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Which continuous-time model is most appropriate for exchange rates?\textsuperscript{*}

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Abstract

This paper determines the most appropriate ways to model diffusion and jump features of high-frequency exchange rates in the presence of intraday periodicity in volatility. We show that periodic volatility prevents conventional tests from accurately identifying the frequency and location of jumps. We propose a two-stage correction for periodicity that improves the properties of the test statistics. The most plausible model for 1-minute exchange rate data features Brownian motion and finite activity jumps but not infinite activity jumps. Brownian motion and Poisson jumps account for 85\% and 15\% of the quadratic variation, respectively, and these proportions appear to be stable through time. The low proportion of jump variation is consistent with the high liquidity and high reliance on public information of currency markets compared to stock markets. The empirical results also indicate that microstructure noise biases but does not unduly impair the statistical tests for jumps and diffusion behavior at the 1-minute frequency.

Keywords: Exchange rates, Brownian motion, Volatility, Jumps, Intraday periodicity, High-frequency data

\textit{JEL:} C15, F31, G01

1. Introduction

What statistical model best describes the evolution of asset prices? Bachelier (1900) provided a very early attempt to answer this question when he modeled stock returns with “Brownian motion.” More recently, the seminal work of Merton (1976) turned researchers’ attention to modeling asset price jumps. The efficient markets hypothesis implies that asset returns exhibit limited predictability but that asset prices react rapidly to news surprises to prevent risk-adjusted profit opportunities. Thus, asset prices are likely to exhibit both continuous changes (diffusion) and discontinuous responses (jumps) to news. Decomposing returns into jumps and a diffusion with time-varying volatility is important because these two components imply different modeling and hedging strategies (Bollerslev and Todorov, 2011a,b). For example, although persistent time-varying diffusion volatility would help forecast future volatility,
jumps might contain no predictive information about volatility or even distort volatility forecasts (Neely, 1999; Andersen et al., 2007a). Therefore, one must jointly model jumps and volatility to better explain asset price dynamics.

Our paper investigates which continuous-time model best describes intraday exchange rate fluctuations. We address the following questions: Does an appropriate model need Brownian motion or infinite activity jumps? Are jumps present in the exchange rates? If so, do exchange rate jumps have finite or infinite activity? If jumps are present, how frequent are they? What fraction of the quadratic variation in foreign exchange returns is due to Brownian motion, big and small jumps, respectively?

The first class of models that we consider is the Brownian Semi-Martingale with Finite-Activity Jumps (BSMFAJ) class. This set of models incorporates a diffusion—Brownian—component to capture the continuous variation of the price process and a jump component to account for price discontinuities. These models exhibit finite-activity jump intensity—a finite number of jumps in any time interval. The compound Poisson process, which exhibits relatively rare and large jumps, is one example of this class. Andersen et al. (2007b) cite several authors who argue that the BSMFAJ class realistically models many asset prices.

BSMFAJ models may inappropriately restrict the jump dynamics of some asset prices, however. In particular, BSMFAJ models allow only relatively rare and large jumps, despite evidence of many small jumps in equities (Aït-Sahalia and Jacod, 2011; Lee and Hannig, 2010). Therefore, Madan et al. (1998) and Carr et al. (2002) have introduced a new class of models with infinite-activity Lévy processes for jumps: the Brownian Semi-Martingale with Infinite-Activity Jumps (BSMIAJ) class (Cont and Tankov, 2004). The diffusion component of a BSMIAJ model captures the continuous price variation; and the jump component captures both rare and large jumps—potentially caused by important macro news (Dungey and Hvozdyk, 2012; Dungey et al., 2009; Lahaye et al., 2011)—as well as smaller, more frequent discontinuities that create risk for high-frequency trading strategies (Aït-Sahalia and Jacod, 2012). Jumps can arrive infinitely often in BSMIAJ models, which might better describe the data than the combination of Brownian motion and Poisson jumps (see Geman, 2002; Li et al., 2012, 2008).

The most appropriate model has important economic implications. For instance, a Brownian Semi-Martingale (BSM) model—with only continuous components—generates different risk measures (and premia) than BSMFAJ or BSMIAJ models (Bollerslev and Todorov, 2011a,b; Drechsler and Yaron, 2011). Consumption-based asset pricing models (C-CAPM) rely on the continuous part of the returns and hence characterize only the diffusion risk (see Back, 1991). As Mahen et al. (2013) show, jumps significantly contribute to the equity risk premium.

Few papers have addressed related issues in exchange rate applications. Todorov and Tauchen (2010) and Cont and Mancini (2011) are most closely related to our paper; these works have also studied the characteristics of foreign-exchange-data-generating processes, focusing on jump activity and jump variation, respectively. Specifically, both papers use the Blumenthal-Getoor index to estimate jump

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1Semimartingale models are quite useful in financial economics and continuous-time finance because they rule out arbitrage opportunities (see Back, 1991).


3It is worth noting that BSMIAJ models can exclude the finite-activity jump component so that only the infinite-activity jump component represents the price discontinuities.

4See also Lustig and Verdelhan (2007), who present empirical evidence on the link between aggregate consumption growth and exchange rate dynamics.
activity and thus to argue that Brownian motion is present in foreign exchange data. We confirm this with power variation measures. We also confirm Cont and Mancini (2011)’s finding of finite jump intensity.

Our work extends and complements these studies in several ways: Methodologically, this paper characterizes the properties of foreign exchange returns with the flexible testing methodology of Aït-Sahalia and Jacod (2012), coupled with a 10-year sample of 1-minute data on three exchange rates that provides better test properties than the 5-minute data used in Todorov and Tauchen (2010) and Cont and Mancini (2011). Our important methodological innovation, however, is to explicitly account for intraday periodicity in volatility. This allows us to identify the correct set of jumps. More specifically, an empirical analysis that fails to account for such periodic patterns incorrectly estimates the location, size and frequency of jumps.

These methods yield new insights into the foreign-exchange-data-generating process. For example, Todorov and Tauchen (2010) conclude that the Brownian-plus-Poisson-jumps model might be misspecified, although they stress that this inference depends on the sampling frequency and testing technique. Using higher frequency data and better tests, we show that a Brownian plus compound Poisson jumps model is indeed the most plausible model. While Brownian motion drives 85% of exchange rate variation, jumps contribute significantly (15%). We also reject the hypothesis that microstructure noise renders the test statistics uninformative, although the noise does bias their empirical values.

We organize our paper as follows: Section 2 presents the methodology. Section 3 extends this technique to account for intraday periodicity. In Section 4, we describe the exchange rate data and report our empirical results. Section 5 concludes.

2. The base methodology

2.1. Theoretical background

In line with previous literature, we assume that the log-price $X(t)$ follows a semimartingale such that

$$dX(t) = \mu(t)dt + \sigma(t)dW(t) + \text{JUMPS}(t),$$

where $dX(t)$ denotes the logarithmic price increment, $\mu(t)$ is a continuous, locally bounded, variation process, $\sigma(t)$ is a strictly positive and càdlàg (right-continuous with left limits) stochastic volatility process, and $W(t)$ is a standard Brownian motion. The JUMPS component of (1) potentially represents both finite- and infinite-activity jumps. That is,

$$\text{JUMPS}(t) := \kappa(t) dq(t) + h(t) dL(t),$$

where $q(t)$ denotes a counting process (e.g., Poisson process), $L(t)$ is a pure Lévy jump process (e.g., a Cauchy process), and $\kappa(t)$ and $h(t)$ denote the jump sizes of the counting and Lévy processes, respectively. In the absence of the JUMPS component, (1) is a BSM model.

Assumptions and notation. We assume that the log-price process $X(t)$ in (1) is observed at discrete points in time. In the absence of noise, the continuously compounded $i$th intraday return of a trading day $t$ is given by $r_{t,i} = X(t+i\Delta) - X(t+(i-1)\Delta)$, with $i = 1, \ldots, M$ and trading days $t = 1, \ldots, T$. Let
\(M \equiv \lceil 1/\Delta \rceil\) denote the number of intraday observations over the day; \(\Delta = 1/M\) is the time between consecutive observations, the inverse of the observation frequency. Note that in the presence of noise, we observe \(X(t)\) with an error. That is,

\[Z(t) = X(t) + e(t),\]

where \(e(t)\) is the additive noise term as in A"ıt-Sahalia et al. (2012). Thus, the \(i\)th intraday return of a trading day \(t\)—in the presence of noise—is now given by

\[r_{t,i} \equiv Z(t+i\Delta) - Z(t+(i-1)\Delta).\]

Some of the test statistics used in this paper are functions of “truncated power variations,” a class of statistics that depend on three parameters: \(p\), the power exponent, \(u\), the truncation parameter, and \(k\), the sampling frequency parameter. A"ıt-Sahalia and Jacod (2010) show that one can infer the properties of the data generating process from the probability limits of truncated power variations under various combinations of these three parameters. We now define the power variation statistics introduced in A"ıt-Sahalia and Jacod (2010).

**Definitions.** Denote the truncated power variation as \(\hat{B}(p, u, \Delta)_t\). That is,

\[\hat{B}(p, u, \Delta)_t := \frac{1}{\Delta} \sum_{i=1}^{1/\Delta} |r_{t,i}|^p 1_{\{|r_{t,i}| \leq u\}}, \tag{3}\]

where \(u = \alpha \Delta^\omega\) is the truncation threshold and \(\alpha > 0\) is expressed in units of standard deviations of the continuous part of the process for a constant \(\omega \in (0, 1/2)\). The truncation function on the right-hand side of (3) can be used to exclude large changes (i.e., jumps) from the calculation, which allows us to calculate moments of the diffusion process while excluding jumps. But \(\hat{B}(p, \infty, \Delta)_t\) takes into account the contribution of jumps and so can be used to calculate moments of the full return process. We can also define reverse truncation to compute moments of large returns (i.e., jumps), excluding small returns.\(^5\) That is,

\[\hat{U}(p, u, \Delta)_t = \sum_{i=1}^{1/\Delta} |r_{t,i}|^p 1_{\{|r_{t,i}| > u\}}. \tag{4}\]

To count the number of increments larger than \(u\), we fix \(p = 0\), for example. Then, (4) becomes

\[\hat{U}(0, u, \Delta)_t = \sum_{i=1}^{1/\Delta} 1_{\{|r_{t,i}| > u\}}. \tag{5}\]

In summary, these truncated power variation functions compute statistics of diffusion returns and “large” returns (i.e., jumps) that allow us to test the hypotheses described in the next section.

2.2. Hypotheses and test statistics

This subsection follows A"ıt-Sahalia and Jacod (2012) in presenting the statistics to test four hypotheses:

\[
\text{(A)} \quad H_0: \text{Brownian motion is present} \quad \text{vs.} \quad H_1: \text{Brownian motion is not present}
\]

\(^5\)We distinguish jumps, which are detected by our jump tests, from large intraday returns, which might be informally considered jumps.
We test these hypotheses by comparing the distributions of the following statistics, calculated each day in the sample, with their probability limits under the alternative hypotheses.

- To test for Brownian motion:
  \[
  \hat{S}_W(p, u, k, \Delta)_t = \frac{\hat{B}(p, u, \Delta)_t}{\hat{B}(p, u, k\Delta)_t} \xrightarrow{p} \begin{cases} 
  k^{1-p/2}, & X \text{ has Brownian motion on } [0, t] \\
  1, & X \text{ has no Brownian motion on } [0, t] \\
  k, & \text{noise dominates on } [0, t].
  \end{cases} \tag{6}
  \]

- To test for jumps (ASJ test):
  \[
  \hat{S}_J(p, k, \Delta)_t = \frac{\hat{B}(p, \infty, k\Delta)_t}{\hat{B}(p, \infty, \Delta)_t} \xrightarrow{p} \begin{cases} 
  1, & X \text{ has jumps on } [0, t] \\
  k^{p/2-1}, & X \text{ is continuous on } [0, t] \\
  1/k, & \text{noise dominates on } [0, t].
  \end{cases} \tag{7}
  \]

- To test for finite-activity jumps:
  \[
  \hat{S}_{FA}(p, u, k, \Delta)_t = \frac{\hat{B}(p, u, k\Delta)_t}{\hat{B}(p, u, \Delta)_t} \xrightarrow{p} \begin{cases} 
  k^{p/2-1}, & X \text{ has finitely many jumps on } [0, t] \\
  1, & X \text{ has infinitely many jumps on } [0, t] \\
  1/k, & \text{noise dominates on } [0, t].
  \end{cases} \tag{8}
  \]

- To test for infinite-activity jumps:
  \[
  \hat{S}_{IA}(p, p', u, \gamma, \Delta)_t = \frac{\hat{B}(p', \gamma u, \Delta)_t \hat{B}(p, u, \Delta)_t}{\hat{B}(p', u, \Delta)_t \hat{B}(p, \gamma u, \Delta)_t} \xrightarrow{p} \begin{cases} 
  \gamma^{p'-p}, & X \text{ has infinitely many jumps on } [0, t] \\
  1, & X \text{ has finitely many jumps on } [0, t],
  \end{cases} \tag{9}
  \]

where \( \gamma > 1 \) and \( p' \) is another parameter for power variation such that \( p' > p > 2. \) Note that while (6) requires \( p < 2, \) the test statistic in (7) identifies the presence of jumps for \( p > 2. \) In addition to investigating the hypotheses (A)-(D), we check whether market microstructure noise affects the results. Specifically, if microstructure noise renders the test statistics in (6) to (8) uninformative, then \( \hat{S}_W, \hat{S}_J, \) and \( \hat{S}_{FA} \) will converge to \( k, 1/k, \) and \( 1/k, \) respectively.\(^6\)

The test statistics in (8) and (9) identify whether jumps most likely exhibit finite or infinite activity, but these statistics do not directly estimate the degree of jump intensity. We thus estimate the degree of activity of jumps with the Blumenthal-Getoor index \( \beta.\)\(^7\) Table 1 shows the activity level of various stochastic processes.

\[ \text{[ Insert Table 1 about here] } \]

\(^{6}\)Aït-Sahalia and Jacod (2012) state that \( \hat{S}_{FA} \) is likely to be more robust than \( \hat{S}_{IA}, \) due to its simpler construction.\(^7\) Other approaches to measure jump activity include the works of Todorov and Tauchen (2010), Belomestny (2010), and Carr et al. (2002), among others. For the technical definition and details of the Blumenthal-Getoor index, we direct the reader to pages 2203–2204 of Aït-Sahalia and Jacod (2009a).
The parameter $\beta$ measures the frequency of jump activity on a scale from 0 to 2. Activity that occurs finitely often—such as Poisson jumps—has an activity level of $\beta = 0$. Activity that occurs infinitely has a positive $\beta$. For example, Cauchy and Normal Inverse Gaussian (NIG) jumps imply that $\beta = 1$; Brownian motion is extremely active with $\beta = 2$. We use two $\beta$ estimators proposed by Aït-Sahalia and Jacod (2009a). That is,

$$\hat{\beta}(\omega, \alpha, \alpha', u, u')_t = \frac{\log(\hat{U}(0, u, \Delta)_t/\hat{U}(0, u', \Delta)_t)}{\log(\alpha'/\alpha)}, \quad (10)$$

where $u' = \alpha'\Delta\omega$ is another truncation parameter for the fixed values $0 < \alpha < \alpha'$. The first estimator, given by (10) is based on computing reverse truncated power variation (5) at two truncation levels, $\alpha$ and $\alpha'$. The second estimator, given by

$$\hat{\beta}'(\omega, \alpha, k, u)_t = \frac{\log(\hat{U}(0, u, \Delta)_t/\hat{U}(0, u, k\Delta)_t)}{\omega\log k}, \quad (11)$$

is based on sampling at two time scales $\Delta$ and $k\Delta$ for $k = 2$. Truncation is necessary to eliminate the diffusion component so that the test can focus on jump activity.

3. An extension: the role of intraday periodicity

The previous section presents the tools to identify the most appropriate continuous-time model for exchange rates. This section extends this methodology to account for intraday periodicity. To do so, we first illustrate the presence of periodicity in the real intraday data, then we use simulations to establish that intraday periodicity in volatility distorts the properties of test statistics. Finally, we propose modifying the truncated power variation statistic and the ASJ jump test statistic—in (7)—to correct them for the influence of intraday periodicity.

3.1. Motivation

As Baillie and Bollerslev (1991) and Andersen and Bollerslev (1997) document, foreign exchange volatility shows strong intraday periodic effects caused by regular trading patterns, such as openings and closings of the three major markets: Asia, Europe and North America. Figure 1 displays distinct U-shaped patterns in the autocorrelation functions (ACFs) for the 1-minute absolute returns.9

[ Insert Figure 1 about here ]

Figure 2 illustrates the periodicity by showing mean absolute EUR/USD, USD/JPY and GBP/USD returns over the (1440) 1-minute intervals.

[ Insert Figure 2 about here ]

The figure indicates that volatility is low during the Far East market hours, from 16:00 EST (21:00 GMT) to 24:00 EST (05:00 GMT), but activity picks up as Europe begins to trade around 2:00 EST (7:00 GMT), and Far East market activity begins to wane. The most active period of the day is during

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8 Gamma and variance-gamma (VG) processes are exceptions.
9 We describe the data in Section 4.1.
the overlap of the European and North American markets, between 7:00 EST and 11:00 EST. Volatility then starts to decline, as first the European and then U.S. markets wind down. Asian markets reopen at around 16:00 EST (21:00 GMT).\textsuperscript{10}

Previous studies have shown that periodicity matters (i) for estimating and forecasting intraday volatility (e.g., \textcite{andersen1998b}), (ii) for studying the impact of news on volatility (e.g., \textcite{dominguez2006, hautsch2011}), (iii) for estimating co-volatility (\textcite{boudt2011a}), and (iv) for detecting intraday jumps (\textcite{boudt2011b}). This paper shows that periodic volatility is also important for continuous-time model selection.

3.2. The impact of periodicity on the realized power variation

The truncation parameter ($u$) in the power variation statistics determines the cutoff between diffusion returns and “large” returns—i.e., jumps—and so should depend on the conditional diffusion volatility. In other words, jumps are defined as returns that are too large to have come from a diffusion process. If diffusion variance changes over time, one must model that variation to recognize jumps. When conditional volatility is high (low), the cutoff between diffusion returns and jumps should be larger (smaller). The test statistics in (6)–(9) assume that the appropriate truncation threshold is constant over the trading day, however. This section shows that intraday periodicity in volatility distorts the properties of the test statistics and proposes a correction.

To demonstrate the impact of periodic volatility on the test statistics, as well as the efficacy of our correction, we generate simulated data under three processes. Appendix A details the simulation setup. In Case I, volatility is only stochastic but there is no periodic component. In Case II, volatility is both stochastic and periodic. We hold everything else (such as data-generating process and parameter values) constant to permit us to analyze the marginal impact of the periodic volatility. In Case III, we correct the testing procedures to account for periodic volatility. For brevity, we report only the main results.

\textit{Case I (no periodicity).} In this case—in the absence of intraday periodicity—we estimate jumps as the intraday returns larger than $u = \alpha \Delta \overline{\sigma}$ (in (5)) with $\overline{\sigma} = 0.47$ and $\alpha = 2$ standard deviations of the diffusion.\textsuperscript{11}

[ Insert Figure 3 about here ]

The upper panel of Figure 3 plots the total number of estimated (spurious) large increments—per 1-minute interval—in the absence of periodicity. The figure delivers a clear message: in the absence of jumps and with constant volatility, a constant threshold truncates observations uniformly throughout the day. This results in an approximately uniform distribution of estimated intraday jumps.

\textit{Case II (with periodicity).} The middle panel of Figure 3 shows that truncated power variation discards too many “large” returns when periodic volatility is high and too few when periodic volatility is low. That is, all of the estimated “large” increments come from the high-volatility portions of the day.

\textsuperscript{10}This intraday pattern is consistent with those reported in the literature. See, e.g., \textcite{andersen1998b}.

\textsuperscript{11}\textcite{ait-sahalia2012, ait-sahalia2009b} show that the test statistics converge in probability, as in (6)–(9), for $\overline{\sigma} < 0.50$ and $\alpha > 0$ (e.g., in a range from 2 to 5). Within these ranges, the \textcite{ait-sahalia2009b} simulations indicate that the choice of $\overline{\sigma} = 0.47$ and $\alpha = 2$ provides the test with good finite sample properties for a data-generating process similar to that of foreign exchange data.
Case III (correction). To solve the problem in Case II, we replace the threshold \( u \) in (4) and (5) with a time-varying threshold \( \tilde{u} \) that is proportional to the intraday periodicity in volatility \( f_{t,i} \), i.e.,

\[
\tilde{u} = \hat{f}_{t,i}^{\text{WSD}} \alpha \Delta^\varpi,
\]

where \( \alpha, \Delta, \) and \( \varpi \) are defined as in Section 2.1. We estimate \( f_{t,i} \) with the weighted standard deviation (WSD), a robust-to-jumps estimator proposed by Boudt et al. (2011b) (see Appendix B). \(^{12}\)

The lower panel of Figure 3 plots the large increments retained by \( U(0, \tilde{u}, \Delta)_t \). Visual inspection suggests that our proposed periodicity-robust threshold, \( \tilde{u} \), appears to truncate observations in a uniform pattern throughout the day, resulting in an approximately uniform distribution for estimated jumps.

3.3. The impact of periodicity on the size of the ASJ jump test statistic

The experiments in the previous section show that intraday periodicity in volatility induces a false daily pattern in the frequency of observations truncated by estimators, such as that in (5), with constant threshold \( u \). This pattern of truncated observations carries through to the distribution of estimated large intraday returns. To examine whether this circadian pattern also affects the size of the jump test statistic \( \widehat{S}_J(p, k, \Delta)_t \), given in (7), we again simulate data and compute test statistics for the three cases.

Case I (no periodicity). To evaluate the size of the jump test under constant volatility, we follow Aït-Sahalia and Jacod (2009b) and first standardize the statistic \( \widehat{S}_J(p, k, \Delta)_t \) by a (consistent) estimator, \( \widehat{V}_t^c \), for the asymptotic variance. The standardized \( \widehat{S}_J(p, k, \Delta)_t \) is defined as

\[
\frac{\widehat{S}_J(p, k, \Delta)_t - k^{p/2-1}}{\sqrt{\widehat{V}_t^c}},
\]

where \( \widehat{V}_t^c \) is based on the truncated power variation \( \hat{A}(p, \Delta) \). That is,

\[
\widehat{V}_t^c = \frac{\Delta M(p, k) \hat{A}(2p, u, \Delta)_t}{\hat{A}(p, u, \Delta)_t^2},
\]

where

\[
\hat{A}(p, u, \Delta) := \frac{(1/M)^{1-p/2}}{m_p} \sum_{i=1}^{1/\Delta} |r_{t,i}|^p 1_{\{|r_{t,i}| \leq u\}},
\]

with

\[
M(p, k) = \frac{1}{m_p} \left( k^{p-2} (1 + k) m_{2p} + k^{p-2} (k - 1) m_{2p}^2 - 2k^{p/2-1} m_{k,p}\right),
\]

where \( m_p \) and \( m_{k,p} \) are defined as

\[
m_p = \mathbb{E}(|U|^p), \quad \tag{17}
\]

\[
m_{k,p} = \mathbb{E}(|U|^p | U + \sqrt{k - 1} V|^p), \quad \tag{18}
\]

\(^{12}\)To ease the comparison with the base threshold \( u \) (see e.g., Equation (3)), we omit the subscript \( t, i \)—denoting the time-variation associated with periodicity—on \( \tilde{u} \).
for $U$ and $V$, two independent $\mathcal{N}(0,1)$ variables. With the choice of $p = 4$, Aït-Sahalia and Jacod (2009b) show that $M(4, k) = 16k(2k^2 - k - 1)/3$. That is, for $k = 2$, $M(4, 2) = 160/3$. Under the null of no jumps (i.e., Hypothesis (B)), the standardized test statistic in (13) is asymptotically (as $\Delta \to 0$) standard normal.

[Insert Table 2 about here]

The rightmost panel of Table 2 reports the probability of rejection of the ASJ jump test (via $\hat{S}_J$) under the null hypothesis of no jumps (i.e., Hypothesis (B)). The test is nearly correctly sized in the absence of periodicity in volatility, particularly at the highest sampling frequency (first row – last three columns). The rejection rates for the null of no jumps are 5.9%, 6.4% and 8.0% for the 30-second, 1-minute, and 5-minute data, respectively.

Case II (with periodicity). Intraday periodicity significantly increases the rejection rates of the jump test to 9.2%, 9.5%, and 10.7%, at 30-second, 1-minute and 5-minute frequencies (right-hand panel, second row of Table 2). Nevertheless, periodic volatility has little impact on the mean test statistics (middle three columns), which are close to the theoretical predictions (i.e., 2). This suggests that periodicity affects mostly the variance of the standardized test statistic.

Case III (correction). To correct the test statistic for intraday periodicity, we first estimate the periodic component of volatility by $\hat{f}_{WSD}^{\text{intra}}$ (see Appendix B) and then replace the base threshold $u$ in (15) by a periodicity-robust threshold $\tilde{u}$, as in (12). Our two-stage correction reduces overrejection at all sampling frequencies, but more so at the highest frequencies, 30-seconds and 1-minute, for which the rejection rates decline to 7.3% and 8.0%, respectively (third row of Table 2).

4. Empirical application

This section describes the data and reports the results of our investigation into the presence of Brownian motion and jump intensity in actual exchange rate data. Throughout this section, we correct the test statistics and estimators (i.e., (6) to (11)) for the presence of intraday periodicity, as previously described.

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13 Following Aït-Sahalia and Jacod (2009b), we compute $\hat{S}_J$ for each trading day, consisting of 24 hours of foreign exchange activity.

14 The probability limits of the test statistics (e.g., (6)−(9)) rely on the ratio of two power variations. Under the null hypothesis of no jumps, Aït-Sahalia and Jacod (2009b) argue that one may choose the parameters of $u$ a priori without affecting the properties of the power variation measures. Our estimation procedures rely on this theoretical argument and we preserve all suitable conditions on the cutoff level $u$, such as $\varpi < 0.50$, $\alpha > 0$ (see Aït-Sahalia and Jacod, 2010). In Equation (12), $\tilde{u}$ adds only a time-varying component (via intraday periodicity estimates) and thus does not impact the approximation of power variation measures. As a consequence, the probability limits of the statistics are not affected when we replace $u$ by $\tilde{u}$.

15 For brevity, we report only the simulation results under the constant volatility model, but a stochastic volatility structure (e.g., GARCH-type) does not change the results.

16 Before applying the aforementioned tests to real data, we assessed the properties of the tests on simulated data with appropriate parameter values (e.g., $p$, $u$, $k$) and cutoff levels (e.g., $\alpha$, $\varpi$). To select parameter values with good properties for our empirical analysis, we simulated data and test statistics with ranges of values for the power parameter $p$, sampling frequency parameter $k$ and cutoff level $u$. The selected parameter values implied Monte Carlo probability limits that were close to the asymptotic probability limits for each test statistic (Equations (6)−(9)) and estimator (Equations (10)−(11)) at various sampling frequencies (e.g., 30-seconds, 5-minutes). For brevity, we do not report these Monte Carlo exercises but they are available upon request.
4.1. Data

We use 1-minute data for the EUR/USD, USD/JPY, and GBP/USD exchange rates from January 1, 2000, to March 12, 2010. Disk Trading provides the last mid-quotes (average of the logarithms of bid and ask quotes) of 1-minute intervals throughout the 24-hour trading day. Following Andersen and Bollerslev (1998a), one trading day extends from 16:01 EST on day \( t \) − 1 to 16:00 EST on day \( t \).

As is usual in the literature, we omit trading days with too many missing values or low trading activity because they provide poor estimates of volatility. Similarly, we deleted weekends plus certain fixed and irregular holidays, trading days for which there are more than 360 missing values at the 1-minute frequency (corresponding to more than one fourth of the data), and trading days with too many empty intervals and consecutive prices. These criteria leave 2483, 2479 and 2472 days, respectively, for the EUR/USD, USD/JPY and the GBP/USD exchange rates.

4.2. Results

4.2.1. Brownian motion: present or not

We first ask whether an appropriate model of high-frequency exchange rates needs Brownian motion. To answer this question, we use our proposed time-varying threshold, \( \bar{u} \), to compute \( \bar{S}_W(p, \bar{u}, k, \Delta) \) in (6) for each day in the sample. Figure 4 plots the histograms of these daily statistics for the three exchange rates.

The figure clearly shows that Brownian motion is present in the data. Under the null hypothesis that Brownian motion is present (Hypothesis (A)), we expect the theoretical distribution of \( \bar{S}_W(p, \bar{u}, k, \Delta) \) to be centered around \( k^{1-p/2} \): that is, 1.4142 and 1.7321 with \( p = 1, k = 2 \) and \( p = 1, k = 3 \), respectively. Under the alternative hypothesis of no Brownian motion, the statistics should converge in probability to 1. For all exchange rates, the mean of the empirical distributions of the non-standardized daily \( \bar{S}_W(p, \bar{u}, k, \Delta) \) are around 1.5 \( (k = 2) \) and 1.9 \( (k = 3) \)—well above the limit under the alternative of no Brownian motion (i.e., 1). Thus, the test confirms the results of Todorov and Tauchen (2010) and Cont and Mancini (2011) that the data-generating model should include Brownian motion.

4.2.2. Jumps: present or not

We then ask whether jumps are present in the exchange rate data. Table 3 reports the results of the Aït-Sahalia and Jacod (2009b) jump test—based on (7)—applied to EUR/USD, USD/JPY, and GBP/USD rates for \( p = 4 \) and combinations of truncation and scaling parameters \( (\omega = 0.47 \text{ and } k = 2, 3) \).

[ Insert Table 3 about here ]
The test statistics in Equation (7) will converge in probability to 1 under the null hypothesis that jumps are present. Under the alternative of no jumps and no microstructure noise, the test statistics should converge to $k^p/2 - 1$. With $p = 4$, this means that the statistics should converge in probability to $k$ (2 or 3) if there are no jumps and no noise. Table 3 shows that the mean statistics are not very close to the probability limit under either hypothesis, however. One explanation could be that noise tends to bias the statistic upwards from 1 in the presence of jumps. To check our expectation, we have conducted simulations (unreported) suggesting that noise in the data will generally bias the empirical value of these test statistics toward the probability limit which exists in the case when noise dominates. We observe this pattern in the data. The $\hat{S}_J(p, k, \Delta)_t$ are typically bigger than 1 but much less than $k$. For $k = 2$, $\hat{S}_J(p, k, \Delta)_t$ ranges from 1.3 to 1.45; for $k = 3$, $\hat{S}_J(p, k, \Delta)_t$ ranges from 1.6 to 1.95. In other words, the evidence is consistent with the presence of jumps coupled with a moderate amount of noise that biases the test statistics upwards (i.e., Hypothesis (B)).

4.2.3. Jumps: finite or infinite activity

Having concluded from Table 3 and Figure 5 that jumps are present, we now ask if exchange rate jumps exhibit finite or infinite-activity. We first focus on the finite activity test (i.e., Hypothesis (C)), and set $k = 2$ and $p = 4$ to compute $\hat{S}_{FA}$ in (8). Recall that, for these parameter choices, we expect the test statistic to converge to 2 if jumps have finite-activity and to 1 if jumps have infinite activity (Equation (8)). If noise dominates the test, $\hat{S}_{FA}$ should converge to $1/k$ i.e., 0.5 for $k = 2$. Therefore, noise will tend to bias the test statistic downwards under either hypothesis.

4.2.3. Jumps: finite or infinite activity

Having concluded from Table 3 and Figure 5 that jumps are present, we now ask if exchange rate jumps exhibit finite or infinite-activity. We first focus on the finite activity test (i.e., Hypothesis (C)), and set $k = 2$ and $p = 4$ to compute $\hat{S}_{FA}$ in (8). Recall that, for these parameter choices, we expect the test statistic to converge to 2 if jumps have finite-activity and to 1 if jumps have infinite activity (Equation (8)). If noise dominates the test, $\hat{S}_{FA}$ should converge to $1/k$ i.e., 0.5 for $k = 2$. Therefore, noise will tend to bias the test statistic downwards under either hypothesis.

Figure 6 displays the empirical distribution of $\hat{S}_{FA}(p, \bar{\alpha}, k, \Delta)_t$ for parameters $\alpha = 6$ (left panels) and $\alpha = 10$ (right panels)—that is, low $\alpha$ and high $\alpha$—as in Ait-Sahalia and Jacod (2011). For all exchange rate data, the histograms in the figure show that the means of the distributions of $\hat{S}_{FA}(p, \bar{\alpha}, k, \Delta)_t$ range

\[\text{for brevity, we present the results only for } \bar{\alpha} = 0.47. \text{ The results with other parameter values (e.g., } \bar{\alpha} = 0.48) \text{ are similar.}\]
from about 1.44 to 1.74. That is, they are between 1 and 2, which are the theoretical limits under the alternative of infinite activity and null of finite activity, respectively. Because some noise will bias the test statistics downward and most of the means of the test statistics are closer to 2 than 1, we conclude that finite activity jumps better characterize the data.

As an alternative to the finite activity test, we also consider the null hypothesis that jumps have infinite activity (i.e., Hypothesis (D)). Under the null hypothesis of infinite jump activity and no microstructure noise, the test statistic, \( \tilde{S}_{IA}(p, p', \tilde{u}, \gamma, \Delta) \), should converge in probability to 2 (for \( \gamma = 2, p' = 4, p = 3 \)). The alternative hypothesis of finite activity jumps with no noise implies that \( \tilde{S}_{IA}(p, p', \tilde{u}, \gamma, \Delta) \) will converge in probability to 1 (see Equation (9)).

The lower panel of Table 4 shows that \( \tilde{S}_{IA}(p, p', \tilde{u}, \gamma, \Delta) \) is close to 1.2 for all series (lower panel), which is more consistent with finite-activity jumps. We therefore conclude that both tests favor the hypothesis that finite jump activity better characterizes 1-minute exchange rate data. This evidence supports the observation of Aït-Sahalia and Jacod (2011) that jumps occurring at very high frequencies—of the order of seconds—tend to be aggregated at lower frequencies. As our results indicate, 1-minute data hence become more compatible with finite jump activity.

4.2.4. How active are exchange rate jumps?

The empirical results of Sections 4.2.2 and 4.2.3 indicate that finite-activity jumps are present in exchange rates. We now analyze the degree of activity of exchange rate jumps.

Table 5 reports the estimates of the jump activity index \( \beta \) (see Equations (10) and (11)) for the EUR/USD, USD/JPY and GBP/USD returns. Recall that finite-activity jumps—such as Poisson jumps—have an activity level of \( \beta = 0 \), while infinite-activity jumps have positive \( \beta \)s. To be consistent with our (unreported) simulation study, we choose the cutoff level \( \bar{\sigma} = 0.47 \) and values of \( \alpha \) ranging from 6 to 10 standard deviations of the diffusion process.\(^20\) The table indicates that \( \hat{\beta}'(\bar{\sigma}, \alpha, k, \tilde{u}) \) and \( \hat{\beta}(\bar{\sigma}, \alpha, \alpha', \tilde{u}, \tilde{u}) \), in (11) and (10), are rather small. They range from 0.07 to 0.10—for \( \alpha = 6 \), indicating a low degree of jump activity, consistent with a compound Poisson process. Increasing \( \alpha \) (from 6 to 10) moves the index estimates closer to 0, producing more evidence for large and infrequent jumps, and reduces standard errors. This is expected because the truncation level increases linearly with \( \alpha \), \( \tilde{u} = \hat{f}_{t,i}^{WSD, \alpha} \Delta \bar{\sigma} \), retaining larger and less-frequent jumps. Table 5 confirms that 1-minute exchange rate data display finite jump activity.

Figure 7 summarizes our test results for the characteristics of the EUR/USD returns, displaying the limits of the test statistics: (i) under the null hypothesis, (ii) under the alternative hypothesis, (iii) when market microstructure noise dominates. Consistent with our previous discussions, we reject the null of no Brownian motion (top panel), reject the null of no jumps (middle panel), and find evidence

\(^20\) These simulation results are available upon request. Moreover, the empirical results with other cutoff levels (e.g., \( \bar{\sigma} = 0.40 \)) are similar and thus we omit them for brevity.
in favor of finite-activity jumps (lower panel). Microstructure noise does not dominate the results for any of the testing procedures.\textsuperscript{21}

4.2.5. The relative magnitude of the components

We can disentangle the quadratic variation ($QV$) of exchange rate returns into its continuous and jump components. That is,

\[
\begin{aligned}
\text{Components} & \quad \Rightarrow \quad \% QV \text{ due to the continuous component} \\
B(p,\hat{\Delta}_t) & \quad \Rightarrow \quad \% QV \text{ due to big jumps} \\
\hat{U}(p,\epsilon,\Delta_t) & \quad \Rightarrow \quad \% QV \text{ due to small jumps}
\end{aligned}
\]

where $QV$ is the sum of the integrated variance and the cumulative squared jumps.\textsuperscript{22} To split the $QV$ into its components, we follow Aït-Sahalia and Jacod (2012) in fixing $p = 2$ so that both jump and diffusion components are present. The decomposition of $QV$ also depends on the jump size cutoff parameter $\epsilon$ in (19). A larger value of $\epsilon$ will attribute more variation to small jumps and less to “big” jumps because it will pick out fewer “big” jumps.\textsuperscript{23}

[ Insert Table 6 about here ]

Table 6 reports each component’s % contribution to the total quadratic variation ($QV$) for each exchange rate and each jump size cutoff. Brownian motion ($QV_{\text{Brownian}}$) drives around 85% of $QV$ for each exchange rate. Jumps create about 15% of $QV$. The jump cutoff size, $\epsilon$, determines the relative proportions explained by big/small jumps. For a low level of the cutoff (0.09), big jumps explain 10–14% of $QV$. For the highest level of the cutoff, the contribution of big jumps declines to 3–4%.\textsuperscript{24}

[ Insert Figure 8 about here ]

Figure 8 plots the times series of the fraction of $QV$ attributable to the continuous and jump components of EUR/USD rate for every trading day in our sample. %$QV$ of both Brownian and jump components tend to be rather stable over time (first and second panels). Although the proportion of $QV$ due to big jumps temporarily increases following the collapse of Lehman Brothers (September 15, 2008), the $QV$ components appear to be fairly stable during the financial crisis.\textsuperscript{25}

\textsuperscript{21}The spectrographic analyses of the USD/JPY and GBP/USD returns are quite similar, and hence we omit those results for brevity.

\textsuperscript{22}The drift term (e.g., $\mu(t)$ in (1)) does not affect $QV$; Andersen and Benzoni (2008) discuss this issue. Further, the integrated variance, the variance of the continuous component, of the underlying log-price process in Model (1) can be further defined as $IV_t \equiv \int_{1-1}^{t} \sigma^2(s)ds$, which is latent because $\sigma^2(s)$ is not directly observable. An unbiased estimation of $IV_t$ depends on whether the jump component of the log-price process has finite or infinite-activity. See Barndorff-Nielsen and Shephard (2004), Mancini (2009), and Bollerslev and Todorov (2011a) who present alternative estimators for $IV_t$.

\textsuperscript{23}Aït-Sahalia and Jacod (2012) note that the cutoff between big and small jumps is used for mathematical reasons and is entirely arbitrary.

\textsuperscript{24}In unreported simulation results, we tested whether the base ratio $\hat{B}(p,u,\Delta_t)/\hat{U}(p,\infty,\Delta_t)$ in (19) is around 1 under the null hypothesis of no jumps. At the 1-minute sampling frequency, we found that Monte Carlo mean values are indeed around 0.99 when we choose the truncation levels $\alpha = 0.47$ and $\alpha = 3$. For lower truncation values such as $\alpha = 2$, the ratio becomes 0.89, and hence it tends to underestimate the proportion of $QV$ due to the continuous component. The volatility process (constant or GARCH) does not affect these simulation results. We thus set $\alpha = 0.47$ and $\alpha = 3$ when applying the $QV$ statistic to the real exchange rate data. These results are available upon request.

\textsuperscript{25}The results are similar for other exchange rates and thus we omit them for brevity. Further, we do not specifically analyze here how crisis periods affect the spectrum of exchange rate returns. Recently, Dungey et al. (2011) propose alternative test statistics to study this issue.
4.2.6. Intraday periodicity and jump detection

In Section 3.3, the simulation evidence reveals that the ASJ jump test tends to overreject the null of no jumps in the presence of periodic volatility. We now compare the jump detection performance of the basic and periodicity-robust ASJ jump tests on real data. Table 7 reports the results of the ASJ test.

The table shows that the periodicity-robust ASJ test, based on the time-varying threshold \( \tilde{u} \) (WSD), identifies 15 to 30\% fewer jumps than the base test, which relies on a constant threshold \( u \). For instance, at the 10\% significance level (column \( j = 0.10 \)) and for \( \omega = 0.47 \), the base test—i.e., no correction for periodicity in volatility—detects 189 daily EUR/USD jumps whereas the corrected tests identify 156 jumps. That is, the uncorrected test finds too many jumps in the real data.

The left panels of Figure 9 plot the total number of large exchange rate returns per (1-minute) intraday period that are retained by (5)’s indicator function. Without a correction for intraday patterns in volatility, most of the estimated large increments occur during the high-volatility periods when European and North American markets are open.

We repeat the same exercise after correcting for periodic intraday volatility patterns. The right panels of Figure 9 illustrate that the correction removes the cyclical patterns in jump detection. That is, detected jumps tend to occur more uniformly after the correction for intraday periodicity in volatility, though jumps do tend to occur during the relatively illiquid period between North American and Asian markets.

5. Conclusion

Our paper applies recently developed power variation statistics (Ait-Sahalia and Jacod, 2012) to determine the most appropriate features of continuous-time exchange rate models using a 10-year span of 1-minute exchange rate data on 3 exchange rates.

We use simulations to demonstrate that the truncated power variation statistics have undesirable properties in the presence of the periodic volatility patterns typical of exchange rates. Specifically, uncorrected truncated power variation identifies far too many (few) large intraday returns during periods of high (low) volatility and the uncorrected jump test detects too many jumps, unconditionally.

We propose a correction procedure that filters out these time-of-day effects and greatly improves the properties of jump test statistics in the presence of periodic volatility. Simulations demonstrate that the corrected diffusion and jump tests have good properties in 1-minute data, even in the presence of microstructure noise.

These corrected statistics imply that Brownian motion and finite-activity jumps, such as those in Poisson processes, are very plausible features of continuous-time models of exchange rates. They are much more plausible than models without jumps or with infinite-activity jumps. Brownian motion creates most (85\%) exchange rate variation over short time scales but jumps also contribute substantially (15\%). This jump proportion is lower than that for individual stocks (Ait-Sahalia and Jacod, 2012) but higher than that documented in Barndorff-Nielsen and Shephard (2006) for 5-minute exchange rate
series. The lower proportion of jump variation in exchange rates—compared to individual stocks—is consistent with the greater liquidity of exchange rate markets and the greater importance of public information in the latter.

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Appendix A. Monte Carlo study of the periodicity analysis

Case I. We consider three sampling frequencies: 30-seconds (i.e., \( M = 2880 \)), 1-minute (i.e., \( M = 1440 \)), and 5-minutes (i.e., \( M = 288 \)). We set \( T = 250 \), and the number of replications is 1000. The DGP is a BSM diffusion model with constant volatility:

\[
\begin{align*}
    dX(t) &= \sigma(t) dW(t) \\
    \sigma(t) &= \sigma_{\text{constant}} = \sqrt{0.30}.
\end{align*}
\]

Cases II and III. We now specify \( \sigma(t) \) as a multiplicative process of the periodicity function \( f(\tau(t)) \), a flexible Fourier form (FFF), which depends only on the time of the day \( \tau(t) \), and a constant volatility process:

\[
\begin{align*}
    dX(t) &= \sigma(t) dW(t) \\
    \sigma(t) &= f(\tau(t)) \sigma_{\text{constant}} \\
    \sigma_{\text{constant}} &= \sqrt{0.30} \\
    \log f(\tau(t)) &= \sum_{j=1}^{4} (\gamma_j \cos(2\tau(t) j \pi) + \delta_j \sin(2\tau(t) j \pi)),
\end{align*}
\]

where the cos and sin terms depend on the time of the day and we use the following estimated parameters:

\[
(\gamma_1, \delta_1, \ldots, \gamma_4, \delta_4) = (-0.24422, -0.49756, -0.054171, 0.073907, -0.26098, 0.32408, -0.11591, -0.21442).
\]

Remark. The results of the simulation exercises above are unchanged by a stochastic volatility structure (i.e., GARCH-type diffusion process). We omit these results for brevity.

Appendix B. Volatility dynamics and periodicity estimation

Let us consider a Brownian model (BSM) without any JUMPS component but with intraday periodicity in volatility. If \( \Delta \) is sufficiently small, then returns are conditionally normally distributed with mean zero and variance equal to the integral of underlying volatility over the short interval:

\[
\sigma_{t,i}^2 = \int_{t+(i-1)\Delta}^{t+i\Delta} \sigma^2(s) ds,
\]

(B.1)
that is \( r_{t,i} \approx \sigma_{t,i} z_{t,i} \), where \( z_{t,i} \sim N(0,1) \). We assume that the high-frequency return variance \( \sigma_{t,i}^2 \) in (B.1) has a periodic component \( f_{t,i}^2 \) which represents the intraday periodic features. That is,

\[
\sigma_{t,i} = s_{t,i} f_{t,i},
\]

(B.2)

where \( s_{t,i} \) is the stochastic intradaily volatility, constant over the day but varying from one day to another.\(^{26}\) The periodic factor, \( f_{t,i} \), is a deterministic function of time within a day. One can estimate \( s_{t,i} \) using the square root of the realized volatility on day \( t \) by setting \( p = 2, u = \infty \) and \( k = 1 \) in (3). That is,

\[
\hat{s}_{t,i} = \sqrt{\frac{1}{M} \hat{B}(2, \infty, \Delta)},
\]

(B.3)

In the presence of jumps (e.g., BSMFAJ model), the square root of Barndorff-Nielsen and Shephard (2004)’s bipower variation estimates the diffusion variance, \( s_{t,i} \), better than (B.3). That estimator is

\[
\hat{s}_{t,i} = \sqrt{\frac{1}{M-1} BV_t},
\]

(B.4)

with

\[
BV_t \equiv \mu_1^{-2} \frac{M}{(M-1)} \sum_{i=2}^{M} |r_{t,i}||r_{t,i-1}|,
\]

(B.5)

where \( \mu_1 \equiv \sqrt{2/\pi} \approx 0.79788 \). Under this representation, the standardized high-frequency return \( r_{t,i} = r_{t,i}/\hat{s}_{t,i} \sim N(0, f_{t,i}^2) \) as \( \Delta \to 0 \). This result suggests estimating the periodicity factor \( f_{t,i} \) in (B.2) using an estimator of the scale of the standardized returns. That is,

\[
\hat{r}_{t,i} = \frac{r_{t,i}}{\sqrt{BV_t}},
\]

(B.6)

where \( BV_t \) is given in (B.5). Boudt et al. (2011b) recommend the use of the Shortest Half scale estimator—proposed by Rousseeuw and Leroy (1988)—because this estimator remains consistent in the presence of infinitesimal contaminations by jumps in the data. To define the Shortest Half scale estimator, we denote the corresponding order statistics \( \tau_{(1):t,i}, \ldots, \tau_{(n_{t,i}):t,i} \) such that \( \tau_{(1):t,i} \leq \tau_{(2):t,i} \leq \ldots \leq \tau_{(n_{t,i}):t,i} \). The shortest half scale is the smallest length of all “halves” consisting of \( h_{t,i} = \lfloor n_{t,i}/2 \rfloor + 1 \) contiguous order observations. These halves equal \( \{\tau_{(1):t,i}, \ldots, \tau_{(h_{t,i}):t,i}\}, \ldots, \{\tau_{(n_{t,i}-h_{t,i}+1):t,i}, \ldots, \tau_{(n_{t,i}):t,i}\} \), and their length is \( \tau_{(h_{t,i}):t,i} - \tau_{(1):t,i}, \ldots, \tau_{(n_{t,i}):t,i} - \tau_{(h_{t,i}):t,i} \), respectively. The corresponding scale estimator (corrected for consistency under normality) equals the minimum of these lengths:

\[
\text{ShortH}_{t,i} = 0.741 \cdot \min\{\tau_{(h_{t,i}):t,i} - \tau_{(1):t,i}, \ldots, \tau_{(n_{t,i}):t,i} - \tau_{(n_{t,i}-h_{t,i}+1):t,i}\}. \tag{B.7}
\]

The Shortest Half estimator for the periodicity factor of \( r_{t,i} \) equals

\[
\hat{f}_{\text{ShortH}} = \frac{\text{ShortH}_{t,i}}{\sqrt{\frac{1}{M} \sum_{j=1}^{M} \text{ShortH}_{t,i}^2}}. \tag{B.8}
\]

The shortest half dispersion is highly robust to jumps, but it has only a 37% relative efficiency under normality of the \( \tau_{t,i}’s \). Boudt et al. (2011b) show that the standard deviation applied to the returns weighted by their outlyingness under the ShortH estimate offers a better trade-off between the efficiency

\(^{26}\)See e.g., Andersen and Bollerslev (1998a), Hecq et al. (2011), and Visser (2010) who also assume that \( s_{t,i} \) is constant over the day but can vary from day to another.
of the standard deviation under normality and robustness to jumps. That is,

\[
\hat{f}_t^{WSD} = \frac{WSD_{t,i}}{\sqrt{\frac{1}{M} \sum_{j=1}^{M} WSD_{t,j}^2}},
\]

where

\[
WSD_{t,j} = \sqrt{1.081 \cdot \frac{\sum_{i=1}^{n_{t,j}} w\left(\frac{\left|\tau_{t,j} / \hat{f}_{t,j}\right|^2}{\left|\tau_{t,j} / \hat{f}_{t,j}\right|^2}\right)}{\sum_{i=1}^{n_{t,j}} w\left(\left|\tau_{t,j} / \hat{f}_{t,j}\right|^2\right)}}.
\]

(B.9)

Because the weighting is applied to the squared standardized returns, which are extremely large in the presence of jumps, Boudt et al. (2011b) recommend the use of the hard rejection with threshold equal to the 99% quantile of the \(\chi^2\) distribution with one degree of freedom, that is

\[
w(z) = \begin{cases} 
1 & \text{if } z \leq 6.635 \\
0 & \text{else}
\end{cases}
\]

(B.11)

The factor 1.081 in Equation (B.10) further ensures the consistency of the estimator under normality. The Weighted Standard Deviation (WSD) in (B.9) has a 69% efficiency under normality of the \(\tau_{t,i}\)’s.

References


Table 1: Theoretical jump activity index values for various random processes

<table>
<thead>
<tr>
<th>Jump index</th>
<th>Jump process</th>
<th>Jump activity</th>
<th>Jump frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta = 0$</td>
<td>Poisson process</td>
<td>finite</td>
<td>low</td>
</tr>
<tr>
<td>$\beta = 0$</td>
<td>Gamma process</td>
<td>infinite</td>
<td>low</td>
</tr>
<tr>
<td>$\beta = 0$</td>
<td>VG process</td>
<td>infinite</td>
<td>low</td>
</tr>
<tr>
<td>$\beta = 0.5$</td>
<td>IG process</td>
<td>infinite</td>
<td>low</td>
</tr>
<tr>
<td>$\beta = 1$</td>
<td>Cauchy process</td>
<td>infinite</td>
<td>moderate</td>
</tr>
<tr>
<td>$\beta = 1$</td>
<td>NIG process</td>
<td>infinite</td>
<td>moderate</td>
</tr>
<tr>
<td>$\beta \in (0, 2)$</td>
<td>Stable process</td>
<td>infinite</td>
<td>(low to high)</td>
</tr>
<tr>
<td>$\beta \in (0, 2)$</td>
<td>GH process</td>
<td>infinite</td>
<td>(low to high)</td>
</tr>
<tr>
<td>$\beta \in (0, 2)$</td>
<td>CGMY process</td>
<td>infinite</td>
<td>(low to high)</td>
</tr>
<tr>
<td>$\beta = 2$</td>
<td>Brownian Motion</td>
<td>(limit case)</td>
<td>extreme</td>
</tr>
</tbody>
</table>

Notes: The table presents the index level of jump activity $\beta$ for various jump processes and Brownian Motion as a limit case. VG: Variance-Gamma, IG: Inverse Gaussian, NIG: Normal Inverse Gaussian, GH: Generalized Hyperbolic, CGMY: process proposed by Carr et al. (2002).

Table 2: The impact of periodicity on the actual size of the jump test at the 5% level

Panel A. Constant Volatility Model

<table>
<thead>
<tr>
<th>Perodicity?</th>
<th>Correction?</th>
<th>Mean $S_{J}(p, k, \Delta)_{t}$</th>
<th>Rejection rates</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>30-sec 1-min 5-min</td>
<td>30-sec 1-min 5-min</td>
</tr>
<tr>
<td>Case I</td>
<td>No</td>
<td>No</td>
<td>2.0006 2.0015 2.0082</td>
</tr>
<tr>
<td>Case II</td>
<td>Yes</td>
<td>No</td>
<td>2.0035 2.0047 2.0194</td>
</tr>
<tr>
<td>Case III</td>
<td>Yes</td>
<td>Yes</td>
<td>2.0035 2.0047 2.0194</td>
</tr>
</tbody>
</table>

Notes: The table shows the size of the ASJ jump test under the null hypothesis of no jumps under three cases, described in the text. The mean $S_{J}(p, k, \Delta)_{t}$ statistic (see Equation (7)) should converge in probability to 2. We estimate the periodic component $f_{t,i}$ using WSD.
Table 3: Testing the presence of jumps in the exchange rate data

<table>
<thead>
<tr>
<th>EUR/USD</th>
<th>Mean $\hat{S}_j(p, k, \Delta)_t$</th>
<th>Number of detected daily jumps</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Asymptotic</td>
<td>Empirical</td>
</tr>
<tr>
<td>k</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
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</tr>
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<td></td>
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<tr>
<td>GBP/USD</td>
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<td></td>
</tr>
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</tr>
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<td>3</td>
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</tbody>
</table>

Notes: The table shows the number of detected daily jumps using the Aït-Sahalia and Jacod (2009b) jump test. The column labeled “Asymptotic” presents asymptotic probability limits under the null hypothesis of no jumps. Under the alternative, the statistics should converge in probability to 1. The sample covers January 1, 2000 to March 12, 2010 with 1-minute data. $j$ denotes the significance levels of the test. We set $\alpha = 4$, $p = 4$, and $\tilde{\omega} = 0.47$ in the testing procedure.

Table 4: Testing whether exchange rate jumps have finite or infinite activity

<table>
<thead>
<tr>
<th>$\hat{S}_{FA}(p, \tilde{u}, k, \Delta)_t$</th>
<th>Asymptotic</th>
<th>Empirical</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha = 6$</td>
<td>$\alpha = 7$</td>
</tr>
<tr>
<td>EUR/USD</td>
<td>2</td>
<td>1.6237</td>
</tr>
<tr>
<td>USD/JPY</td>
<td>2</td>
<td>1.4429</td>
</tr>
<tr>
<td>GBP/USD</td>
<td>2</td>
<td>1.5869</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\hat{S}_{IA}(p, p', \tilde{u}, \gamma, \Delta)_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>EUR/USD</td>
</tr>
<tr>
<td>USD/JPY</td>
</tr>
<tr>
<td>GBP/USD</td>
</tr>
</tbody>
</table>

Notes: The table presents the empirical mean values of the finite activity test statistic $\hat{S}_{FA}(p, \tilde{u}, k, \Delta)_t$ (upper panel) and infinite activity test statistic $\hat{S}_{IA}(p, p', \tilde{u}, \gamma, \Delta)_t$ (lower panel). The sample covers January 1, 2000 to March 12, 2010 with 1-minute data. The sampling frequency is 1-minute. For $\hat{S}_{FA}(p, \tilde{u}, k, \Delta)_t$, we set $p = 4$. For $\hat{S}_{IA}(p, p', \tilde{u}, \gamma, \Delta)_t$, we set $p' = 4$, $p = 3$, and $\gamma = \alpha'/\alpha = 2$. For both tests, we choose $\tilde{\omega} = 0.47$, and $k = 2$. In the top panel, the statistics should converge in probability to 2 under the null of finite jumps (the column labeled “Asymptotic”) and to 1 under the alternative of infinite jumps. In the bottom panel, the statistics should converge in probability to 2 under the null of infinite jumps and to 1 under the alternative of finite jumps.
Table 5: Estimates of jump activity levels from exchange rate returns

<table>
<thead>
<tr>
<th></th>
<th>$\alpha = 6$</th>
<th>$\alpha = 8$</th>
<th>$\alpha = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>EUR/USD</td>
<td>0.103</td>
<td>0.015</td>
<td>0.006</td>
</tr>
<tr>
<td></td>
<td>(0.534)</td>
<td>(0.223)</td>
<td>(0.138)</td>
</tr>
<tr>
<td>USD/JPY</td>
<td>0.103</td>
<td>0.031</td>
<td>0.010</td>
</tr>
<tr>
<td></td>
<td>(0.525)</td>
<td>(0.312)</td>
<td>(0.160)</td>
</tr>
<tr>
<td>GBP/USD</td>
<td>0.089</td>
<td>0.017</td>
<td>0.006</td>
</tr>
<tr>
<td></td>
<td>(0.527)</td>
<td>(0.211)</td>
<td>(0.117)</td>
</tr>
</tbody>
</table>

\[ \hat{\beta}(\varpi, \alpha, \alpha', \bar{u}, \bar{u}')_t \]

<table>
<thead>
<tr>
<th></th>
<th>$\alpha = 6$</th>
<th>$\alpha = 8$</th>
<th>$\alpha = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>EUR/USD</td>
<td>0.074</td>
<td>0.013</td>
<td>0.006</td>
</tr>
<tr>
<td></td>
<td>(0.394)</td>
<td>(0.161)</td>
<td>(0.106)</td>
</tr>
<tr>
<td>USD/JPY</td>
<td>0.075</td>
<td>0.019</td>
<td>0.008</td>
</tr>
<tr>
<td></td>
<td>(0.391)</td>
<td>(0.187)</td>
<td>(0.131)</td>
</tr>
<tr>
<td>GBP/USD</td>
<td>0.069</td>
<td>0.010</td>
<td>0.003</td>
</tr>
<tr>
<td></td>
<td>(0.387)</td>
<td>(0.132)</td>
<td>(0.081)</td>
</tr>
</tbody>
</table>

Notes: The table reports the estimates of $\hat{\beta}(\varpi, \alpha, k, \bar{u})_t$ (upper panel) and $\hat{\beta}(\varpi, \alpha, \alpha', \bar{u}, \bar{u}')_t$ (lower panel) for the EUR/USD, USD/JPY, and GBP/USD exchange rates. The sample covers January 1, 2000 to March 12, 2010. We choose $k = 2$, $\alpha'/\alpha = 1.5$, and fix $p = 0$. Standard errors of the estimators are reported in parenthesis. The test statistics should converge in probability to zero for finite-activity processes.

Table 6: Quadratic variation ($QV$) for the EUR/USD, USD/JPY and GBP/USD

<table>
<thead>
<tr>
<th></th>
<th>$\epsilon = 0.09$</th>
<th>$\epsilon = 0.13$</th>
<th>$\epsilon = 0.19$</th>
</tr>
</thead>
<tbody>
<tr>
<td>EUR/USD</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$QV^{\text{Brownian}}$</td>
<td>86.47%</td>
<td>86.47%</td>
<td>86.47%</td>
</tr>
<tr>
<td>$QV^{\text{big jumps}}$</td>
<td>12.25%</td>
<td>6.46%</td>
<td>3.62%</td>
</tr>
<tr>
<td>$QV^{\text{small jumps}}$</td>
<td>1.28%</td>
<td>7.07%</td>
<td>9.92%</td>
</tr>
<tr>
<td>USD/JPY</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$QV^{\text{Brownian}}$</td>
<td>85.68%</td>
<td>85.68%</td>
<td>85.68%</td>
</tr>
<tr>
<td>$QV^{\text{big jumps}}$</td>
<td>14.08%</td>
<td>7.45%</td>
<td>4.09%</td>
</tr>
<tr>
<td>$QV^{\text{small jumps}}$</td>
<td>0.24%</td>
<td>6.87%</td>
<td>10.23%</td>
</tr>
<tr>
<td>GBP/USD</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$QV^{\text{Brownian}}$</td>
<td>85.71%</td>
<td>85.71%</td>
<td>85.71%</td>
</tr>
<tr>
<td>$QV^{\text{big jumps}}$</td>
<td>10.22%</td>
<td>5.27%</td>
<td>2.89%</td>
</tr>
<tr>
<td>$QV^{\text{small jumps}}$</td>
<td>4.07%</td>
<td>9.02%</td>
<td>11.41%</td>
</tr>
</tbody>
</table>

Notes: The table shows the fraction of quadratic variation ($QV$) attributable to the continuous and jump components for the exchange rates. The sample covers January 1, 2000 to March 12, 2010. We fix $p = 2$, $\alpha = 3$ and $\varpi = 0.47$. The truncation rate is $\bar{u} = \int_{t, \iota} \hat{\theta}_{\text{wsp}} \alpha \sqrt{BV_t \Delta^\varpi}$. 

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Table 7: Testing for jumps when volatility is periodic

<table>
<thead>
<tr>
<th>Correction?</th>
<th>EUR/USD</th>
<th>USD/JPY</th>
<th>GBP/USD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega = 0.47$</td>
<td>$j = 0.10$</td>
<td>$j = 0.05$</td>
<td>$j = 0.05$</td>
</tr>
<tr>
<td>No</td>
<td>189 [0.08] 162 [0.07] 112 [0.05]</td>
<td>136 [0.05] 101 [0.04] 75 [0.03]</td>
<td>170 [0.07] 146 [0.06] 103 [0.04]</td>
</tr>
<tr>
<td>WSD</td>
<td>156 [0.06] 125 [0.05] 82 [0.03]</td>
<td>115 [0.05] 91 [0.04] 66 [0.03]</td>
<td>141 [0.06] 116 [0.05] 74 [0.03]</td>
</tr>
</tbody>
</table>

| $\omega = 0.48$ | $j = 0.10$ | $j = 0.05$ | $j = 0.05$ |
| No | 198 [0.08] 169 [0.07] 119 [0.05] | 144 [0.06] 113 [0.05] 78 [0.03] | 182 [0.07] 154 [0.06] 116 [0.05] |
| WSD | 169 [0.07] 136 [0.05] 91 [0.04] | 124 [0.05] 96 [0.04] 69 [0.03] | 150 [0.06] 130 [0.05] 81 [0.03] |

Notes: The table shows the results of the base and periodicity-robust Ait-Sahalia and Jacod (2009b) jump tests. We use the non-parametric WSD method to correct the test for periodic volatility. The table reports the number of detected daily jumps and the jump proportion in brackets (100×#jumps/#days). $j$ denotes the significance levels of the test. We report the results for $k = 2$. The sample period and other parameter values are same as in Table 3 (i.e., $\alpha = 4$ and $p = 4$).

![ACF of the absolute EUR/USD, USD/JPY, and GBP/USD returns at 1-minute sampling frequency. The number of lags corresponds to 5 days of 1-minute observations.](image)
Figure 2: Mean absolute 1-min EUR/USD (left), USD/JPY (middle), and GBP/USD (right) returns on whole sample. The X-axis represents the intraday periods in EST, from 16:01 EST of day $t - 1$ to 16:00 EST of day $t$.

Figure 3: The three panels display the number of estimated large increments retained by $I\{\cdot\}$ per intraday period (see Equation (5)) for Cases I, II and III, respectively, as described in the text. The number of simulations is 1000 with $T = 250$ and $M = 1440$ (i.e., 1-minute sampling frequency). In the upper and middle panels, the truncation rate is $u = \alpha \Delta \varpi$. In the lower panel, the truncation rate is $\tilde{u} = f_{i,t} \alpha \Delta \varpi$. We set $\varpi = 0.47$ and $\alpha = 2$. The details of the simulation study are available upon request.
Brownian Motion: Present or Not?

Figure 4: The figure shows the empirical distribution of the non-standardized test statistic $\hat{S}_W(p, \tilde{u}, k, \Delta_t)$ in Equation (6) for $k = 2$ (left panels) and $k = 3$ (right panels). The sample covers January 1, 2000 to March 12, 2010. We set $p = 1$ and $\alpha = 8$. The sampling frequency is 1-minute. Under the null that Brownian motion is present in the data, the test statistic in the left (right) hand panels should converge in probability to $1.41$ ($1.73$). Under the alternative, the test statistic should converge to $1$. 
Figure 5: The figure shows the empirical distributions of the non-standardized (upper panels) and standardized (lower panels) test statistics $S_J(p, k, \Delta_t)$ in Equation (7) for $k = 2$, $p = 4$, $\alpha = 4$. The sample covers January 1, 2000 to March 12, 2010. The sampling frequency is 1-minute. Under the null that there are no jumps, the test statistic in the upper panel should converge in probability to 2. Under the alternative, the test statistic should converge to 1.

Figure 6: The figure shows the empirical distribution of the non-standardized test statistic $S_{FA}(p, \bar{u}, k, \Delta_t)$ in Equation (8) for the truncation parameter $\alpha = 6$ (left panels) and $\alpha = 10$ (right panels). We set $k = 2$, $p = 4$. The sample covers January 1, 2000 to March 12, 2010. The sampling frequency is 1-minute. Under the null of finite activity jumps, the test statistic should converge in probability to 2. Under the alternative, the test statistic should converge to 1.
Brownian Motion: Present or Not?

Ho: Brownian Motion - Empirical histogram of $S_W$

- Brownian Motion Present
- No Brownian
- Noise Dominates

Jumps: Present or Not?

Ho: no jumps - Empirical histogram of $S_J$

- No Jumps
- Jumps Present
- Noise Dominates

Jumps: Finite or Infinite?

Ho: finite activity - Empirical histogram of $S_{FA}$

- Finite Activity Jumps
- Infinite Activity Jumps
- Noise Dominates

Figure 7: Characteristics of the EUR/USD returns. The sample covers January 1, 2000 to March 12, 2010. The sampling frequency is 1-minute. Vertical bars denote the asymptotic probability limit of the statistics under the respective null hypotheses.

Quadratic variation of the EUR/USD

collapse of Lehman Brothers

Figure 8: The relative magnitudes of the continuous and jump components of the EUR/USD exchange rate. We fix $p = 2$, $\varpi = 0.47$, $\epsilon = 0.9$ and $\alpha = 3$. The sampling frequency is 1-minute.
Figure 9: The figure shows the number of estimated jumps retained by $J(\bullet)$ per intraday period for EUR/USD, USD/JPY and GBP/USD returns. In the left panels, the truncation rate is constant over the day, $u = \alpha \sqrt{BV_t} \Delta \omega$. In the right panels, the truncation rate varies with intraday periodicity, $\tilde{u} = f_{WSD} \alpha \sqrt{BV_t} \Delta \omega$. 

\[ \]