Externalities, Endogenous Productivity, and Poverty Traps

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Externalities, Endogenous Productivity, and Poverty Traps*

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Abstract

We present a version of the neoclassical model with an endogenous industry structure. We obtain multiple steady-state equilibria with an arbitrarily small degree of increasing returns to scale. While the most productive firms operate across all the steady states, in a poverty trap less productive firms operate as well. This results in lower average firm productivity and total factor productivity. A calibrated version of our model displays sizable differences in TFP and output across steady state equilibria.

JEL: L16, O11, O33, O40

Keywords: endogenous productivity, multiple equilibria, poverty traps

*Any views expressed are our own and do not necessarily reflect the views of the Federal Reserve Bank of St. Louis or the Federal Reserve System. Corresponding author: Riccardo DiCecio, dicecio@stls.frb.org.
1 Introduction

The role of an endogenous industry structure as a powerful amplifying mechanism is well established in the literature, building on the seminal work of Hopenhayn (1992). Endogeneity of the industry structure allows to obtain countercyclical markups and indeterminacy (Jaimovich, 2007). It can rationalize the business cycle characteristics of entry, profits, and markups, while accounting for the standard business cycle moments (Bilbiie, Ghironi, and Melitz, 2012). It provides an amplification mechanism for differences in prices of investment goods to translate into large differences in output across countries (Armenter and Lahiri, 2012). Finally, the endogenous response of the industry structure to various institutional and policy failures, such as barriers to entry, induces misallocation of inputs across firms.

We show that endogenizing the industry structure can lead to an extreme form of amplification: We obtain multiple, locally stable steady states with an arbitrarily small degree of increasing returns to scale. We introduce heterogeneous firms à la Hopenhayn (1992) in a standard neoclassical model. Many ex-ante identical potential firms obtain a random productivity draw upon payment of an entry cost. Only firms productive enough to pay an overhead labor cost choose to operate. As long as the productivity distribution features a disproportionately high number of similarly unproductive firms, arbitrarily small demand externalities (increasing returns) induce multiple steady state equilibria. Consider a steady state with a high productivity cutoff and a large capital stock. The high cutoff implies that firms’ average productivity is high. A large capital stock and high productivity imply that the wage is high, as is the operating cost. A high operating cost makes low productivity firms unprofitable, effectively cleansing the pool of firms. This justifies the cutoff being high in the first place. Since only high productivity firms are operating, TFP is high. Conversely, in a steady state where capital is low and lower productivity firms are operating, the wage is low and lower profits are sufficient to cover the operating cost. Low productivity firms sully

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2Galí (1995) obtains multiple equilibria and poverty traps in a model where large increasing returns stem from endogenous markups. However, subsequent empirical evidence suggests that the degree of increasing returns is small.
the pool of producers, leading to lower TFP and capital.\footnote{In a good equilibrium high productivity firms produce more than in a bad equilibrium, despite facing a higher wage and the same interest rate. This is optimal because firms face a higher demand for their goods, which offsets higher factor prices.}

A calibrated version of our model is able to generate sizeable differences across two stable steady states. For example, with 25 percent increasing returns and a 65 percent variable capital share,\footnote{A capital share this high is meant to capture the fact that our model does not include explicitly other forms of capital besides physical capital.} TFP in the high steady state is 45 percent larger than in the low steady state. The corresponding difference in levels of output across steady states is larger than 600 percent.

Empirical motivation for our work stems from the studies of the determinants of cross-country income differences of Klenow and Rodriguez-Clare (1997), Hall and Jones (1999), and Caselli (2005). These authors find that income differences can be attributed, at least in part, to differences in TFP. Previous studies of poverty trap models with endogenous TFP pointed to the failure of adopting the most productive technology as the cause of low TFP and income in poor countries.\footnote{See, for example, Murphy, Shleifer, and Vishny (1989) and Ciccone and Matsuyama (1996). For comprehensive reviews of the literature on poverty traps, see Matsuyama (2005) and Azariadis and Stachurski (2005).} However, there is evidence pointing to the fact that differences in TFP across economies are related to the lowest level of firms’ productivity. Comin and Hobijn (2010) take a comprehensive look at the uses of various technologies as determinants of TFP and find that the key is not when new, better technologies are adopted, but when old, obsolete ones are relinquished. Also, the empirical evidence on the importance of international knowledge spillovers summarized in Klenow and Rodriguez-Clare (2005) suggests that all countries can easily access frontier technologies. Banerjee and Duflo (2005) cite the McKinsey Global Institute (2001) report on India, which finds that while larger production units (firms) use relatively new technologies, smaller (in home) production units have low productivity.

Finally, Barseghyan and DiCecio (2011) analyze a model similar to the one presented here and show that the observed differences in entry costs across countries generate sizeable differences in output and TFP.\footnote{The model in Barseghyan and DiCecio (2011) has a unique steady state. The key difference is that the distribution of productivity is calibrated to match the distribution of firms and employment by size in the U.S. and it does not feature a “mass point” of similar, low-productivity firms.}
and Mukoyama (2012) show that entry regulations and firing costs have important effects on cross-country differences in TFP and output. Poschke (2010) argues that small differences in entry costs, by affecting firms’ technology choices, can explain a substantial part of the TFP differences across similarly developed economies.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 studies its steady state and dynamics properties. Section 4 the quantitative implications of a calibrated version of the model. We conclude in Section 5. We provide proofs in an appendix.

2 The Model

Our model is a variant of the neoclassical growth model. The model departs from the standard framework by having a richer structure of the production side of the economy. We model firms following Lucas (1978), Jo-ivanovic (1982), and Hopenhayn (1992). Firms are heterogenous: each firm has monopoly power over the good it produces, and firms have different productivity levels. Two features of the production side of the economy are crucial for the results of the paper:

1. a sunk entry cost;

2. an operating cost: in addition to capital and labor used directly in production, firms pay for a fixed amount of overhead labor.

A part of the entry costs stems from satisfying different official regulatory requirements (see Djankov, La Porta, Lopez-de-Silanes, and Shleifer, 2002). In addition, in some countries, entry requires significant side payments to local officials.7 Entry cost may also include expenses related to acquisition of firm-specific capital,8 acquisition of appropriate technology,9 and market research.

The operating cost typically refers to overhead labor and expenses that are lumpy in nature (e.g., renting a physical location). According to the

7 In the case of Peru, this is documented by De Soto (1989).
8 Ramey and Shapiro (2001) show that in some instances the specificity of firm capital is so extreme that the sale price of such capital after a firm has been dissolved is only a small fraction of the original cost.
9 See, for example, Atkeson and Kehoe (2005).
findings of Domowitz, Hubbard, and Petersen (1988), in U.S. manufacturing plants, the overhead labor accounts for 31 percent of total labor. Ramey (1991) suggests that overhead labor is about 20 percent. The preferred estimate of overhead inputs in Basu (1996) is 28 percent.

We also assume that firms learn their productivity only after a sunk entry cost is paid. This assumption reflects very high uncertainty faced by entering firms and is a stylized fact documented, for example, by Klette and Kortum (2004).

2.1 Households

There is a continuum of households that supply a fixed amount of labor, consume, invest, and own all firms in the economy. The problem of the representative household is given by

$$\max_{t=0}^\infty \beta^t U(C_t), \ \beta \in (0, 1)$$

s.t. $C_t + I_t = r_t K_t + w_t + \Pi_t + T_t$,

$I_t = K_{t+1} - (1 - \delta) K_t$,

where $C_t$ denotes consumption, $I_t$ is investment, $K_t$ denotes the total household capital, $r_t$ is the rental rate on capital, and $w_t$ is the wage. $\Pi_t$ is the firms’ profits, and $T_t$ is a lump-sum transfer from the government; $\beta$ and $\delta \in (0, 1)$ are the discount rate and depreciation rate, respectively. We assume a constant elasticity of substitution utility function with elasticity $\sigma > 0$.

2.2 Firms

2.2.1 Final Good Producers

The final consumption good in this economy is produced by perfectly competitive firms, according to the following production function:

$$Y_t = \left[ \int_0^{\mu_t} \left[ y_t(i) \right]^{\frac{1}{\lambda}} di \right]^\lambda,$$

\hspace{1cm} \footnote{We assume that the household inelastically supplies one unit of labor.}
where $\mu_t$ is the number of intermediate goods produced in the economy, $\lambda$ is a constant which is greater than 1, and $y_t(i)$ is the quantity of the intermediate good $i$. Let $p_t(i)$ be the price of the $i^{th}$ intermediate good relative to the final good. Then, the maximization problem of the final good producer can be written as

$$\Pi_t^{FF} = \max \left[ \int_0^{\mu_t} [y_t(i)]^{\frac{1}{\lambda}} di \right] - \int_0^{\mu_t} p_t(i)y_t(i)di,$$

and the first-order optimality condition implies that the demand function for the $i^{th}$ intermediate good is given by

$$p_t(i) = \left[ \frac{y_t(i)}{Y_t} \right]^{-\frac{\lambda-1}{\lambda}}.$$

### 2.2.2 Intermediate Goods Producers

A firm in the intermediate goods sector lives for one period and is profit maximizing. All firms are ex-ante identical. There is a sunk entry cost, $\kappa$. Once the entry cost is paid, a firm gains the ability to produce an intermediate good. The firm has monopoly power over the good it produces. Next, the firm draws a productivity parameter $A(j)$, where $j$ is drawn from an i.i.d. uniform distribution over $[0,1]$. The production function for the good $j$ is given by

$$[A(j)]^{1-\gamma} [k(j)^{\alpha}n(j)^{1-\alpha}]^\gamma,$$

where $k(j)$ and $n(j)$ denote capital and labor, respectively. The productivity parameter differs among the firms. A firm with a higher index has a higher productivity parameter, i.e., $A(j) > A(i)$ for $j > i$. In addition, function $A(j)$ is assumed to be continuous, and $A(0) = 0$. The parameter $\gamma \in (0, \lambda)$ determines the degree of returns to scale in variable inputs.\(^{12}\) The parameter $\alpha$ is between zero and 1.

If a firm decides to produce, it must incur an operating cost in terms of wages paid to $\phi$ units of overhead labor. Consider the decision of a firm born

---

\(^{11}\)We assume that $\kappa$ is denominated in consumption units and that all entry-cost payments are rebated to the households in a lump-sum fashion. Alternatively, one can model $\kappa$ as a sunk investment, i.e., in units of capital. Such a formulation would not change any of our results, but it would make the exposition more cumbersome.

\(^{12}\)This is what Lucas (1978) calls managers’ span of control.
in time $t$ with a draw $j$. If it decides to produce, its profits are

$$\pi_t^P(j) = \max_{k_t(j), n_t(j)} p_t(j) y_t(j) - r_t k_t(j) - w_t [n_t(j) + \phi]$$

s.t. $y_t(j) = [A(j)]^{1-\gamma} \left[k_t(j) n_t^{1-\alpha}(j)\right]^\gamma, \quad p_t(j) = \left[y_t(j) \frac{1}{Y_j}\right]^{1-\alpha}$. \(3\)

The decision to produce or not depends on whether $\pi_t^P(j)$ is positive. Therefore, the $j^{th}$ firm’s profits, $\pi_t^F(j)$, are given by

$$\pi_t^F(j) = \max\{\pi_t^P(j), 0\}. \quad 4$$

Free entry implies that, in equilibrium, firms’ expected profits must be equal to the entry cost, $\kappa$:

$$\int_0^1 \pi_t^F(j) dj = \kappa. \quad 5$$

### 2.2.3 Firms’ average productivity

We derive the equilibrium relationship between the firms’ average productivity and the operating cost. First, we determine the lowest productivity level necessary for a firm to decide to produce. The existence of economy-wide competitive factor markets implies that in equilibrium, the gross profits, capital, and labor ratios of any two firms are equal to their (scaled) productivity ratio:

$$\frac{p_t(j) y_t(j)}{p_t(i) y_t(i)} = \frac{k_t(j)}{k_t(i)} = \frac{n_t(j)}{n_t(i)} = \frac{a(j)}{a(i)}, \ \forall i, j, \quad 6$$

where $a(j) \equiv A(j)^{\frac{1-\gamma}{\lambda}}$. The first-order conditions of problem (3) imply that profits from producing are equal to the firm’s share of the gross profits $(1 - \frac{\gamma}{\lambda})$ minus the operating cost\(^{13}\)

$$\pi_t^F(j) = \left(1 - \frac{\gamma}{\lambda}\right) p_t(j) y_t(j) - \phi w_t.$$  

Clearly $\pi_t^F(j)$ is increasing in $j$ and, since $a(j) = 0$, there exists a cutoff firm, $J_t$, which is indifferent between producing or not:

$$\left(1 - \frac{\gamma}{\lambda}\right) p_t(J_t) y_t(J_t) = \phi w_t. \quad 7$$

\(^{13}\)Later on, with some abuse of terminology, we will refer to $(1 - \frac{\gamma}{\lambda})p_t(j)y_t(j)$ as firms’ gross profits.
Firms with indices higher than \( J_t \) will produce, and those with lower indices will not. Thus, firms’ zero profit condition in (5) can be written as

\[
\kappa = \phi w_t \int_{J_t}^{1} \left[ \frac{a(j)}{a(J_t)} - 1 \right] dj. \tag{8}
\]

The previous equation defines the cutoff \( J_t \) as a function of the operating cost \( \phi w_t \). An increase in the cutoff \( J_t \) has two effects: Profits of every firm decline, and the number of producing firms as a fraction of entering firms declines. Therefore, the right-hand side of (8) is decreasing in \( J_t \) and increasing in the fixed cost \( \phi w_t \). Hence, the cutoff is increasing in the operating cost and average firm productivity, \( \bar{a}(J_t) = \int_{J_t}^{1} \frac{a(j) dj}{1-J_t} \), is an increasing function of the operating cost.

### 2.2.4 Entry and the number of operating firms

Entry in this model refers to the number of firms that pay the entry cost, \( \kappa \). The number of entering firms differs from the number of operating firms because only a fraction of entrants will have productivity high enough to operate: the pool of producers consists only of firms which have an index higher than \( J_t \). In particular, let \( \nu_t \) denote the number of entering firms and \( \mu_t \) the number of operating firms. Then

\[
\mu_t = \nu_t \int_{J_t}^{1} dj. \tag{9}
\]

### 2.3 Aggregate Output and TFP

Aggregate capital, labor, and output can be expressed as

\[
K_t = \nu_t \int_{J_t}^{1} k_t(j) dj, \tag{10}
\]

\[
N_t = \nu_t \int_{J_t}^{1} [n_t(j) + \phi] dj, \tag{11}
\]

\[
Y_t = \left[ (\mu_t \bar{a}(J_t))^{(\lambda-\gamma)} u_t^{(1-\alpha)\gamma} \right] K_t^{\alpha\gamma} (N_t)^{(1-\alpha)\gamma}, \tag{12}
\]

where \( u_t \) is the fraction of labor used in production. Finally, the rental rate on capital, the wage rate, and the equation determining the cutoff \( J_t \) can be
written as

\[ r_t = \alpha \frac{\gamma}{\lambda} \frac{Y_t}{K_t}, \]  
\[ w_t = (1 - \alpha) \frac{\gamma}{\lambda} \frac{Y_t}{u_t N_{tt}}, \]  
\[ (1 - \frac{\gamma}{\lambda} \frac{a(J_t)}{a(J_t)} \frac{Y_t}{(1 - u_t)N_t} = w_t. \]

2.4 Closing the Model

The resource constraint is given by

\[ C_t + K_{t+1} = Y_t + (1 - \delta)K_t. \]  

The only role the government has in the model is to collect the entry fees \( \nu_t \kappa \) from firms and rebate them lump-sum to the households:

\[ T_t = \nu_t \kappa. \]

Profits and the labor market clearing condition are

\[ \Pi_t = \Pi_t^{FF}, \]
\[ N_t = 1. \]

The definition of equilibrium is standard.

3 Steady States, Dynamics, and Some Extensions

In this section we analyze the existence and stability of the steady states, and discuss some extensions to our basic model. The main finding is that there can be multiple stable steady states for an arbitrarily low degree of increasing returns to scale.

Intuitively, if there are multiple steady states, their existence is due to the endogenous productivity mechanism embedded in the model. Equation [8] relates the cutoff \( J \) to the operating cost, \( \phi w_t \). The integral on the right-hand side of this equation is decreasing in \( J_t \). Thus, a higher operating cost
translates into a higher cutoff and vice versa. In an economy where the operating cost is high, higher (gross) profits are required to cover this cost. Only high productivity firms can generate such profits. Therefore, the lower productivity firms are forced out from the pool of producers. As the operating cost increases, the entry cost relative to operating cost falls, allowing more firms to enter. However, only the ones with higher productivity firms are profitable enough to operate. This relation between the operating cost and the cutoff provides economic intuition for the existence of multiple steady states. If multiple steady states exist, then one steady state has high capital and only high productivity firms are operating. High capital stock and high productivity imply that the wage rate is high, and so is the operating cost. A high operating cost, in turn, justifies why only high productivity firms are operating. Finally, since productivity is high, a high capital stock is necessary to equate the return on capital to $1/\beta$. Conversely, in a “low” steady state, the capital stock and firms’ average productivity are low. This implies a low operating cost which allows lower productivity firms to operate. Since firms’ average productivity is low, the capital stock must be low to have the return on capital equal to $1/\beta$. A firm productive enough to be active in different steady states produces more in a good steady state than in a bad one, despite a higher wage and the same interest rate. This is optimal because it faces a higher demand for its goods, which offsets the contractionary pressure of higher factor prices.

3.1 Steady States

We present the argument formally in Propositions 1 and 2; we provide proofs in Appendix A. First, note that the number of firms is proportional to the total amount of labor used to cover the fixed cost:

$$\mu_t = \frac{1 - u_t}{\phi} N_t.$$

Therefore, aggregate output is given by

$$Y_t = TFP_t K_t^{\alpha \gamma},$$

(20)

where total factor productivity is

$$TFP_t = \phi^{\gamma - \lambda} [\bar{\alpha} (J_t)]^{\lambda - \gamma} (1 - u_t)^{\lambda - \gamma} u_t^{(1 - \alpha) \gamma}.$$

(21)
There are two components of TFP: firms’ average productivity \([\bar{a} (J_t)]^{\lambda - \gamma}\) and the term \(u_t ((1-\alpha)^{\gamma} (1 - u_t) + \gamma,\) which we call the labor allocation component. Firms’ average productivity is increasing in \(J_t\). The labor allocation component is a function of \(J_t\) as well, though not necessarily monotonic. The effect of \(J_t\) on average productivity dominates and \(TFP_t\) is increasing in \(J_t\).

The following proposition allows us to present the model economy in a more familiar, neoclassical framework.

**Proposition 1** The aggregate production function in (12) and total factor productivity (21) are increasing in the cutoff \(J_t\): the cutoff \(J_t\), the wage \(w_t\), and the aggregate output \(Y_t\) are all increasing functions of capital \(K_t\). The rate of return on capital \(R_t = (r_t + 1 - \delta)\) is a function of \(K_t\).

**Proof.** See Appendix A.

The proposition above implies that the dynamics of the economy can be characterized by the following system of difference equations,

\[
\begin{align*}
(c_{t+1}/c_t)^{1/\sigma} &= \beta R(K_{t+1}), \\
C_t + K_{t+1} &= Y(K_t) + (1 - \delta)K_t,
\end{align*}
\]

(22)

plus a transversality condition. We now turn to the existence and multiplicity of steady states.

**Proposition 2** The economy characterized by the system in (22) generically has an odd number of steady states. For any \(\lambda > 1\), there exists a distribution of productivities, \(a (j)\), and a value of \(\kappa\) such that the system (22) has multiple steady-state equilibria.

**Proof.** (sketch) Straightforward manipulations of the first order conditions lead to the following relation between the rate of return on capital and the cutoff \(J\) in steady state:

\[
\kappa = \Phi(J) \eta r^{\frac{\alpha \gamma - 1}{\alpha \gamma}},
\]

(23)

where

\[
\Phi(J) = \left[ \frac{\bar{a}(J)}{\bar{a}(J) + \frac{\lambda - \gamma}{(1-\alpha)\gamma} a(J)} \right]^{\frac{1-\alpha}{\lambda - \gamma}} a(J)^{\frac{\lambda - \gamma}{1-\alpha}} \int_{J}^{1} \left[ \frac{a(j)}{a(J)} - 1 \right] dj
\]

(24)

and \(\eta\) is a constant. Since \(\Phi(J)\) is continuous and \(\lim_{J \to 0^+} \Phi(J) = +\infty\), \(\Phi(1) = 0\), there always exists at least one value of \(J \in (0, 1)\) which satisfies
Figure 1: A graphical depiction of equation (23).
equation (23). To have multiple solutions, it is necessary for the function $\Phi$ to be non-monotone (see Figure 1).

In Appendix A we show that there always exists a function $a(j)$ such that this is the case. With a non-monotone $\Phi(J)$, it is trivial to find a value of $\kappa$ such that equation (23) has multiple solutions. Note that equation (23) implies that if $\Phi(J)$ is increasing, so is $r(K)$. The necessary condition for the existence of multiple steady states is that for some values of $K$ the return on capital must be increasing. The properties of the function $\Phi$ mimic those of firms’ expected profits, i.e., the right-hand side of the zero-profit condition (8). A necessary condition for existence of multiple steady state is that firms’ expected profits are increasing in $J$. An increase in the cutoff $J$ has two opposite effects. On one hand the wage rate increases, increasing expected profits. On the other hand, a higher $J$ implies a lower value of the integral on the right-hand side of (8). For expected profits to rise with $J$, the increase in the wage has to dominate the fall in the value of the integral. A sufficient condition for this is that $\partial a(J)/\partial J \gg \partial a(J)/\partial J$: A relatively high derivative of the average productivity guarantees a strong positive effect on TFP and the wage rate, while a relatively low derivative of the function $a(J)$ implies a mild negative response of the integral term. A function $a(j)$ which is sufficiently flat on some interval and increases rapidly for higher values of $j$ has this property.

Given Propositions 1 and 2 it is easy to establish that the “high $J$” economy has higher capital stock, higher output, higher total factor productivity, and higher average productivity for firms.

3.2 Dynamics

The following proposition characterizes the behavior of the economy around the steady state(s).\footnote{An analysis of the global dynamics of our model is beyond the scope of this paper, and we refer the reader to Galí (1995) and Slobodyan (2005).}

**Proposition 3** Steady states with an odd index are saddles. Steady states with an even index can be classified as follows:

1. *source*, if $Y' - \delta > \frac{\sigma CR'}{R}$ and $\left( (Y' - \delta) - \frac{\sigma CR'}{R} \right)^2 > 4 \frac{\sigma CR'}{R}$;

2. *unstable spiral*, if $Y' - \delta > \frac{\sigma CR'}{R}$ and $\left( (Y' - \delta) - \frac{\sigma CR'}{R} \right)^2 < 4 \frac{\sigma CR'}{R}$;
3. sink, if \( Y' - \delta < \frac{\sigma CR'}{R} \) and \( [(Y' - \delta) - \frac{\sigma CR'}{R}]^2 > 4 \frac{\sigma CR'}{R} \); 

4. stable spiral, if \( Y' - \delta < \frac{\sigma CR'}{R} \) and \( [(Y' - \delta) - \frac{\sigma CR'}{R}]^2 < 4 \frac{\sigma CR'}{R} \).

**Proof.** See appendix A. 

For the parameter values we consider in the rest of the paper, we obtain three steady states, with the odd steady state unstable (cases 1 and 2 in Proposition 3). In comparing output and TFP across steady states we will focus on the two stable steady states.

### 4 Properties of the Model

In this section we discuss some quantitative properties of our model by computing differences in output and TFP in a calibrated version of the model. Also, we perform sensitivity analysis on the degree of increasing returns to scale and capital share parameters. We conclude that the differences between high and low steady states are sizable.

#### 4.1 Model Calibration

Our model contains seven parameters (\( \beta, \delta, \lambda, \gamma, \alpha, \phi, \kappa \)), plus any additional parameters determining the function \( a(j) \). The model’s implications are robust to the choice of \( \beta \) and \( \delta \) for the commonly used values of \( \beta \in (0.94, 0.99) \) and \( \delta \in (0.08, 0.12) \). Therefore, we set \( \beta = 0.95 \) and \( \delta = 0.10 \). The parameters \( \lambda, \gamma, \) and \( \alpha \) deserve more consideration.

The first parameter, \( \lambda \), governs the degree of increasing returns to scale in the economy. There has been a large debate in the recent literature on the magnitude of increasing returns in the economy. While earlier researchers (most notably, Hall [1988]) suggested that there are large increasing returns to scale in the economy, subsequent work has shown that the returns to scale can be best described as constant or at most moderately increasing. The latest estimates of \( \lambda \) are probably those constructed by Laitner and Stolyarov [2004]. Their preferred point estimate is \( \lambda = 1.1 \), with confidence interval (1.03, 1.2). These figures are close to the estimates of Bartelsman, Caballero, and Lyons [1994], Burnside [1996], Burnside, Eichenbaum, and Rebelo [1995], Basu [1996], Basu and Fernald [1997], and Harrison [2003]. Hence, we calibrate our model with \( \lambda = 1.1 \).
The next parameter, $\gamma$, represents the share of output that goes to capital and labor used directly in production, for a given value of $\lambda$. Note that in the model there is a difference between aggregate returns to scale and firm level returns to scale. While at the aggregate level there are increasing returns to scale, at the firm level, as long as $\gamma < 1$, the returns to scale in variable inputs are decreasing. In our model, heterogenous productivity leads to a heterogenous degree of returns to scale in all inputs. For high productivity firms, the decreasing returns to scale in variable inputs dominate the increasing returns to scale effect of the fixed cost; for low productivity firms, it is the opposite. These observations are broadly consistent with empirical findings of Basu (1996), and Basu and Fernald (1997). As a benchmark, we consider $\gamma = 0.85\lambda$, which is the preferred value of Atkeson and Kehoe (2005) and is very close to the estimated value of 0.84 in Basu (1996).

The choice of the next parameter, $\alpha$, depends on the interpretation of entrepreneurs’ income share, $(1 - \gamma/\lambda)$. If part of it is considered capital income, then the share of output that remunerates variable capital, $rK/Y$, is less than the overall share. A commonly used rule is to apportion to capital a fraction $\alpha$ of the entrepreneurs’ income share. Summing up, the capital share of output is $[rK/Y + \alpha(1 - \gamma/\lambda)] = \alpha$. We set $rK/Y = 0.4$, which implies $\alpha = 0.47$.

We have shown that for some functions $a(j)$ there will be multiple stable steady states. The key property of the function $a(j)$ that generates multiplicity of equilibria is that $a_{J}$ strongly dominates $a_{J}$ for some $J$. A function that has this property is one that is sufficiently flat on some interval $(J_{1}, J_{2})$. The larger this interval is, the farther apart the stable steady states are from each other. In terms of firms’ productivity distribution, this translates into the lower steady state having a large number of firms with nearly the same low productivity. Hence, we parameterize the function $a(j)$ as follows:

$$a(j) = \begin{cases} 
  j, & \text{if } j \leq J_{1} \\
  J_{1} + b \left( \frac{j - J_{1}}{J_{2} - J_{1}} \right)^{N_{1}}, & \text{if } J_{1} < j \leq J_{2} \\
  (J_{1} + b) \left( \frac{j}{J_{2}} \right)^{N_{2}}, & \text{if } j > J_{2}.
\end{cases}$$

(25)

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15 See Kim (2004) for a detailed discussion on different ways of modeling increasing returns to scale.

16 See proof of Proposition 2 in Appendix A.
Table 1: Parameter Values: function $a(j)$ and fixed cost

<table>
<thead>
<tr>
<th></th>
<th>$J_1$</th>
<th>$J_2$</th>
<th>$b$</th>
<th>$N_1$</th>
<th>$N_2$</th>
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</tr>
</tbody>
</table>

We normalize $\phi$ to 1\(^{17}\) and we choose $\kappa$ and the five parameters pinning down the productivity distribution ($J_1, J_2, N_1, N_2, b$) so that the distribution of firms by size implied by our model in the two stable steady state is as close as possible to the distributions of firms by size in the average Least Developed Country (LDC) and in the U.S. (see Tybout, 2000, Table 1). Notice how in the average LDC the distribution of firms by size is characterized by a much higher share of small firms than in the U.S.

![Figure 2: Employment Shares: model versus data.](image)

Figure 2 portrays the distributions of firms by size for the U.S. and the average LDC (right column), together with the distributions for the high and

\(^{17}\)Notice that for our results only $\kappa/\phi^{\lambda-\alpha}$ matters (see equations \((23)\) and \((33)\) in Appendix A).
Figure 3: Calibrated function $a(j)$.

low steady states of our model (left column).

Figure 3 reports the function $a(j)$, which minimizes the distance between the model distributions and their empirical counterparts.

For this calibration, TFP and output differ across steady states by a factor of 1.1 and 1.21, respectively.

4.2 Sensitivity Analysis

In this section we conduct sensitivity analysis of the baseline calibration by analyzing how varying the degree of increasing returns to scale and the capital share maps into differences across the high and the low steady states of our model.

Tables 2-5 present the ratios of values of output and TFP levels for the two stable steady states for different parameter values. In the first column of Tables 2-5 we report the worst-case scenario of no increasing returns to scale, together with the most favorable function $a(j)$. In the remaining columns we maintain $a(j)$ fixed to the calibrated function discussed above, and we
analyze the effect of varying the capital share and the ratio of $\gamma$ to $\lambda$.

| $\lambda$ | 1.0 $|$ 1.01 $|$ 1.05 $|$ 1.1 $|$ 1.15 $|$ 1.2 $|$ 1.25 |
|----------|---------|---------|---------|---------|---------|---------|
| 0.95 $\lambda$ | 1.05 $|$ 1.03 $|$ 1.03 $|$ 1.03 $|$ 1.03 $|$ 1.03 $|$ 1.04 |
| $\gamma$ | 0.9 $\lambda$ | 1.12 $|$ 1.06 $|$ 1.06 $|$ 1.07 $|$ 1.07 $|$ 1.08 $|$ 1.08 |
| 0.85 $\lambda$ | 1.19 $|$ 1.10 $|$ 1.10 $|$ 1.11 $b$ $|$ 1.12 $|$ 1.13 $|$ 1.14 |
| 0.8 $\lambda$ | 1.28 $|$ 1.14 $|$ 1.16 $|$ 1.17 $|$ 1.18 $|$ 1.20 $|$ 1.22 |

$^a$ Theoretical upper bound for $\lambda \to 1$. $^b$ Benchmark calibration.

Table 2: Relative TFP for different values of $\gamma$ and $\lambda$ ($rK/Y = 0.4$).

| $\lambda$ | 1.0 $|$ 1.01 $|$ 1.05 $|$ 1.1 $|$ 1.15 $|$ 1.2 $|$ 1.25 |
|----------|---------|---------|---------|---------|---------|---------|
| 0.95 $\lambda$ | 1.09 $|$ 1.05 $|$ 1.05 $|$ 1.06 $|$ 1.06 $|$ 1.07 $|$ 1.07 |
| $\gamma$ | 0.9 $\lambda$ | 1.20 $|$ 1.10 $|$ 1.11 $|$ 1.12 $|$ 1.14 $|$ 1.15 $|$ 1.17 |
| 0.85 $\lambda$ | 1.33 $|$ 1.17 $|$ 1.19 $|$ 1.21 $b$ $|$ 1.23 $|$ 1.26 $|$ 1.30 |
| 0.8 $\lambda$ | 1.50 $|$ 1.25 $|$ 1.28 $|$ 1.32 $|$ 1.37 $|$ 1.42 $|$ 1.48 |

$^a$ Theoretical upper bound for $\lambda \to 1$. $^b$ Benchmark calibration.

Table 3: Relative output for different values of $\gamma$ and $\lambda$ ($rK/Y = 0.4$).

| $rK/Y$ | 0.36 $|$ 1.19 $|$ 1.10 $|$ 1.10 $|$ 1.11 $|$ 1.12 $|$ 1.13 $|$ 1.13 |
|---------|---------|---------|---------|---------|---------|---------|---------|
| 0.4 $|$ 1.19 $|$ 1.10 $|$ 1.10 $|$ 1.11 $b$ $|$ 1.12 $|$ 1.13 $|$ 1.14 |
| 0.55 $|$ 1.19 $|$ 1.10 $|$ 1.11 $|$ 1.12 $|$ 1.13 $|$ 1.14 $|$ 1.16 |
| 0.6 $|$ 1.20 $|$ 1.11 $|$ 1.12 $|$ 1.13 $|$ 1.14 $|$ 1.16 $|$ 1.18 |
| 0.65 $|$ 1.22 $|$ 1.12 $|$ 1.13 $|$ 1.15 $|$ 1.19 $|$ 1.26 $|$ 1.45 |

$^a$ Theoretical upper bound for $\lambda \to 1$. $^b$ Benchmark calibration.

Table 4: Relative TFP for different value of $\lambda$ and $rK/Y$ ($\gamma = 0.85$).

The limiting case of $\lambda$ equal to 1 has the least favorable implications for the existence of multiple steady states, because the model essentially collapses to the standard neoclassical model. It is important to see how large
Theoretical upper bound for $\lambda$. The condition for the existence of multiple steady states translates to $a(J)$ being flat over some interval. In this case, the extremes of this interval correspond to the two steady-state values of $J$. This implies that the ratios of total factor productivity, capital, and output levels in the two stable steady states are bounded:

$$\frac{TFP_H}{TFP_L} \leq \left( \frac{1 - rK/Y}{\gamma - rK/Y} \right)^{1-rK/Y},$$

$$\frac{K^H}{K^L} \leq \frac{1 - rK/Y}{\gamma - rK/Y},$$

$$\frac{Y^H}{Y^L} \leq \frac{1 - rK/Y}{\gamma - rK/Y}.$$  

Table 5: Relative output for different values of $\lambda$ and $rK/Y$ ($\gamma = 0.85$).

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>1</th>
<th>1.01</th>
<th>1.05</th>
<th>1.1</th>
<th>1.15</th>
<th>1.2</th>
<th>1.25</th>
</tr>
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<tbody>
<tr>
<td>0.36</td>
<td>1.31</td>
<td>1.16</td>
<td>1.17</td>
<td>1.19</td>
<td>1.21</td>
<td>1.23</td>
<td>1.26</td>
</tr>
<tr>
<td>0.4</td>
<td>1.33</td>
<td>1.17</td>
<td>1.19</td>
<td>1.21</td>
<td>1.23</td>
<td>1.26</td>
<td>1.30</td>
</tr>
<tr>
<td>$rK/Y$</td>
<td>0.45</td>
<td>1.38</td>
<td>1.19</td>
<td>1.21</td>
<td>1.24</td>
<td>1.28</td>
<td>1.32</td>
</tr>
<tr>
<td>0.55</td>
<td>1.43</td>
<td>1.22</td>
<td>1.25</td>
<td>1.29</td>
<td>1.34</td>
<td>1.40</td>
<td>1.48</td>
</tr>
<tr>
<td>0.6</td>
<td>1.50</td>
<td>1.26</td>
<td>1.30</td>
<td>1.35</td>
<td>1.43</td>
<td>1.54</td>
<td>1.70</td>
</tr>
<tr>
<td>0.65</td>
<td>1.75</td>
<td>1.39</td>
<td>1.48</td>
<td>1.66</td>
<td>1.99</td>
<td>2.87</td>
<td>7.27</td>
</tr>
</tbody>
</table>

* Theoretical upper bound for $\lambda \to 1$.  

When our economy approaches constant returns to scale, the endogenous TFP mechanism alone is quite powerful and it can generate differences in TFP and output across steady states of up to 28 and 50 percent, respectively.

In the studies of the long-run behavior of an economy, using the proper measure of capital share of output is of crucial importance. For example, for the unified theory of Parente and Prescott (2005) to be successful, the capital share of output should be between 0.55 and 0.65. The magnitude of this share depends on the definition of investment (capital). In the context of this paper it is proper to define investment as “any allocation of resources that is designed to increase future productivity” (see Parente and Prescott, 2000). That is, investment should include maintenance and repair, research and development, software, investment in organizational capital, and investment
in human capital. Parente and Prescott (2000) find that including these items in investment implies that the capital share of output is larger than 1/2 and can reach as high as 2/3.

The capital share is important for two reasons. First, there is a standard neoclassical effect: The higher the capital share is, the higher the effect of TFP is on the economy. For two economies differing only in their TFP, the steady state capital ratio relates to the TFP ratio as follows:

\[
\frac{K^H}{K^L} = \left(\frac{TFP^H}{TFP^L}\right)^{\frac{1}{1-\alpha}}.
\]

The higher the share of capital is, the higher the difference in steady state capital is between the two economies.

Second, the capital share directly impacts TFP, because it enters into the definition of TFP in (21) and into the definition of the function \(\Phi(J)\) in (24). Because of the highly non-linear nature of TFP and \(\Phi\) as functions of the cutoff \(J\), it is not possible to derive analytically the effect of an increase in the capital share on the resulting TFP differences across the steady states. However, when \(\lambda\) tends to 1 the theoretical upper bound on these differences gets larger as the capital share grows (see equation (27) above). For all numerical experiments (Table 4), the increase in the capital share of output increases the TFP differences. Combined with the “neoclassical effect” described above, this leads to even larger differences in output and in capital across the steady states (Table 5).

Differences across steady states increase in \(\lambda\) and \(rK/Y\). When both \(rK/Y\) and \(\lambda\) are high, the resulting differences in output are large, reaching as much as 627 percent.

5 Conclusions

In this paper we show that allowing for an endogenous industry structure along the lines of Hopenhayn (1992) can yield to multiple locally stable steady states. This implies that large differences in productivity and output may persist. Differently from the previous literature, we establish the existence of multiple steady states for an arbitrarily small degree externalities (increasing returns to scale).

For details and references see the original paper. A large portion of the unmeasured capital is organization capital. Findings of Atkeson and Kehoe (2005) imply that the value of organizational capital in the US manufacturing sector is larger than the value of physical capital.
We calibrate the model using standard parameter values and a distribution of productivity across firms that matches the distribution of firms by size across developed and LDC countries. Our calibrated economy displays large differences in TFP across stable steady states and even larger differences in output levels.

References


A   Proofs of Propositions

A.1   Proof of Proposition 1

Equations (14) and (15) imply that the fraction of labor used in production $u_t$ is a function only of the cutoff $J_t$:

$$u_t = \frac{\ddot{a}(J_t)}{\ddot{a}(J_t) + \frac{\lambda - \gamma}{(1-\alpha)\gamma} a(J_t)}.$$ 

Substituting this expression of $u_t$ into equation (21), we obtain:

$$TFP(J_t) = \phi^{\gamma-\lambda} \left[ \frac{\frac{\lambda - \gamma}{(1-\alpha)\gamma} a(J_t)}{\ddot{a}(J_t) + \frac{\lambda - \gamma}{(1-\alpha)\gamma} a(J_t)} \right]^{\lambda-\gamma} \times$$

$$\times \left[ \frac{\ddot{a}(J_t)}{\ddot{a}(J_t) + \frac{\lambda - \gamma}{(1-\alpha)\gamma} a(J_t)} \right]^{(1-\alpha)\gamma} \left[ \ddot{a}(J_t) \right]^{\lambda-\gamma}$$

(29)

Differentiating the previous expression,

$$\text{signum}(TFP_J) = \text{signum} \left[ \frac{(\lambda - \gamma)(1-\alpha)\gamma a(J_t)}{(1-\alpha)\gamma a + (\lambda - \gamma) a} \frac{\partial a(J_t)}{\partial J} + \frac{\partial a(J_t)}{\partial J} \left( \frac{1}{\ddot{a}} - \frac{1}{\ddot{a}(J_t) + \frac{\lambda - \gamma}{(1-\alpha)\gamma} a(J_t)} \right) \right],$$

(30)

where

$$TFP_J = \frac{\partial TFP(J)}{\partial J}, \quad a_J = \frac{\partial a(J)}{\partial J}, \quad \ddot{a}_J = \frac{\partial \ddot{a}(J)}{\partial J}.$$ 

The terms in parenthesis in (30) are positive and they are multiplied by positive terms. Hence, $TFP_J > 0$.

Using the firms’ first-order condition in (14) and the zero profit condition in (8) we get that the following relation between the cutoff $J_t$ and capital $K_t$:

$$\kappa = (1 - \alpha)^{\gamma} \lambda \left[ \frac{\lambda - \gamma}{(1-\alpha)\gamma} \right]^{\lambda-\gamma} \left[ \frac{\ddot{a}(J_t)}{\ddot{a}(J_t) + \frac{\lambda - \gamma}{(1-\alpha)\gamma} a(J_t)} \right]^{(1-\alpha)\gamma + (\lambda - \gamma) - 1}$$

$$[a(J_t)]^{\lambda-\gamma} K_t^{\alpha\gamma} \left[ \frac{1}{a(J_t)} \int_{J_t}^1 a(j) dj - (1 - J_t) \right]$$

25
For a given $K_t$ the left-hand side of this equation varies with $J_t$ from $+\infty$ to zero. Moreover, one can easily show that the left-hand side is decreasing in $J_t$. Thus, there exists a unique $J_t$ which solves the equation. In addition, it is increasing in $K_t$. Because $J_t$ is increasing in $K_t$, so is output $Y_t$ and wage $w_t$. In addition, since, for a given $K_t$, output $Y_t$ is uniquely determined, so is the $R_t$; i.e., $R_t$ is a function of $K_t$. ■

A.2 Proof of Proposition 2

We use equations (13) and (20) to express $K_t$ as a function of $r_t$ and $J_t$. By substituting this expression of $K_t$ into equation (29) and by using equation (14), we get

$$ r_t \frac{1-a}{\alpha} \kappa = \eta \cdot \Phi(J_t) $$

(31)

where

$$ \Phi(J) \equiv \left[ \frac{\bar{a}(J)}{\bar{a}(J) + \frac{\lambda-\gamma}{(1-\alpha)\gamma}a(J)} \right]^{\frac{\lambda-1}{1-\alpha}} a(J) \frac{\lambda-\gamma}{1-\alpha} \int J_a(J) - 1 \right] dj, $$

(32)

and $\eta$ is a constant:

$$ \eta = \phi \frac{\alpha-\lambda}{\lambda} \left( 1 - \frac{\gamma}{\lambda} \right) \frac{\alpha \gamma}{1-\alpha} \left( \frac{\lambda - \gamma}{\lambda} \right) \frac{\lambda - \gamma}{1-\alpha \gamma} - 1. $$

(33)

Since $\Phi(J)$ is continuous and $\Phi(0) = \infty, \Phi(1) = 0$, there always exists a $J^*$ that satisfies the equation below:

$$ \left[ 1/\beta - (1 - \delta) \right]^{\frac{1-a}{\alpha}} \kappa = \eta \cdot \Phi(J^*). $$

(34)

We have to show that for any $J^*$ satisfying equation (34) there exists a pair $(c^*, K^*)$, both positive, such that $R(K^*) = 1/\beta$ and $c^* = Y(K^*) - \delta K^*$. This is an immediate consequence of proposition 1.

If there is more than one $J^*$ satisfying equation (34), then there will be multiple steady states. Note that for given parameters $\lambda, \gamma$, and $\alpha$, the shape of the function $\Phi(J)$ is entirely determined by the shape of function $a(j)$: If $a(j)$ is such that $\Phi_J > 0$ then (34) can have multiple solutions. To conclude the proof, we must show that there exists a function $a(j)$ such that $\Phi_J > 0.$
The sign of $\Phi_J$ can be checked as follows:

$$\text{signum} \left( \Phi_J \right) = \text{signum} \left\{ \frac{\lambda - 1}{1 - \alpha \gamma} \frac{\hat{a}_J}{a} - \frac{\hat{a}_J + \frac{\lambda - \gamma}{(1 - \alpha \gamma)} \hat{a}_J}{(1 - \alpha \gamma) a} \right\} + \left[ \frac{\lambda - \gamma}{1 - \alpha \gamma} - \frac{1 - J}{j_j(a(j) - a(J))a} \right] \frac{\hat{a}_J}{a} \right\}.$$

Consider a function

$$a(j) = \begin{cases} j, & \text{if } j \leq J_1 \\ J_1 + b_1 \left( \frac{j - J_1}{J_2 - J_1} \right)^{N_2}, & \text{if } J_1 < j \leq J_2 \\ b_2 \left( \frac{j}{J_2} \right)^{N_2}, & \text{if } j > J_2. \end{cases} \quad (35)$$

where $b_i, N_i > 0$. The constant and $b_2$ must satisfy the following restrictions to guarantee continuity:

$$b_2 = (J_1 + b_1).$$

Taking limits for $N_1 \to \infty$:

$$\lim_{N_1 \to \infty} a(J) = J_1$$

$$\lim_{N_1 \to \infty} a_J(J) = 0$$

$$\lim_{N_1 \to \infty} \bar{a}(J) = \frac{1}{1 - J} \left[ (J_2 - J_1) J_1 + \frac{J_1 + b_1}{1 + N_2} \left( \frac{1}{J_2} N_2 - J_2 \right) \right] > J_1 > 0$$

$$\lim_{N_1 \to \infty} \bar{a}_J(J) = \frac{\lim_{N_1 \to \infty} \bar{a}(J) - J_1}{(1 - J)} > 0$$

Therefore, as long as $\lambda > 1$, $\lim_{N_1 \to \infty} \Phi_J > 0$. It follows that there exists a finite $N_1$ for which $\Phi_J > 0$. If $\Phi_J > 0$, then $\Phi$ will have at least one local minimum and one local maximum, say $\Phi$ and $\Phi$. For any $\kappa \in \left( \frac{[1/\beta - (1 - \delta)]^{\alpha / \gamma} \eta \Phi}{[1/\beta - (1 - \delta)]^{\alpha / \gamma} \eta \Phi} \right)$ our model has multiple steady-state equilibria.$\blacksquare$

### A.3 Proof of Proposition 3

Linearizing (22) about a steady state:

$$\begin{bmatrix} \dot{K}_{t+1} \\ \dot{C}_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{Y' + 1 - \delta}{\sigma R' C (Y' + 1 - \delta)} -1 - \frac{\sigma R' C}{\sigma R' C} \end{bmatrix} \begin{bmatrix} K_t \\ C_t \end{bmatrix}$$
### Table A.1: Steady-state stability for different parameters configurations

<table>
<thead>
<tr>
<th>Condition</th>
<th>Steady state stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R' &lt; 0$</td>
<td>saddle</td>
</tr>
<tr>
<td>$Y' - \delta &gt; 0$</td>
<td>source</td>
</tr>
<tr>
<td>$(Y' - \delta) - \frac{\sigma C R'}{R} &gt; 0$</td>
<td>unstable</td>
</tr>
<tr>
<td>$(Y' - \delta) - \frac{\sigma C R'}{R} &lt; 4 \frac{\sigma C R'}{R}$</td>
<td>spiral</td>
</tr>
<tr>
<td>$\frac{\sigma C R'}{R} &gt; Y' - \delta &gt; 0$</td>
<td>sink</td>
</tr>
<tr>
<td>$(Y' - \delta) - \frac{\sigma C R'}{R} &gt; 4 \frac{\sigma C R'}{R}$</td>
<td>stable</td>
</tr>
<tr>
<td>$(Y' - \delta) - \frac{\sigma C R'}{R} &lt; 4 \frac{\sigma C R'}{R}$</td>
<td>spiral</td>
</tr>
</tbody>
</table>

The eigenvalues of the transition matrix are given by:

$$\xi_{1,2} = 1 + \frac{(Y' - \delta) - \frac{\sigma C R'}{R} \pm \sqrt{[(Y' - \delta) - \frac{\sigma C R'}{R}]^2 - 4 \frac{\sigma C R'}{R}}}{2}.$$  

If $R' < 0$ (odd steady states) both eigenvalues are real and $\xi_1 < 1 < \xi_2$. If $R' > 0$ (even steady states) there are four possible cases:

1. $Y' - \delta > \frac{\sigma C R'}{R} \land [(Y' - \delta) - \frac{\sigma C R'}{R}]^2 > 4 \frac{\sigma C R'}{R} \Rightarrow \xi_{1,2} \in \mathbb{R}, \|\xi_{1,2}\| > 1$;
2. $Y' - \delta > \frac{\sigma C R'}{R} \land [(Y' - \delta) - \frac{\sigma C R'}{R}]^2 < 4 \frac{\sigma C R'}{R} \Rightarrow \xi_{1,2} \in \mathbb{C}, \|\xi_{1,2}\| > 1$;
3. $Y' - \delta < \frac{\sigma C R'}{R} \land [(Y' - \delta) - \frac{\sigma C R'}{R}]^2 > 4 \frac{\sigma C R'}{R} \Rightarrow \xi_{1,2} \in \mathbb{R}, \|\xi_{1,2}\| < 1$;
4. $Y' - \delta < \frac{\sigma C R'}{R} \land [(Y' - \delta) - \frac{\sigma C R'}{R}]^2 < 4 \frac{\sigma C R'}{R} \Rightarrow \xi_{1,2} \in \mathbb{C}, \|\xi_{1,2}\| < 1$. 

\[\blacksquare\]