\inf}(\omega)$ and $P^\tau_{\sup}(\omega)$ equal the aggregate price forecast (20) ensuring rational expectations. The next Proposition gives the condition of stability under learning, i.e for the Ω sequences $\left(P^\tau_{\inf}(\omega), P^\tau_{\sup}(\omega)\right)$ to converge toward the aggregate price forecast (20) when τ tends toward infinity.

Proposition 1 *Assume that $\phi(\omega) > 0$ for any $\omega = 1, \ldots, \Omega$. If $\alpha\phi(\omega) > 1$ for some ω, then the rational expectations equilibrium is unstable in the dynamics with eductive learning. If $\alpha\phi(\omega) < 1$ for every ω, then the rational expectations equilibrium is stable in the dynamics with eductive learning if and only if*

$$\sum_{w=1}^{\Omega} \pi(w) \frac{(1-\alpha)\phi(w)}{1-\alpha\phi(w)} < 1. \tag{25}$$

Proof. The Ω equations (23) define the Ω lowest prices $(P_{\text{inf}}^{\tau+1}(\omega))$ in each state ω ($\omega = 1, \ldots, \Omega$) at step $\tau+1$ ($\tau \geq 0$) as a sole function of the Ω prices $(P_{\text{inf}}^{\tau}(\omega))$ at step τ. Moreover the fixed point $(P_{\text{inf}}(1), \ldots, P_{\text{inf}}(\Omega))$ of (23) is stable under learning if and only if the fixed point $(P_{\text{sup}}(1), \ldots, P_{\text{sup}}(\Omega))$ is stable in the Ω-dimensional system (24). This allows us to restrict our attention to the Ω-dimensional system (23). Given (18), the Ω equations (23) rewrite in matrix form $\mathbf{p}_{\text{inf}}^{\tau+1} = \mathbf{M}\mathbf{p}_{\text{inf}}^{\tau} + \boldsymbol{\eta}$, where $\mathbf{p}_{\text{inf}}^{\tau}$ is the $\Omega \times 1$ vector $(P_{\text{inf}}^{\tau}(1), \ldots, P_{\text{inf}}^{\tau}(\Omega))$, $\boldsymbol{\eta}$ is the $\Omega \times 1$ vector $(\eta(1), ..., \eta(\Omega))$ and $\mathbf{M}$ is the $\Omega \times \Omega$ matrix $\alpha\mathbf{\Phi} + (1-\alpha)\mathbf{\Phi}\mathbf{\Pi}$ (with $\mathbf{\Phi}$ the diagonal $\Omega \times \Omega$ matrix whose $\omega\omega$-th entry is $\phi(\omega)$, and $\mathbf{\Pi}$ the $\Omega \times \Omega$ stochastic matrix whose $\omega\omega'$-th entry is $\pi(\omega')$). The REE is the only fixed point of the dynamics with learning. It is stable in this dynamics iff the spectral radius $\rho(\mathbf{M})$ of $\mathbf{M}$ has modulus less than 1.[6] The proof then hinges on the fact that for any $\Omega \times \Omega$ matrix $\mathbf{M} = (m_{ij})$, with every $m_{ij} \geq 0$, and any $\Omega \times 1$ vector $\mathbf{x} = (x_\omega)$ with every $x_\omega > 0$, we have

$$\min_\omega \frac{(\mathbf{Mx})_\omega}{x_\omega} \leq \rho(\mathbf{M}) \leq \max_\omega \frac{(\mathbf{Mx})_\omega}{x_\omega},$$

where $(\mathbf{Mx})_\omega$ stands for the ω-th component of the Ω-dimensional vector $\mathbf{Mx}$. See Lemma 3.1.2. in Bapat and Raghavan (1997) for instance. Let

$$Q(\mathbf{x}, \omega) = \frac{(\mathbf{Mx})_\omega}{x_\omega} = \phi(\omega)\left[\alpha + (1-\alpha)\frac{1}{x_\omega}\sum_{w=1}^{\Omega}\pi(w)\, x_w\right],$$

for any ω. Assume first that $\alpha\phi(\omega) > 1$ for some ω, e.g. $\alpha\phi(\Omega) > 1$. Then, consider the vector $\mathbf{x} = (\varepsilon, \ldots, \varepsilon, 1)'$ where $\varepsilon > 0$. When ε tends toward 0, $Q(\mathbf{x}, \omega)$ tends to $+\infty$ for every $\omega < \Omega$, and $Q(\mathbf{x}, \Omega) \geq \alpha\phi(\Omega) > 1$. Hence, for ε small enough, $\min_\omega Q(\mathbf{x}, \omega) > 1$ and thus $\rho(\mathbf{M}) > 1$. This shows that the REE is unstable if $\alpha\phi(\omega) > 1$ for some ω. Assume now that $\alpha\phi(\omega) < 1$ for any ω. Let

$$E = \sum_{w=1}^{\Omega}\pi(w)\frac{(1-\alpha)\phi(w)}{1-\alpha\phi(w)}.$$

Consider the $\Omega \times 1$ positive vector $\mathbf{x}$ whose ω-th component is

$$x_\omega = \frac{1}{E}\frac{(1-\alpha)\phi(\omega)}{1-\alpha\phi(\omega)} > 0.$$

[6] We do not consider the case where $\mathbf{M}$ has an eigenvalue equal to 1.

If $E \geq 1$, then $Q(\mathbf{x}, \omega) > 1$ for any ω, so that $\min_\omega Q(\mathbf{x}, \omega) \geq 1$, and the equilibrium is unstable. If, on the contrary, $E < 1$, then $Q(\mathbf{x}, \omega) < 1$ for any ω, so that $\max_\omega Q(\mathbf{x}, \omega) < 1$, and the equilibrium is stable. This shows the result. ■

Although some traders are perfectly informed about the actual state of the world, stability properties of the learning dynamics depend on the whole set of possible states. Indeed, uninformed traders have to predict price forecasts of informed traders in any possible state, which urges informed agents to take care about all the states of the world through the restrictions they draw on the behavior of uninformed agents. As a result, expectations coordination may fail in a state ω where $\phi(\omega) < 1$. Namely, if the equilibrium is unstable under learning, then there is necessarily one unstable price $p^*(\omega)$, and so the average equilibrium price

$$\sum_{w=1}^{\Omega} \pi(w) \, p^*(w)$$

is also unstable. Since uninformed agents can not guess this average price, informed agents can not predict the actual price, whatever the actual state is. From this fact, one could believe that the presence of uninformed traders has a destabilizing effect in the learning process. Nevertheless, an increase in the proportion of informed traders tends to increase $\alpha\phi(\omega)$ in all states, which also favors instability. Therefore, the consequences of the interaction between the underlying economic fundamentals and the information structure onto the learning process are ambiguous. The purpose of the following corollary of Proposition 1 is to disentangle these two dimensions.

Corollary 2 *Assume that $\phi(\omega) > 0$ for any $\omega = 1, \ldots, \Omega$.*

If $\phi(\omega) < 1$ for any $\omega = 1, \ldots, \Omega$, then the rational expectations equilibrium is stable in the dynamics with learning given by (23) and (24).

If $\inf_\omega \phi(\omega) < 1 < \sup_\omega \phi(\omega)$, then provided that $\bar{\phi} < 1$, there exists a threshold proportion α^*, $0 < \alpha^* < 1$, of informed traders such that stability of the equilibrium obtains if and only if $\alpha < \alpha^*$. The threshold α^* is a decreasing function of each $\phi(\omega)$. If $\bar{\phi} \geq 1$, then stability under learning never obtains.

If $\phi(\omega) > 1$ for any $\omega = 1, \ldots, \Omega$, then the equilibrium is unstable under learning.

Proof. Assume first that $\phi(\omega) < 1$ for any $\omega = 1, \ldots, \Omega$. For every$\omega$, $\alpha\phi(\omega) < 1$ and $(1-\alpha)\phi(\omega) / (1 - \alpha\phi(\omega)) < 1$. Proposition 1 then implies that the REE is stable.

Assume now that $\inf_{\omega}\phi(\omega) < 1 < \sup_{\omega}\phi(\omega)$. If $\alpha > 1/\sup_{\omega}\phi(\omega)$, then it directly follows from Proposition that the REE is unstable. If $\alpha \leq 1/\sup_{\omega}\phi(\omega)$, then $\alpha\phi(\omega) < 1$ for everyω, and, from Proposition 1 again, the condition for stability is (25). Let

$$F(\alpha) = \sum_{w=1}^{\Omega} \pi(w) \frac{\phi(w)}{1-\alpha\phi(w)} - \frac{1}{(1-\alpha)} \tag{26}$$

$F(\cdot)$ is a continuous and increasing function of α on the interval $[0, 1/\sup_{\omega}\phi(\omega)]$ ($F'(\alpha) > 0$ whatever α is). Hence, there is at most one value α such that $F(\alpha) = 0$ on this interval. Finally, $F(0) = \bar{\phi} - 1$, and $F(\alpha)$ tends to $+\infty$ when α tends to $1/\sup_{\omega}\phi(\omega)$ from below. Thus, one distinguishes between two cases. Firstly, if $\bar{\phi} \geq 1$, then $F(\alpha) \geq F(0) > 0$ for any $\alpha \in [0, 1/\sup_{\omega}\phi(\omega)]$, and the stability condition (25) is never satisfied. Secondly, if $\bar{\phi} < 1$, then there exists a unique solution α^* ($\alpha^* > 0$) to $F(\alpha) = 0$ in the interval $[0, 1/\sup_{\omega}\phi(\omega)]$. $F(\alpha) < 0$, i.e. the stability condition (25) is satisfied, iff $\alpha < \alpha^*$. Furthermore, $F(\alpha^*) = 0$ implicitly defines α^* as a function of the collection $(\phi(1), \ldots, \phi(\Omega))$. It is straighforward to verify that $F(\cdot)$ increases in every $\phi(\omega)$. Thus, α^* decreases in every $\phi(\omega)$.

Assume finally that $\phi(\omega) > 1$ for any ω. Then, $\bar{\phi} > 1$, and we have already seen that $F(\alpha) > 0$ for any $\alpha \in [0, 1/\sup_{\omega}\phi(\omega)]$. As a result, the stability condition (25) is never satisfied. The point follows. ■

The intuition for this result again hinges on the sensitivity of individual price forecasts to the perceived behavior of other agents that is summarized in our framework by the aggregate price forecast. Assume indeed that an agent $i \in [0,1]$ expects the aggregate price forecast to undergo a marginal change $dP^e(\omega)$ in some state ω ($\omega = 1, \ldots, \Omega$). If such an agent is informed, then he reacts (or he is expected to react) to this change by adjusting his own price forecast for an amount $dp_i^e(\omega)$ equal to $\phi(\omega)\,dP^e(\omega)$ in the state of the world is ω. If, on the contrary, such an agent uninformed, then he modifies his own price forecast for an amount $\pi(\omega)\,\phi(\omega)\,dP^e(\omega)$ which is clearly less than $\phi(\omega)\,dP^e(\omega)$ as soon as this agent does not hold for sure that the actual state of the world is ω. Thus, the forecasting behavior of an uninformed agent is less sensitive to others' price forecasts than an informed agent. His behavior is accordingly easier to predict, which favors the coordination of expectations. Of course, since the inertia that is due to the presence of agents who are not aware of the actual state of the world transmits to the price, stability under learning of the REE obtains whenever the influence of traders' beliefs onto the economy is low enough, as in the main strand of the literature. However, the inertia channel is to be found not in the direct effect of price forecasts, as measured by the forecast weight, but instead at some upstream level where the influence of traders beliefs about the aggregate behavior of others onto price forecasts is low enough.

3.3 Sunspot equilibria as a coordination device

It is often argued that sunspots could be used as a coordination device to which agents could refer when they form their price forecasts. The purpose of this section is to examine whether sunspots may favor coordination of expectations on rational expectations equilibria. It will be shown that this is not the case. Indeed, on the one hand, sunspots matter if and only if the REE defined in (17) is unstable in the dynamics with learning; in this sense, sunspots disturb the expectations coordination process under the conditions exhibited in Proposition 1. On the other hand, any equilibrium where the actual price depends not only on the state of the underlying fundamentals but also on extraneous sunspot uncertainty is unstable under learning.

In order to define a sunspot equilibrium, we consider a stochastic sunspot variable that can take Σ different values ($S = 1, \ldots, \Sigma$) and that is not correlated with the underlying fundamentals. Although its actual value is not perfectly known when agents choose their expectations, every agent $i \in [0, 1]$ observes a private signal $s_i = 1, \ldots, \Sigma$ imperfectly correlated with S. Private signals are assumed to be independently and identically distributed accross traders (conditionnally to S), whatever S is. In sunspot event S, every agent observes the signal s_i with a probability $\Pr(s_i \mid S)$ that is independent on i. Thus, in sunspot event S, there are $\Pr(s \mid S)$ agents who observe the signal s (for every $s = 1, \ldots, \Sigma$). $\Pr(S \mid s)$ stands for the probability that sunspot event is S when private signal is s. ($\Pr(S \mid s)$ and $\Pr(s \mid S)$ are linked by Bayes' law.)

Suppose that all the agents expect the price $p^e(\omega, S)$ to arise if the state of fundamentals is ω and the sunspot state is S. In this event, there are $\alpha \Pr(s \mid S)$ informed agents whose price forecast is

$$\sum_{S'=1}^{\Sigma} \Pr(S' \mid s) p^e(\omega, S'), \quad \text{for } s = 1, \ldots, \Sigma.$$

There are also $(1-\alpha)\Pr(s \mid S)$ uninformed agents who expect

$$\sum_{S'=1}^{\Sigma} \Pr(S' \mid s) \sum_{w=1}^{\Omega} \pi(w)\, p^e(w, S'), \quad \text{for } s = 1, \dots, \Sigma.$$

As a result, the aggregate price forecast $P^e(\omega, S)$ in this state of the world writes

$$\sum_{S'=1}^{\Sigma} \sum_{s=1}^{\Sigma} \Pr(s \mid S) \Pr(S' \mid s) \left[\alpha p^e(\omega, S') + (1-\alpha) \sum_{w=1}^{\Omega} \pi(w)\, p^e(w, S') \right].$$

Let

$$\mu(S'|S) = \sum_{s=1}^{\Sigma} \Pr(s \mid S) \Pr(S' \mid s)$$

be the average probability (across agents) of sunspot state S' if the actual sunspot is S. Then, in the event (ω, S), the actual price $p(\omega, S)$ determined by (1) is

$$p(\omega, S) = \phi(\omega) \sum_{S'=1}^{\Sigma} \mu(S'|S) \left[\alpha p^e(\omega, S') + (1-\alpha) \sum_{w=1}^{\Omega} \pi(w)\, p^e(w, S') \right] + \eta(\omega). \tag{27}$$

By definition, a REE is a price vector $p^*(\omega, S)$ such that $p^e(\omega, S) = p(\omega, S) = p^*(\omega, S)$ for every (ω, S) in (27). The fundamental solution is the (generically) unique REE such that $p^*(\omega, S)$ is independent on S. We say that sunspots matter in equilibrium when there are REE satisfying $p^*(\omega, S) \neq p^*(\omega, S')$ for some ω and $S \neq S'$. The following result shows that existence of sunspot equilibria is closely linked with the stability properties of the fundamental solution in the dynamics with learning.

Proposition 3 *There exist sunspot equilibria iff the REE is unstable.*

Proof. Let us rewrite conditions (27) in matrix form. To this aim, let $\mathbf{p}(S)$ be the Ω-dimensional vector whose ω-th component is $p(\omega, S)$. Let $\mathbf{p}$ be the $\Omega\Sigma$-dimensional vector $(\mathbf{p}(1), \dots, \mathbf{p}(\Sigma))$, and $\mathbf{p}^e$ the corresponding $\Omega\Sigma$-dimensional vector of expected prices. Let S be the $\Sigma \times \Sigma$ stochastic matrix whose SS'-th entry is $\mu(S', S)$. Recall that $\boldsymbol{\eta}$ is the Ω-dimensional vector $(\eta(1), \dots, \eta(\Omega))'$. Then, with $\mathbf{M}$ defined in Proposition 1, we have

$$\mathbf{p} = (\mathbf{S} \otimes \mathbf{M})\, \mathbf{p}^e + \mathbf{1}_\Sigma \otimes \boldsymbol{\eta}. \tag{28}$$

A REE is a price vector satisfying $\mathbf{p} = \mathbf{p}^e$ in (28). The fundamental solution is a REE $\mathbf{p}^*$ such that $p^*(\omega, S) = p^*(\omega, S')$ for any ω, S, S'. Let $e(S)$, $S = 1, \ldots, \Sigma$, be the Σ eigenvalues of the stochastic matrix $\mathbf{S}$. $e(S) \in [-1, 1]$ and there is a unique S with $e(S) = 1$. Let $\mu(\omega)$ be the ω-th eigenvalue of $\mathbf{M}$. Then, the $\Omega\Sigma$ eigenvalues of $S \otimes \mathbf{M}$ are $e(S)\mu(\omega)$ for any pair (ω, S). Recall that $\rho(\mathbf{M}) = \sup_\omega |\mu(\omega)|$. If $\rho(\mathbf{M}) < 1$, then all the eigenvalues of $S \otimes \mathbf{M}$ have modulus less than 1, and so $S \otimes \mathbf{M} - \mathbf{I}_{2\Omega}$ is invertible and there is a unique REE. If $\rho(\mathbf{M}) \geq 1$, then there exist stochastic matrices such that $e(S) = 1/\rho(\mathbf{M})$ for some S. In this case, the matrix $S \otimes \mathbf{M}$ has an eigenvalue equal to 1, and so there are infinitely many solutions $\mathbf{p}$ to (28), i.e. infinitely many sunspot equilibria and the fundamental solution $\mathbf{p}^*$. ■

As shown in Proposition 1, stability under learning requires a low proportion of informed traders. Thus, a large proportion of informed agents is needed for sunspot equilibria to exist. The intuition is clear, in view of the fact that the behavior of uninformed agents does not vary according to the state of the world (see Evans, Honkapohja and Sargent, 1993, for a similar view).

A further issue is whether sunspots may allow traders to coordinate their forecasts when the fundamental equilibrium $\mathbf{p}^*$ is unstable under learning. We turn attention to this point and we define the learning dynamics on a SSE.

Consider that an agent i expects the price $p_i^e(\omega, s)$ in the state of the world (ω, S). Define $\Gamma(s|S) = \sum_{n=1}^{n=s} \Pr(n \mid S)$ for every $s = 1, ..., \Sigma$ and $\Gamma(0|S) = 0$. Then, in a given state (ω, S), the aggregate price forecast $P^e(\omega, S)$ is (with the suitable labelling of private signals and agents)[7]:

$$\sum_{s=1}^{\Sigma} \int_{\alpha\Gamma(s-1|S)}^{\alpha\Gamma(s|S)} p_i^e(\omega, s) di + \sum_{s=1}^{\Sigma} \int_{\alpha+(1-\alpha)\Gamma(s-1|S)}^{\alpha+(1-\alpha)\Gamma(s|S)} \left[\sum_{w=1}^{w=\Omega} p_i^e(w) \right] di.$$

As a result, if it is initially CK that $P^e(\omega, S) \in P^0(\omega, S) = \left[P^0_{\inf}(\omega, S), P^0_{\sup}(\omega, S)\right]$ for any pair (ω, S), then it is CK that $P^e(\omega, S) \in P^1(\omega, S) = \left[P^1_{\inf}(\omega, S), P^1_{\sup}(\omega, S)\right]$,

[7] Label private signals such that all the informed traders $i \in [0, \alpha \Pr(1 \mid S)]$ receive signal $s = 1$, while all the informed traders $i \in [\alpha \Pr(s-1 \mid S), \alpha \Pr(s \mid S)]$ receive signal s $(s = 2, \ldots, \Sigma)$. Label uninformed traders such that an agent i of this type receives signal $s = 1$ if $i \in [\alpha \Pr(\Sigma \mid S), \alpha \Pr(\Sigma \mid S) + (1-\alpha) \Pr(1 \mid S)]$, and he receives signal s $(s = 2, \ldots, \Sigma)$ if $i \in [\alpha \Pr(\Sigma \mid S) + (1-\alpha) \Pr(s-1 \mid S), \alpha \Pr(\Sigma \mid S) + (1-\alpha) \Pr(s \mid S)]$.

with $P^1_{\inf}(\omega, S)$ and $P^1_{\sup}(\omega, S)$ respectively given by

$$\phi(\omega) \sum_{S'=1}^{\Sigma} \mu(S', S) \left[\alpha P^0_{\inf}(\omega, S') + (1-\alpha) \sum_{w=1}^{\Omega} \pi\left(w\right) P^0_{\inf}(w, S') \right] + \eta\left(\omega\right),$$

$$\text{and} \quad \phi(\omega) \sum_{S'=1}^{\Sigma} \mu(S', S) \left[\alpha P^0_{\sup}(\omega, S') + (1-\alpha) \sum_{w=1}^{\Omega} \pi\left(w\right) P^0_{\sup}(w, S') \right] + \eta\left(\omega\right).$$

Iterating this argument defines a coordination process on a SSE analogously to the process defined inthe preceding sections.

As the following result shows, sunspots can not be used as a coordination device.

Proposition 4 *Any sunspot equilibrium is unstable under learning.*

Proof. The dynamics with learning is governed by the $\Omega\Sigma \times \Omega\Sigma$ matrix $\mathbf{S} \otimes [\alpha \mathbf{\Phi} + (1-\alpha) \not\succ \mathbf{\Phi}] = \mathbf{S} \otimes \mathbf{M}$. Since S is a stochastic matrix, the spectral radius of $S \otimes \mathbf{M}$ is equal to $\rho\left(\mathbf{M}\right)$. A sunspot equilibrium is stable in the dynamics with learning if and only if $\rho\left(\mathbf{M}\right) < 1$, but then there does not exist sunspot equilibria. This shows the result. ■

4 Informational Efficiency

The question is to know whether the uninformed agents learn the state ω from the price, once it is made public. If the REE is stable, then the answer is obvious: any agent is able to deduce the state ω from the observed price. Indeed, the observed price is a REE price $p^*\left(\omega\right)$. As all these prices are different, the price always reveals the underlying state. Analogously, in the case where the REE is not stable, any price is compatible with any state ω so that nothing can be to deduced from the observed price. Thus, the price transmits information about the state whenever the equilibrium is stable under learning.

Beyond this fact, one may wonder whether agents can discover ω from the price when they go through a finite number of steps of the eductive reasoning only. In other words, the issue is whether the full CK assumptions are needed for the price to reveal the state. We now investigate this question. Formally, we know from the preceding section that, after τ steps of the eductive process, it is CK that the actual price in state ω belongs to $P^{\tau}(\omega)$. Hence, the price will reveal ω with certainty after τ steps whenever $P^{\tau}(\omega)$ does not intersect any other set $P^{\tau}(\omega')$ (even if the sets $P^{\tau}(\omega)$ and $P^{\tau}(\omega')$ do not reduce to a unique price). Notice that if $P^{\tau}(\omega) \cap P^{\tau}(\omega')$ is not empty, it can either be the case that the actual price do not reveal the state to the uninformed agents (if the price belongs to $P^{\tau}(\omega) \cap P^{\tau}(\omega')$) or it can be the case that the price reveals the state to the uninformed agents (if the price belongs to $P^{\tau}(\omega) - P^{\tau}(\omega')$ or to $P^{\tau}(\omega') - P^{\tau}(\omega)$).

Thus, we consider that the number of steps that are necessary for the sets $P^{\tau}(\omega)$ to be disjoint measures the degree of *ex-post* revelation of the state ω by the price that is allowed by the learning process: the smaller this number of steps, the more efficient the learning process.

For analytical simplicity, we restrict attention to the case where ϕ does not depend on ω (η only depends on ω). The following proposition gives the explicit condition under which the price can reveal the state ω when the eductive process is converging. In particular, α has a positive impact on learning.

Remark 1. In the case where the forecast weight is the same in every state of the world, i.e. $\phi(\omega) = \phi$ for any $\omega = 1, \ldots, \Omega$, so that uncertainty about economic fundamentals only bears on the scale factor, i.e. $\eta(\omega) \neq \eta(\omega')$ for $\omega \neq \omega'$ ($\omega' = 1, \ldots, \Omega$), Proposition 1 implies that the equilibrium is stable under learning if and only if $\phi < 1$. In this case, stability properties are not affected by informational asymmetries, as shown by Heinemann (2002). However, the speed of convergence toward the equilibrium is then inversely related to $\alpha\phi$, so that an increase in the proportion α of informed agents slows down learning.

Proposition 5 *Assume that the REE is stable (i.e., $\phi < 1$). Consider that the price reveals the underlying state ω after τ steps of learning iff the sets $P^{n}(\omega)$ do not intersect. Define N as the smallest integer n satisfying:*

$$p^0_{\sup} - p^0_{\inf} < \frac{\alpha}{\phi^n} \frac{1 - (\alpha\phi)^n}{1 - \alpha\phi} \inf_{\omega, \omega'} |\eta(\omega') - \eta(\omega)| . \tag{29}$$

Then, the price reveals the underlying state ω after n steps of learning if and only if $n \geq N$. Furthermore, N decreases with α.

If $\phi < 1$, then, for any proportion of informed agents, there exists a threshold N above which uninformed agents are able to deduce the current state ω from the price observed at the end of period, whatever this price is. *The current state will be revealed with certainty to uninformed agents even when agents have proceeded to N step only of "eductive" learning.*

This happens when (i) the difference $\inf_{\omega,\omega'} |\eta(\omega') - \eta(\omega)|$ is large (which corresponds to a large difference $|p^*(\omega') - p^*(\omega)|$), (ii) the anchorage assumption is informative, i.e. $p^0_{\sup} - p^0_{\inf}$ is small enough, and (iii) the economic system is not very sensitive to price forecasts, i.e. the effect ϕ of price forecasts onto equilibrium prices is small enough.

Proof. Recall from Equations 23 and 24 that

$$
\begin{aligned}
p^{\tau+1}_{\inf}(\omega) &= \alpha\left[\phi p^{\tau}_{\inf}(\omega) + \eta(\omega)\right] + (1-\alpha)\left[\phi \bar{p}^{\tau}_{\inf} + \bar{\eta}\right], \\
p^{\tau+1}_{\sup}(\omega) &= \alpha\left[\phi p^{\tau}_{\sup}(\omega) + \eta(\omega)\right] + (1-\alpha)\left[\phi \bar{p}^{\tau}_{\sup} + \bar{\eta}\right].
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\bar{p}^{\tau+1}_{\inf} &= \phi \bar{p}^{\tau}_{\inf} + \bar{\eta} = \phi^{\tau+1}\bar{p}^{0}_{\inf} + \frac{1-\phi^{\tau+1}}{1-\phi}\bar{\eta}, \\
\bar{p}^{\tau+1}_{\sup} &= \phi \bar{p}^{\tau}_{\sup} + \bar{\eta} = \phi^{\tau+1}\bar{p}^{0}_{\sup} + \frac{1-\phi^{\tau+1}}{1-\phi}\bar{\eta}.
\end{aligned}
$$

Hence, combining these four equations, we have

$$
p^{\tau+1}_{\sup}(\omega') - p^{\tau+1}_{\inf}(\omega) = \alpha\phi\left[p^{\tau}_{\sup}(\omega') - p^{\tau}_{\inf}(\omega)\right] + \alpha\left[\eta(\omega') - \eta(\omega)\right] + (1-\alpha)\phi^{\tau+1}\left[\bar{p}^{0}_{\sup} - \bar{p}^{0}_{\inf}\right],
$$

Hence, denoting $dp^{\tau+1} = p^{\tau+1}_{\sup}(\omega') - p^{\tau+1}_{\inf}(\omega)$, we have

$$
\begin{aligned}
dp^{\tau+1} &= \alpha\phi dp^{\tau} + \alpha\left[\eta(\omega') - \eta(\omega)\right] + (1-\alpha)\left[\bar{p}^{0}_{\sup} - \bar{p}^{0}_{\inf}\right]\phi^{\tau+1}, \\
&= (\alpha\phi)^{\tau+1} dp^{0} + \alpha\left[\eta(\omega') - \eta(\omega)\right]\frac{1-(\alpha\phi)^{\tau+1}}{1-\alpha\phi} + (1-\alpha)\left[\bar{p}^{0}_{\sup} - \bar{p}^{0}_{\inf}\right]\phi^{\tau+1}\frac{1-\alpha^{\tau+1}}{1-\alpha}.
\end{aligned}
$$

The sets $P^{\tau}(\omega)$ do not intersect as soon as $dp^{\tau} < 0$ whenever $\eta(\omega') < \eta(\omega)$. This is equivalent to:

$$
\bar{p}^{0}_{\sup} - \bar{p}^{0}_{\inf} < \frac{\alpha}{\phi^{\tau}}\frac{1-(\alpha\phi)^{\tau}}{1-\alpha\phi}\inf_{\omega,\omega'}|\eta(\omega') - \eta(\omega)|.
$$

As the LHS does not depends on τ and the RHS isincreasing in τ, the threshold value N stated in the proposition is the smallest integer satisfying the above inequality. This shows the first part of the proposition.

To show the second part of the proposition, it is enough to show that $\frac{\alpha}{\phi^\tau}\frac{1-(\alpha\phi)^\tau}{1-\alpha\phi}$ increases with α. This follows from:

$$\frac{d}{d\alpha}\left[\frac{\alpha}{\phi^\tau}\sum_{0\leq k\leq\tau-1}(\alpha\phi)^k\right]=\frac{1}{\phi^\tau}\sum_{0\leq k\leq\tau-1}(\alpha\phi)^k+\frac{\alpha}{\phi^\tau}\sum_{0\leq k\leq\tau-1}k\alpha^{k-1}\phi^k>0.$$

This ends the proof. ■

Bayesian learning. Another way of studying informational efficiency in this economy would be to consider repeated plays of the economy. A precise model woud be along the following lines: at date 0, ω is observed (once for all) by informed agents. Then, at dates 1, 2,..., agents "play" the economy described by a modified version of Equation (**??**), namely:

$$p_t(\omega)=\phi(\omega)\left[\int_0^\alpha p_{i,t}^e(\omega)\,di+\int_\alpha^1\bar{p}_{i,t}^e di\right]+\eta(\omega)+\varepsilon_t, \tag{30}$$

where ε_t is an i.i.d. white noise.[8] In this setting, the observation of the sequence of prices allows the uninformed agents to progressively learn the state ω. Then, a well known result is that, in a Bayesian equilibrium of this game, under mild assumptions on ε_t, the price converges in the long run to its full information value $\eta(\omega)/(1-\phi(\omega))$ with probability 1, and the speed of convergence *increases with* α.[9]

5 Extensions

5.1 Higher order uncertainty on the fundamentals

So far it has been assumed not only that uninformed agents use the objective distribution of the states of the world, but also that this fact is CK. In fact, the careful reader will have noticed that our analysis also applies to the case where the subjective probability $\pi_i(\omega)$ that some uninformed trader i assigns to the state of the world ω is private information, but the average probability

$$\pi(\omega)\overset{\text{def}}{=}\frac{1}{(1-\alpha)}\int_\alpha^1\pi_i(\omega)di \tag{31}$$

[8]Without the additional white noise ε_t, uninformed agents would learn s exactly at the end of period 1 at a Bayesian equilibrium.

[9]Computing the equilibrium is routine. See, among others, Desgranges, Geoffard and Guesnerie (2003) for the derivation of a similar equilibrium.

is CK. It may appear difficult, however, to justify such an assumption in a decentralized framework which otherwise stipulates a high level of ignorance. Thus, in this section, we relax this hypothesis to focus attention on the case where each agent is uncertain about others' beliefs about states of the world, i.e. we examine the case of so-called higher order uncertainty on the fundamentals. Then, an informed agent still knows the actual state of the world when he forms his price forecast, and an uninformed agent is still not aware of the actual state, but the probability distribution he uses for forecasting purpose is no longer CK.

For analytical simplicity, we assume that the forecast weight does not vary in accordance with the state of the world, i.e. $\phi(\omega)$ is equal to a constant ϕ ($\eta(\omega)$ only depends on $\omega = 1, \ldots, \Omega$).

As in Section 3.2, the learning algorithm proceeds from some anchorage assumption ensuring CK that the state ω-price forecasts belong to some interval $P^0(\omega)$. This assumption, together with the other assumptions of CK of individual rationality and model, trigger an infinite sequence of steps.

Every step τ ($\tau \geq 0$) is as follows. At the outset of step τ, it is CK that every price forecast $p_i^e(\omega)$ is in a well defined set $P^\tau(\omega) = \left[P_{\text{inf}}^\tau(\omega), P_{\text{sup}}^\tau(\omega)\right]$. It follows that, in state ω, the aggregate forecast is derived from the collection of forecasts $p_i^e(\omega) \in P^\tau(\omega)$ of informed agents and the collection of "average" price forecasts $\sum_{w=1}^{w=\Omega} \pi_i\left(w\right) p_i^e(w)$ formed by uninformed agents. The aggregate forecast is then

$$P^e\left(\omega\right) = \int_0^\alpha p_i^e(\omega) + \int_\alpha^1 \sum_{w=1}^{w=\Omega} \pi_i\left(w\right) p_i^e(w) di.$$

It follows that $P^e\left(\omega\right)$ lies in $\hat{P}^\tau(\omega) = \left[\hat{P}_{\text{inf}}^\tau(\omega), \hat{P}_{\text{sup}}^\tau(\omega)\right]$, where $\hat{P}_{\text{inf}}^\tau(\omega)$ and $\hat{P}_{\text{sup}}^\tau(\omega)$ are respectively the minimal and maximum aggregate forecast for a given aggregate distribution $\pi(\omega)$, namely:

$$\begin{aligned}
\hat{P}_{\text{inf}}^\tau(\omega) &= \alpha P_{\text{inf}}^\tau(\omega) + (1-\alpha) \sum_{w=1}^{w=\Omega} \pi\left(w\right) P_{\text{inf}}^\tau(w) \\
\hat{P}_{\text{sup}}^\tau(\omega) &= \alpha P_{\text{sup}}^\tau(\omega) + (1-\alpha) \sum_{w=1}^{w=\Omega} \pi\left(w\right) P_{\text{sup}}^\tau(w)
\end{aligned}$$

with $\pi(\omega)$ the aggregate probability defined in (31). But, contrarily to what happened in the preceding section, no agent is aware of the values $\pi\left(\omega\right)$.

Precisely, we *assume that agents have no information at all about the distribution from which the aggregate prior distribution* π *is drawn.* The more stringent property about the aggregate price forecast $P^e(\omega)$ that is known with certainty by an agent is then that $P^e(\omega)$ belongs to $\cup_\Pi \hat{P}^n(\omega)$ where Π is the set of probability distributions $(\pi(1), \ldots, \pi(\Omega))$ in the Ω-simplex. This set is actually:

$$\hat{P}^\tau(\omega) = \left[\alpha P^\tau_{\text{inf}}(\omega) + (1-\alpha)\inf_w P^\tau_{\text{inf}}(w), \alpha P^\tau_{\text{sup}}(\omega) + (1-\alpha)\sup_w P^\tau_{\text{sup}}(w)\right].$$

Therefore, taking into account this restriction, every agent understands that the price in state ω lies in the interval $R_\omega\left[\hat{P}^\tau(\omega)\right]$ (given the reaction function 18). We define $P^{\tau+1}(\omega) = R_\omega\left[\hat{P}^\tau(\omega)\right]$. Thus, at the end of step τ, it is CK that the state-ω price forecast belongs to the interval $P^{\tau+1}(\omega) = \left[P^{\tau+1}_{\text{inf}}(\omega), P^{\tau+1}_{\text{sup}}(\omega)\right]$ in state ω, where

$$P^{\tau+1}_{\text{inf}}(\omega) = \phi\left[\alpha P^\tau_{\text{inf}}(\omega) + (1-\alpha)\inf_w P^\tau_{\text{inf}}(w)\right] + \eta(\omega), \tag{32}$$

$$P^{\tau+1}_{\text{sup}}(\omega) = \phi\left[\alpha P^\tau_{\text{sup}}(\omega) + (1-\alpha)\sup_w P^\tau_{\text{sup}}(w)\right] + \eta(\omega). \tag{33}$$

The main consequence of introducing higher order uncertainty on the fundamentals into the modelling is to make the fixed points of (32) and (33) different in general, i.e. $P_{\text{inf}}(\omega) \neq P_{\text{sup}}(\omega)$, with

$$\begin{aligned} P_{\text{inf}}(\omega) &= \phi\left[\alpha P_{\text{inf}}(\omega) + (1-\alpha)\inf_w P_{\text{inf}}(w)\right] + \eta(\omega), \\ P_{\text{sup}}(\omega) &= \phi\left[\alpha P_{\text{sup}}(\omega) + (1-\alpha)\sup_w P_{\text{sup}}(w)\right] + \eta(\omega). \end{aligned}$$

It follows that stability under learning merely amounts to convergence of the Ω sequences $\left(P^\tau_{\text{inf}}(\omega), P^\tau_{\text{sup}}(\omega)\right)$ toward $(P_{\text{inf}}(\omega), P_{\text{sup}}(\omega))$ as τ becomes arbitrarily large.

Proposition 6 *The learning dynamics defined by (32) and (33) respectively converge toward $P_{\text{inf}}(\omega)$ and $P_{\text{sup}}(\omega)$ if and only if $\phi < 1$. That is, if $\phi < 1$, agents learn that the price in state ω belongs to an interval $[P_{\text{inf}}(\omega), P_{\text{sup}}(\omega)]$, $\omega = 1, \ldots, \Omega$. Otherwise, if $\phi > 1$, the learning dynamics diverges to infinity.*

Proof. The Ω equations (32) rewrite as

$$\mathbf{p}_{\inf}^{\tau+1} = \phi\left[\alpha \mathbf{I}_{\Omega} + (1-\alpha)\,\mathbf{1}_{\Omega}\right]\mathbf{p}_{\inf}^{\tau},$$

where $\mathbf{1}_{\Omega}$ stands for the $\Omega \times \Omega$ stochastic matrix whose each entry in the $\underline{\omega}$-th column is 1, where $\underline{\omega} = \arg\inf_w P_{\inf}(w)$, and any remaining entry is 0. The Ω eigenvalues of the matrix $\phi\left[\alpha \mathbf{I}_{\Omega} + (1-\alpha)\,\mathbf{1}_{\Omega}\right]$ are ϕ, $\alpha\phi$, ..., $\alpha\phi$. The same analysis applies to the Ω equations (33), which shows the result. ■

In presence of higher order uncertainty, agents may fail to learn the actual state of the world from the price though the learning dynamics is convergent. This event arises iff there are two sets $P(\omega)$ and $P(\omega')$ with a non empty intersection. Otherwise, the price always reveals the underlying state. The next result provides the condition for the price to be informationally efficient.

Proposition 7 *Let $\phi < 1$. Let us rank the scale factors in the order of increasing value, i.e. $\eta(\omega) < \eta(\omega')$ whenever $\omega < \omega'$ $(\omega, \omega' = 1, \ldots, \Omega)$. Then, the price reveals the actual state of the world if and only if*

$$\alpha > 1 - \frac{\inf_{\omega \neq \omega'} |\eta(\omega) - \eta(\omega')|}{\eta(\Omega) - \eta(1)}\left(\frac{1}{\phi} - 1\right).$$

Proof. Let $\phi < 1$. $\inf_w P_{\inf}(w) = P_{\inf}(1)$ and $\sup_w P_{\sup}(w) = P_{\sup}(\Omega)$.

$$\begin{aligned}
P_{\inf}(1) &= \frac{\eta(1)}{1-\phi}, \quad \text{and } P_{\inf}(\omega) = \frac{(1-\alpha)\,\phi}{1-\phi}\frac{\eta(1)}{1-\alpha\phi} + \frac{\eta(\omega)}{1-\alpha\phi} \text{ for } \omega > 1, \\
P_{\sup}(\Omega) &= \frac{\eta(\Omega)}{1-\phi}, \quad \text{and } P_{\sup}(\omega) = \frac{(1-\alpha)\,\phi}{1-\phi}\frac{\eta(\Omega)}{1-\alpha\phi} + \frac{\eta(\omega)}{1-\alpha\phi} \text{ for } \omega < \Omega.
\end{aligned}$$

$P_{\inf}(\omega) < P_{\inf}(\omega')$ and $P_{\sup}(\omega) < P_{\sup}(\omega')$ for $\omega < \omega'$. Thus, no two sets $P(\omega)$ and $P(\omega')$ intersect iff $P_{\sup}(\omega) - P_{\inf}(\omega') < 0$ whenever $\omega < \omega'$, or equivalently

$$\frac{\phi\,(1-\alpha)}{1-\phi}\frac{\eta(\Omega) - \eta(1)}{1-\alpha\phi} < \frac{\eta(\omega') - \eta(\omega)}{1-\alpha\phi},$$

for every $\omega < \omega'$. This inequality rewrites

$$\frac{\phi\,(1-\alpha)}{1-\phi} < \frac{\inf_{\omega,\omega'} |\eta(\omega) - \eta(\omega')|}{\eta(\Omega) - \eta(1)},$$

which leads to the condition stated in the Proposition. ■

Thus,given that the learning dynamics is convergent ($\phi < 1$), observing the price allows uninformed traders to discover the actual state of the world if the proportion of informed traders is above a certain threshold. This threshold increases in the weight forecast ϕ and decreases in the factor

$$\frac{\inf_{\omega,\omega'} |\eta(\omega) - \eta(\omega')|}{\eta(\Omega) - \eta(1)}.$$

To understand this point, notice first that the size of every interval $[P_{\inf}(\omega), P_{\sup}(\omega)]$ decreases in the proportion α of informed traders and increases in ϕ. However, these intervals $P(\omega)$ can intersect even if they are quite small. In particular, this is the case when the REE prices $p^*(\omega)$ (defined in Equation (17)) are close enough to each other because every $p^*(\omega)$ belongs to $P(\omega)$. Then, large fluctuations of REE prices (measured by $|p^*(\omega') - p^*(\omega)|$) should favor informational efficiency. Indeed, increasing the factor

$$\frac{\inf_{\omega,\omega'} |\eta(\omega) - \eta(\omega')|}{\eta(\Omega) - \eta(1)},$$

increases REE prices fluctuations.

5.2 Individual Heterogeneity

So far our analysis has been restricted to the particular case with homogeneous agents. Indeed, the actual price depends on individual price forecasts through the aggregate price forecast only (see the temporary equilibrium relation (**??**)). Then, the sensitivity $\phi(\omega)$ of the actual price depends on the state of the world ω, but not on the identity and type of agents. This section provides some insights about the expectations coordination process in presence of individual heterogeneity. For analytical simplicity, we limit heterogeneity to depend on type. An extension to a more general case will not add any insight.

Example 1 (continued)**.** *The Muth model with heterogeneity accross firms.* Firms differ according to their technologies. Each informed firm $i \in [0, \alpha]$ has a cost function $C^I(q_i) = q_i^2/2c^I$; the cost of an uninformed firm $i \in [\alpha, 1]$ is $C^U(q_i) = q_i^2/2c^U$. Thus, the aggregate production writes

$$c^I \int_0^\alpha p_i^e(\omega) di + c^U \int_\alpha^1 \sum_{w=1}^{\Omega} \pi(w) p_i^e(w) di.$$

With an unchanged aggregate demand $\delta(\omega) - \mu(\omega)p(\omega)$, the temporary equilibrium price $p(\omega)$ in this state is

$$p(\omega) = -\frac{c^I}{\mu(\omega)} \int_0^{\alpha} p_i^e(\omega) di - \frac{c^U}{\mu(\omega)} \int_{\alpha}^{1} \sum_{w=1}^{\Omega} \pi(w) p_i^e(w) di + \frac{\delta(\omega)}{\mu(\omega)}.$$

More generally, we consider the class of economic models whose reduced form is

$$p(\omega) = \phi_I(\omega) \int_0^{\alpha} p_i^e(\omega)\, di + \phi_U(\omega) \int_{\alpha}^{1} \sum_{w=1}^{\Omega} \pi(w) p_i^e(w) di + \eta(\omega), \tag{34}$$

where $\phi_I(\omega)$ and $\phi_U(\omega)$ are real parameters of the same sign. Without loss of generality, we set $\phi_I(\omega) > 0$ and $\phi_U(\omega) > 0$ in every state of the world. A REE is a Ω-dimensional price vector $(p^*(1), ..., p^*(\Omega))$ such that

$$p^*(\omega) = \phi_I(\omega)\, \alpha p^*(\omega) + \phi_U(\omega)(1-\alpha) \sum_{w=1}^{\Omega} \pi(w) p^*(w) + \eta(\omega)$$

for any ω. It corresponds to a situation where it is CK among traders that the price belief of any trader is precisely $(p^*(1), ..., p^*(\Omega))$. Unlike the previous sections, individual heterogeneity makes useless the aggregate price forecast.

Again, the coordination process bears on an Anchorage Assumption weakening of the assumption of CK of expectations. Precisely, we assume that it is CK that any trader expects the price to belong to some interval $P^0(\omega) = \left[p_{\inf}^0(\omega), p_{\sup}^0(\omega)\right]$ in the state of the world ω ($\omega = 1, \ldots, \Omega$). Then, the CK assumptions of individual rationality and model triggers an iterative process whose every step τ ($\tau \geq 0$) is as follows. The starting assumption of step τ is that it is CK that any trader expects the price to belong to some interval $P^{\tau}(\omega) = \left[p_{\inf}^{\tau}(\omega), p_{\sup}^{\tau}(\omega)\right]$ in the state of the world ω ($\omega = 1, \ldots, \Omega$). This implies that it is CK that the price in state ω belongs to an interval $P^{\tau}(\omega) = \left[P_{\inf}^{\tau}(\omega), P_{\sup}^{\tau}(\omega)\right]$, where

$$P_{\inf}^{\tau+1}(\omega) = \phi_I(\omega)\, \alpha P_{\inf}^{\tau}(\omega) + \phi_U(\omega)(1-\alpha) \sum_{w=1}^{\Omega} \pi(w) P_{\inf}^{\tau}(w) + \eta(\omega), \tag{35}$$

$$P_{\sup}^{\tau+1}(\omega) = \phi_I(\omega)\, \alpha P_{\sup}^{\tau}(\omega) + \phi_U(\omega)(1-\alpha) \sum_{w=1}^{\Omega} \pi(w) P_{\sup}^{\tau}(w) + \eta(\omega) \tag{36}$$

It is then CK that any trader expects the state ω-price to be in $P^{\tau+1}(\omega)$. This ends step τ.

The learning dynamics defined by the sequence $P^{\tau}(\omega) = \left[P^{\tau}_{\inf}(\omega), P^{\tau}_{\sup}(\omega)\right]$ admits the REE as only fixed point, i.e. $P^{\tau}_{\inf}(\omega)$ and $P^{\tau}_{\sup}(\omega)$ both converge to $p^{*}(\omega)$. The next result provides conditions for successful learning.

Proposition 8 *Assume that $\phi_I(\omega) > 0$ and $\phi_U(\omega) > 0$ for any $\omega = 1, \ldots, \Omega$. If $\alpha\phi_I(\omega) > 1$ for some ω, then the rational expectations equilibrium is unstable in the dynamics with eductive learning. If $\alpha\phi_I(\omega) < 1$ for every ω, then the rational expectations equilibrium is stable in the dynamics with eductive learning if and only if*

$$\sum_{w=1}^{\Omega} \pi(w) \frac{(1-\alpha)\phi_U(w)}{1-\alpha\phi_I(w)} < 1. \tag{37}$$

Proof. The proof is similar to the one of Proposition 1. Rewriting the two systems of Ω equations (35) and (36) in matrix form shows that the REE is stable iff the spectral radius $\rho(\mathbf{M})$ of the $\Omega \times \Omega$ matrix $\mathbf{M} = \alpha\mathbf{\Phi}^{I} + (1-\alpha)\mathbf{\Phi}^{U}\mathbf{\Pi}$ is less than 1.[10] Here $\mathbf{\Phi}^{I}$ and $\mathbf{\Phi}^{U}$ are two $\Omega \times \Omega$ diagonal matrices whose $\omega\omega$-th entries are $\phi^{I}(\omega)$ and $\phi^{U}(\omega)$, respectively. We define:

$$Q(\mathbf{x}, \omega) = \alpha\phi_I(\omega) + (1-\alpha)\phi_U(\omega)\frac{1}{x_\omega}\sum_{w=1}^{\Omega}\pi(w)\, x_w.$$

The first part of Proposition 8 follows again by appealing to the $\Omega \times 1$ vector $\mathbf{x} = (\varepsilon, ..., \varepsilon, 1)$, where $\varepsilon > 0$ is small enough. The last part follows now by appealing to the $\Omega \times 1$ vector $\mathbf{x}$ whose ω-th component x_ω is

$$\frac{1}{E}\frac{(1-\alpha)\phi_U(\omega)}{1-\alpha\phi_I(\omega)},$$

with

$$E = \sum_{w=1}^{\Omega}\pi(w)\frac{(1-\alpha)\phi_U(w)}{1-\alpha\phi_I(w)}.$$

■

[10] Again, we do not consider the case where $\mathbf{M}$ has an eigenvalue with modulus 1.

In particular, Proposition 8 implies that stability obtains when every $(1-\alpha)\,\phi_U(\omega)+\alpha\phi_I(\omega)$ is less than 1,[11] i.e. the aggregate sensitivity parameter is less than 1. This extends the property that $\sup_\omega \phi(\omega)<1$ is a sufficient condition for stability when agents are homogeneous. Analogously, stability does not obtain when every $(1-\alpha)\,\phi_U(\omega)+\alpha\phi_I(\omega)$ is greater than 1.

Informed traders play a crucial role in the coordination process, as the learning dynamics is unstable as soon as the aggregate effect of their price forecasts onto the actual price $\alpha\phi_I(\omega)$ is greater than 1, independently of the behavior of the uninformed agents. The following two corollaries analyze the effect of the presence of these agents on the learning dynamics.

Corollary 9 *Assume that $\phi_I(\omega)>0$ and $\phi_U(\omega)>0$ for any $\omega=1,\ldots,\Omega$.*

1. *If $\phi_I(\omega)<1$ for any state of the world ω, then there exists $\alpha^*<1$ such that stability obtains if and only if $\alpha>\alpha^*$. Furthermore, $\alpha^*>0$ if and only if $E\phi_U>1$.*

2. *If, on the contrary, $\phi_I(\omega)>1$ for any state of the world ω, then there exists $\alpha^*<1$ such that stability obtains if and only if $\alpha<\alpha^*$. Furthermore, $\alpha^*>0$ if and only if $E\phi_U<1$.*

Proof. In the case $\alpha\sup_\omega \phi_I(\omega)\geq 1$, the equilibrium is unstable. In the case $\alpha\sup_\omega \phi_I(\omega)<1$, the equilibrium is stable iff Condition (37) is met, i.e.

$$F(\alpha)\overset{def}{=}\sum_{w=1}^{\Omega}\pi(w)\,\frac{(1-\alpha)\,\phi_U(w)}{1-\alpha\phi_I(w)}<1.$$

$F(\cdot)$ is continuous in α and $F(0)=E\phi_U$.[12] Straightforward computations show that, if $\phi_I(\omega)<1$ for any ω, then $F'(\alpha)<0$. It follows that $F(\alpha)<1$ iff $\alpha>\alpha^*$. Since $F(1)=0$, $\alpha^*<1$. Lastly, $E\phi_U>1$ iff $\alpha^*>0$.
If, on the contrary, $\phi_I(\omega)>1$ for any ω, then $F'(\alpha)>0$. It follows that $F(\alpha)<1$ iff $\alpha<\alpha^*$. Since $F(1/\sup_\omega \phi_I(\omega))=+\infty$, $\alpha^*<1$. Again, $E\phi_U<1$ iff $\alpha^*>0$. This ends the proof. ■

[11] To see this point, rewrite $(1-\alpha)\,\phi_U(\omega)+\alpha\phi_I(\omega)<1$ as $\frac{(1-\alpha)\phi_U(w)}{1-\alpha\phi_I(w)}<1$ for $\alpha\phi_I(\omega)<1$.

[12] $E\phi_U\overset{def}{=}\sum_{w=1}^{\Omega}\pi(w)\,\phi_U(w)$.

The intuition for this corrollary is clear: small values of the sensitivity parameters $\phi_I(\omega)$ and $\phi_U(\omega)$ favor stability. Namely, many informed agents favor stability (resp. instability) when the $\phi_I(\omega)$ are less (resp. greater) than 1. If the $\phi_U(\omega)$ are small (resp. large) as well (in average), then stability always obtains (resp. never obtains). In particular, when all the $\phi_I(\omega)$ and $\phi_U(\omega)$ are less (resp. larger) than 1, the REE is stable (resp. unstable).

In the intermediate situation where some weight forecasts $\phi_I(\omega)$ are less than 1 while the remaining weights are greater than 1, the role of informed agents is more intricate, as shown in the last corollary of Proposition 8.

Corollary 10 *Let* $\inf_\omega \phi_I(\omega) < 1 < \sup_\omega \phi_I(\omega)$.

1. *If* $E\phi_U < E\phi_I\phi_U$,[13] *then there exists* $\alpha^* < 1/\sup_\omega \phi_I(\omega) < 1$ *such that stability obtains if and only if* $\alpha < \alpha^*$. $\alpha^* > 0$ *if and only if* $E\phi_U < 1$.

2. *If* $E\phi_U \geq E\phi_I\phi_U$ *and* $E\phi_U \leq 1$, *then there exists* α^*, $0 < \alpha^* < 1/\sup_\omega \phi_I(\omega) < 1$, *such that stability obtains if and only if* $\alpha < \alpha^*$.

3. *If* $E\phi_U \geq E\phi_I\phi_U$ *and* $E\phi_U > 1$, *then,* (i) *either the equilibrium is unstable for every* α, (ii) *or there are two values* α_- *and* α_+ *with* $0 < \alpha_- < \alpha_+ < 1/\sup_\omega \phi_I(\omega) < 1$ *such that the equilibrium is stable if and only if* $\alpha \in [\alpha_-, \alpha_+]$. *Precisely, consider a vector* $(\phi_I(1), ..., \phi_I(\Omega))$. *In the space* $\mathbb{R}_+^\Omega$ *of the vectors* $(\phi_U(1), ..., \phi_U(\Omega))$, *there is a neighborhood of the hyperplane* $E\phi_U = 1$ *such that case* (i) *(resp.* (ii)*) obtains when* $(\phi_U(1), ..., \phi_U(\Omega))$ *is outside (resp. inside) this neighborhood. In particular, case* (i) *obtains when* $E\phi_U > \sup_\omega \phi_I(\omega)/(\sup_\omega \phi_I(\omega) - 1)$.

Proof. We write $F'(\alpha) = Q_+ - Q_-$ where[14]

$$
\begin{aligned}
Q_+ &= \sum_{\omega/\phi_I>1} \pi\phi_U \frac{\phi_I - 1}{(1-\alpha\phi_I)^2} \geq 0, \\
Q_- &= -\sum_{\omega/\phi_I<1} \pi\phi_U \frac{\phi_I - 1}{(1-\alpha\phi_I)^2} \geq 0.
\end{aligned}
$$

Q_+ and Q_- are both continuous, increasing and convex.

[13] $E\phi_I\phi_U \stackrel{def}{=} \sum_{w=1}^{\Omega} \pi(w)\phi_I(w)\phi_U(w)$.

[14] We drop the index ω for simplicity.

In the case $E\phi_U < E\phi_U\phi_I$, (that is $F'(0) > 0$) given that $(1-\alpha x)^{-2}$ is increasing in x for every given α, we have:

$$\begin{aligned} Q_+ &\geq \sum_{s/\phi_I>1} \pi\phi_U \frac{\phi_I - 1}{(1-\alpha)^2}, \\ Q_- &\leq -\sum_{s/\phi_I<1} \pi\phi_U \frac{\phi_I - 1}{(1-\alpha)^2}. \end{aligned}$$

It follows that $F'(\alpha) \geq F'(0)/(1-\alpha)^2 > 0$, F is increasing and $F\left(\frac{1}{\max\phi_I}\right) = +\infty$ so that stability obtains iff α is below a certain threshold α^*. Given that $F(0) = E\phi_U$, $\alpha^* > 0$ iff $E\phi_U < 1$. This proves the first point in the corollary.

In the case $E\phi_U > E\phi_U\phi_I$, (that is $F'(0) \leq 0$), at a point where $Q_+ = Q_-$, we have that

$$\begin{aligned} \frac{dQ_+}{d\alpha} &\geq 2\sum_{s/\phi_I>1} \pi\phi_U \frac{1}{(1-\alpha)} \frac{\phi_I - 1}{(1-\alpha\phi_I)^2} = \frac{2Q_+}{(1-\alpha)}, \\ \frac{dQ_-}{d\alpha} &\leq -2\sum_{s/\phi_I<1} \pi\phi_U \frac{1}{(1-\alpha)} \frac{\phi_I - 1}{(1-\alpha\phi_I)^2} = \frac{2Q_-}{(1-\alpha)}, \end{aligned}$$

so that

$$\frac{dQ_-}{d\alpha} \leq \frac{2Q_-}{(1-\alpha)} = \frac{2Q_+}{(1-\alpha)} \leq \frac{dQ_+}{d\alpha},$$

i.e. Q_+ crosses Q_- from below at any intersection point. It follows that there is at most one intersection point. Notice now that

$$Q_+(0) < Q_-(0) \text{ and } Q_-\left(\frac{1}{\sup\phi_I}\right) < Q_+\left(\frac{1}{\sup\phi_I}\right) = +\infty,$$

implying that there is exactly one intersection point (denoted $\alpha_{\min} > 0$) between Q_+ and Q_-. It follows that $F(\alpha)$ is decreasing iff $\alpha \leq \alpha_{\min}$ and $F(\alpha)$ reaches a minimum at $\alpha_{\min}$. As a result, we have that, in the case $F(\alpha_{\min}) < 1$, there exists α_- and α_+ such that stability obtains iff $\alpha \in [\alpha_-, \alpha_+]$, while in the case $F(\alpha_{\min}) > 1$, stability never obtains.

To prove the second point in the corollary, notice that $\alpha_- = 0$ iff $E\phi_U < 1$. To prove the third point, notice first that, for α in $[0, 1/\sup\phi_I]$

$$\left(1 - \frac{1}{\sup\phi_I}\right) E\phi_U < (1-\alpha) E\phi_U < F(\alpha).$$

Fix a vector $(\phi_I(1), ..., \phi_I(\Omega))$. Consider a given vector $\boldsymbol{\phi}_U^1 = \left(\phi_U^1(1), ..., \phi_U^1(\Omega)\right)$ such that $E\phi_U^1 = 1$, $E\phi_U^1 \geq E\phi_U^1\phi_I$. Define $\boldsymbol{\phi}_U = \lambda\boldsymbol{\phi}_U^1$ with $\lambda \geq 1$, and denote $F_\lambda = \lambda F_1$. The value $\alpha_{\min}$ such that $F_\lambda'(\alpha_{\min}) = 0$ does not depend on λ. $F_1(\alpha) < 1$ in a non empty interval. As $F_\lambda(\alpha)$ increases in λ and stability writes $F_\lambda(\alpha) < 1$, there is a value $\lambda_{\max}\left(\boldsymbol{\phi}_U^1\right)$ such that $F_\lambda(\alpha) < 1$ for some α iff $\lambda < \lambda_{\max}\left(\boldsymbol{\phi}_U^1\right)$. Consider now the set $I = \left\{\boldsymbol{\phi}_U / E\phi_U < \lambda_{\max}\left(\frac{1}{E\phi_U}\boldsymbol{\phi}_U\right)\right\}$. This a neighborhood of the hyperplane $E\phi_U = 1$ satisfying the third point. ■

All the results follow from combining two factors: stability is favored by (i) small forecast weights $\phi_U(\omega)$ and $\phi_I(\omega)$ and (ii) a small number α of informed agents. When the forecast weights $\phi_U(\omega)$ of the uninformed are small, either in the absolute sense ($E\phi_U < 1$) or relatively to the weights of informed agents ($E\phi_U < E\phi_I\phi_U$), then the two above factors go in the same direction and the result is clear: the equilibrium is stable iff the proportion $1-\alpha$ of uninformed traders is large enough. Otherwise, the forecast weights $\phi_I(\omega)$ of the informed are small relatively to the $\phi_U(\omega)$, and the above factor (i) creates a stabilizing effect of a large proportion of informed agents. Thus, this stabilizing effect tends to oppose to the destabilizing effect of informed traders (due to factor (ii)). In particular, if the $\phi_U(\omega)$ are not too large (point 3 in the corollary), the stabilizing effect of informed traders can overcome the destabilizing effect: namely, a small α (a large proportion of uninformed agents) makes REE unstable, while it is stable for intermediate values of α ($\alpha \in [\alpha_-, \alpha_+]$). Still, for large values of α, the destabilizing effect (factor (ii)) always dominates and REE is unstable (but this is not true anylonger when $\sup_\omega \phi_I(\omega) < 1$ as shown in the preceding corollary). Lastly, if the $\phi_U(\omega)$ are quite large and the $\phi_I(\omega)$ are not small enough to make the stabilizing effect of a large α dominant, the REE is never stable (points 1 and 2 when $E\phi_U > 1$ and point 3 when $E\phi_U$ is large enough).

5.3 A nonlinear framework

We now analyze how the eductive learning analysis extends to a nonlinear framework. In such a case, it turns out that the forecast weights of informed and uninformed agents differ in the temporary equilibrium relation. Namely, we consider the following temporary equilibrium relation determining the actual state ω-price $p(\omega)$:

$$p(\omega) = F_\omega \left(\int_0^\alpha \varphi_I(p_i^e(\omega))di + \int_\alpha^1 \varphi_U((p_i^e(1), \dots, p_i^e(\Omega)))di \right). \tag{38}$$

$p(\omega)$ depends on the underlying state of economic fundamentals ω, and on an aggregate of individuals decisions $\varphi_I(\cdot)$ and $\varphi_U(\cdot)$. The decision $\varphi_I(\cdot)$ of an informed agent i depends on the price $p_i^e(\omega)$ expected in state ω, whereas the decision $\varphi_U(\cdot)$ of an uninformed agent depends on the Ω expected prices $(p_i^e(1), \dots, p_i^e(\Omega))$. This relation (38) assumes that all the informed agents are identical, and all the uninformed agents are identical, in the sense that $\varphi_I(\cdot)$ and $\varphi_U(\cdot)$ do not depend on the identity of the agent. Thus, this framework encompasses most usual models with homogeneous agents where the price is determined by agents' expectations of its value (see the examples below).

Example 1 (continued). *A nonlinear cobweb model.*
The cost function of firm i is $c(q_i)$, where $c'(\cdot) \geq 0$ and $c''(\cdot) > 0$. In state ω, an informed firm expects the price $p_i^e(\omega)$ to clear the market in the next period, and so produces $q_i = (c')^{-1}(p_i^e(\omega))$. An uninformed firm produces q_i such that

$$q_i = (c')^{-1} \left(\sum_{w=1}^{\Omega} \pi(w) p_i^e(w) \right).$$

Let the aggregate demand function in state ω be $D_\omega(p)$, where $D'_\omega(\cdot) > 0$. The temporary equilibrium price $p(\omega)$ satisfies

$$p(\omega) = D_\omega^{-1} \left(\int_0^\alpha (c')^{-1} (p_i^e(\omega))di + \int_\alpha^1 (c')^{-1} \left(\sum_{w=1}^{\Omega} \pi(w) p_i^e(w) \right) di \right), \tag{39}$$

which fits (38).

Example 3. *A market of a risky asset.*
Consider the market of a risky asset with future value v that is normally distributed with mean $E(\omega)$ and variance σ^2 in every state ω $(\omega = 1, \dots, \Omega)$. Each agent i sends a buy order x_i to the market *before* he observes the price p of the asset (and the future value v). Individual demand x_i is assumed to maximize an Von Neumann Morgenstern expected utility $E[u((v-p)x_i)]$ of the net gain $(v-p)x_i$ (where u satisfy standard assumptions, i.e. $u' > 0 > u''$). In state ω, an informed agent who observes ω faces some residual

uncertainty that is inversely related to the precision $1/\sigma^2$ and he sends a buy order $x_i = x_I(\omega, p_i^e(\omega))$, where x_I is the solution of the program

$$\max E\left[u\left(\left(v - p_i^e(\omega)\right)x_i\right)\right].$$

The order of an uninformed agent $i \in [\alpha, 1]$ is $x_i = x_U(p_i^e(1), ..., p_i^e(\Omega))$, where x_U is the solution of the program

$$\max \sum_{\omega=1}^{\Omega} \pi(\omega) E\left[u\left(\left(v - p_i^e(\omega)\right)x_i\right)|\omega\right].$$

Let the aggregate supply of the asset be $S(\omega, p)$ $(S' > 0)$ at price p. Hence, the actual (market clearing) price $p(\omega)$ in state ω is such that

$$S(p(\omega)) = \int_0^{\alpha} x_I(\omega, p_i^e(\omega))\, di + \int_{\alpha}^{1} x_U(p_i^e(1), ..., p_i^e(\Omega))\, di, \tag{40}$$

which fits (38).

In this non linear framework, a REE is a Ω-dimensional price vector $\mathbf{p}^* = (p^*(1), ..., p^*(\Omega))$ such that

$$p^*(\omega) = F_\omega\left[\alpha\varphi_I(p^*(s)) + (1-\alpha)\varphi_U(\mathbf{p}^*)\right], \tag{41}$$

whatever $\omega = 1, \ldots, \Omega$ is. From now on, assume existence of a REE. Let $\varphi_\omega^* = \alpha\varphi_I(p^*(\omega)) + (1-\alpha)\varphi_U(\mathbf{p}^*)$ stand for the REE aggregate decision in state ω. Let $dp(\omega) = p(\omega) - p^*(\omega)$, and $dp_i^e(\omega) = p_i^e(\omega) - p^*(\omega)$ for $i \in [0,1]$ and $\omega = 1, \ldots, \Omega$. Then, in an arbitrarily small neighborhood of the REE, the temporary equilibrium relation can be approximated as

$$dp(\omega) = F_\omega'(\varphi_\omega^*)\frac{d\varphi_I}{dp_i^e(\omega)}(p^*(\omega))\int_0^{\alpha} dp_i^e(\omega)di + F_\omega'(\varphi_\omega^*)\int_{\alpha}^{1}\sum_{w=1}^{\Omega}\frac{d\varphi_U}{dp_i^e(w)}(\mathbf{p}^*)dp_i^e(w)di,$$

whatever ω is. Since this relation fits (34), all the results of Section 5.2 apply locally, in particular the stability result given in Proposition 8, provided that the suitable sign restrictions are satisfied.[15]

[15] It must be the case that

$$\frac{d\varphi_I}{dp_i^e(s)}(p^*(s))\frac{d\varphi_U}{dp_i^e(s')}(\mathbf{p}^*) > 0$$

for any $s, s' = 1, \ldots, S$.

Nevertheless, unlike in the linear framework examined in Section 5.2, the local forecast weights rely on the equilibrium prices $(p^*(1), ..., p^*(\Omega))$. Thus, Proposition 8 and other results in Section 5.2 must be apply carefully. For instance, these results do not allow us to assert that local stability is favored by introducing uninformed traders into the economy, as changing the value of α affects the REE prices and then the values of the parameters involved in the stability conditions (the $\phi_I(\omega)$ and the $\phi_U(\omega)$ in Section 5.2). Still, there are examples where taking account of the variation of the REE prices does not affect the spirit of our results. In particular, in the above Example 3 in the case of a mean-variance utility, it is easy to check that stability of the REE obtains iff the proportion α of informed agents is below a certain threshold (by computing explicitly the temporary equilibrium relation and the REE).

In the linear framework, the stability of the fundamental REE is necessary and sufficient for the price system to be efficient, i.e. for the actual price to reveal the underlying state of economic fundamentals. This is no longer the case in a nonlinear framework. Indeed, as the following example shows, local instability of the fundamental solution does not prevent efficiency in the price system. Indeed, there exist then sunspot equilibria locally stable in the dynamics with learning. Such sunspot equilibria allow full revelation the underlying economic state of the world to traders; that is, extraneous uncertainty ensures efficiency in the price system.

Example 1 (continued). *Sunspots in the nonlinear cobweb model.*
As in Section 3.3 we shall assume that firm i receives a private signal $\sigma_i = 1, \ldots, S$ imperfectly correlated with the actual sunspot event $s = 1, \ldots, S$ and independent of the underlying state of economic fundamentals $\omega = 1, \ldots, \Omega$. In the state of the world (ω, s), there are $\alpha \Pr(\sigma \mid s)$ informed firms whose receive a signal σ and thus the the aggregate supply of informed firms is

$$\alpha \sum_{\sigma=1}^{S} \Pr(\sigma \mid s) \phi \left[\sum_{\Sigma=1}^{S} \Pr(\Sigma \mid \sigma)\, p^e(\omega, \Sigma) \right]$$

whenever price beliefs are homogeneous accross traders. By the same way the aggregate supply of uninformed firms is

$$(1-\alpha) \sum_{\sigma=1}^{S} \Pr(\sigma \mid s) \phi \left[\sum_{w=1}^{\Omega} \pi(w) \sum_{\Sigma=1}^{S} \Pr(\Sigma \mid \sigma)\, p^e(w, \Sigma) \right].$$

A REE is a set of ΩS prices such that, in state (ω, s), the aggregate demand

$$\delta\left(p(\omega, \sigma)\right) + W(\omega)$$

equals the aggregate supply

$$\begin{aligned} &\alpha \sum_{\sigma=1}^{S} \Pr(\sigma \mid s) \phi \left[\sum_{\Sigma=1}^{S} \Pr\left(\Sigma \mid \sigma\right) p(\omega, \Sigma) \right] \\ &+ (1-\alpha) \sum_{\sigma=1}^{S} \Pr(\sigma \mid s) \phi \left[\sum_{w=1}^{\Omega} \pi(w) \sum_{\Sigma=1}^{S} \Pr\left(\Sigma \mid \sigma\right) p(w, \Sigma) \right], \end{aligned}$$

and price beliefs are self-fulfilling, i.e. $p^e(\omega, s) = p(\omega, s)$ whatever (ω, s) is. The fundamental solution is such that $p(\omega, s) = p(\omega, s')$ for any ω and any s, s' $(s \neq s')$. In a sunspot equilibrium, $p(\omega, s) \neq p(\omega, s')$ for some ω and some s, s' $(s \neq s')$.

Proposition 11 *A sunspot equilibrium exists if* $(-1)^{\Omega S} \det\left(\mathbf{M}(\mathbf{p}^*) \otimes \mathbf{S} - \mathbf{I}_{\Omega S}\right) < 0$.

Proof. Define

$$F(\mathbf{p}; (\omega, S)) = \delta^{-1}\left(\Phi(\mathbf{p}; (\omega, S)) - \delta(\omega)\right) - p(\omega, S).$$

Consider the vector field $F(\mathbf{p})$ whose (ω, S)-th dimension is $F(\mathbf{p}; (\omega, S))$. Note that rational expectations equilibria coïncide with zeros of $F(\cdot)$. Since this vector field points inward at the boudaries of $IR_+^{\Omega S}$, the sum of the indices of the equilibria is equal to $(+1)$. The index of the fundamental solution is equal to (-1) if

$$(-1)^{\Omega S} \det\left(\mathbf{DF}(\mathbf{p}^*) - \mathbf{I}_{\Omega S}\right) < 0,$$

where $\mathbf{DF}(\mathbf{p}^*)$ is the Jacobian matrix of $F(\cdot)$ at the fundamental solution. Otherwise the index of this equilibrium equals $(+1)$. It is straighforward to show that $\mathbf{DF}(\mathbf{p}^*) = \mathbf{M}(\mathbf{p}^*) \otimes \mathbf{S}$, where

$$\begin{aligned} \mathbf{M}(\mathbf{p}^*) = {} & \alpha \begin{pmatrix} \phi'\left(p(1)\right)/\delta(p(1)) & & 0 \\ & \ddots & \\ 0 & & \phi'\left(p(\Omega)\right)/\delta(p(\Omega)) \end{pmatrix} \\ & +(1-\alpha) \begin{pmatrix} \pi(1) & \cdots & \pi(\Omega) \\ \vdots & \ddots & \vdots \\ \pi(1) & \cdots & \pi(\Omega) \end{pmatrix} \begin{pmatrix} \phi'\left(E_w p(w)\right)/\delta(p(1)) & & 0 \\ & \ddots & \\ 0 & & \phi'\left(E_w p(w)\right)/\delta(p(\Omega)) \end{pmatrix} \end{aligned}$$

and

$$\mathbf{S} = \begin{pmatrix} \mu(1,1) & \cdots & \mu(S,1) \\ \vdots & & \vdots \\ \mu(1,S) & \cdots & \mu(S,S) \end{pmatrix}.$$

The result follows. ■

Suppose now that it is CK that the actual price $p(\omega, s)$ in state (ω, s) belongs to some interval $P^0(\omega, s) = \left[p^0_{\text{inf}}(\omega, s), p^0_{\text{sup}}(\omega, s)\right]$ for any (ω, s). Suppose also that individual rationality and the structure of the conomy are both CK. The anchorage assumption triggers a learning process whose step $\tau + 1$ ($\tau \geq 0$) is as follows. First, each firm knows that all the other firms choose price forecasts in $P^\tau(\omega, s)$ in state (ω, s). Hence, every firm knows that the aggregate supply belongs to the interval $\left[\Phi^\tau_{\text{inf}}(\omega, s), \Phi^\tau_{\text{sup}}(\omega, s)\right]$ in state (ω, S). Since $\phi'(\cdot) > 0$,

$$\begin{aligned} \Phi^\tau_{\text{inf}}(\omega, S) &= \Phi\left((\mathbf{p}^\tau_{\text{inf}}; (\omega, S)\right), \\ \Phi^\tau_{\text{sup}}(\omega, S) &= \Phi\Big((\mathbf{p}^\tau_{\text{sup}}; (\omega, S)\Big). \end{aligned}$$

Since $\delta'(\cdot) < 0$, every firm knows that $p(\omega, S) \in P^1(\omega, S)$ for any pair (ω, S), where

$$\begin{aligned} p^{\tau+1}_{\text{inf}}(\omega, S) &= \delta^{-1}\left[\Phi^\tau_{\text{sup}}(\omega, S) - \eta(\omega)\right], \\ p^{\tau+1}_{\text{sup}}(\omega, S) &= \delta^{-1}\left[\Phi^\tau_{\text{inf}}(\omega, S) - \eta(\omega)\right]. \end{aligned}$$

The matrix which governs the dynamics with learning in a neighborhood of the sunspot equilibrium is block diagonal. Stability of the fundamental solution obtains if and only all the eigenvalues φ_j ($j = 1, \ldots, \Omega S$) of $\mathbf{DF}(\mathbf{p}^*)$ have moduli less than 1. Thus, if sunspot equilibria are detected by the Poincaré-Hopf theorem, then the fundamental solution is locally unstable in the dynamics with learning. Indeed, local stability requires that $\varphi_j > -1$, so that $1 - \varphi_j > 0$, which implies that

$$(-1)^{\Omega S} \det\left(\mathbf{DF}(\mathbf{p}^*) - \mathbf{I}_{\Omega S}\right) = (-1)^{\Omega S} \prod_{j=1}^{\Omega S} \left(\varphi_j - 1\right) = \prod_{j=1}^{\Omega S} \left(1 - \varphi_j\right) > 0.$$

Proposition 12 *If some sunspot equilibria are to be detected by the index theorem, then some of them are locally stable in the dynamics with learning.*

Proof. To be proven. ■

If there are sunspot equilibria, then the fundamental solution is locally stable under learning. Thus, traders should not succeed to coordinat their forecasts on this equilibrium. Price do not transmit valuable information. But still, information revealed though sunspots.

6 Conclusion

We have discussed two questions concerning the REE under asymmetric information: the coordination of expectations and the informational efficiency of the price. Our framework encompasses simple versions of some standard macroeconomic models (in particular aggregate supply/aggregate demand model). In this latter model, a standard goal of economic policy is to stabilize *equilibrium* price fluctuations. Our results show that this kind of stabilization policy may be detrimental to alternative stabilizing considerations (either coordination problem or informational efficiency of the price). Further work should precisely describe these aspects in a complete macroeconomic model. It could then contrast these preliminary results with the recent learning literature considering monetary policy (see Evans Honkapohja (2002) for an example).

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