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## **Endogenous Productivity and Multiple Steady States**

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# Endogenous Productivity and Multiple Steady States\*

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## Abstract

We endogenize total factor productivity in a neoclassical model with increasing returns to scale. We obtain multiple steady-state equilibria with an arbitrarily small degree of increasing returns to scale. While the most productive firms operate across all the steady states, in a poverty trap less productive firms operate as well. This results in lower average firm productivity and total factor productivity. A calibrated version of our model displays sizable differences in TFP and output across steady state equilibria.

JEL: L16, O11, O33, O40

Keywords: endogenous productivity, multiple equilibria, poverty traps

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# 1 Introduction

We construct an endogenous total factor productivity (TFP) model that leads to multiple steady-state equilibria and, hence, poverty traps. Our model is a variant of the neoclassical growth model with increasing returns introduced by Benhabib and Farmer (1994), with firms modeled in the tradition of Lucas (1978), Jovanovic (1982), and Hopenhayn (1992). There are many ex-ante identical potential firms which face an entry cost. Firms that choose to enter are entitled to produce an intermediate good with a productivity level drawn independently across firms from a given distribution. Because firms face a fixed operating cost, the decision to operate depends on the level of the firm's productivity. Productivity must be high enough so that the firm generates enough revenue (net of payments to factor inputs) to cover the operating cost. In other words, the operating cost defines a cutoff: firms with productivity above the cutoff choose to operate, the rest of the firms choose not to. The higher the cutoff, the more productive the average firm is.

The existence of multiple steady states depends on small demand externalities, which imply increasing returns to scale at the aggregate level. The main result of our paper is that poverty traps may occur for arbitrarily small increasing returns to scale.<sup>1</sup> Endogenizing TFP allows us to bridge the gap between poverty trap models based on increasing returns and the most recent empirical literature on the degree of returns to scale. An endogenous operating cost provides a powerful amplifying mechanism for increasing returns. We model the operating cost as payments to overhead labor. Since the wage is endogenous, so is the lowest level of productivity used in the economy. This endogeneity may lead to multiple steady states. Consider an economy in a steady state with a high productivity cutoff and a large capital stock. The high cutoff implies that firms' average productivity is high. A large capital stock and high productivity imply that the wage is high, as is the operating cost. A high operating cost makes low productivity firms unprofitable, effectively cleansing the pool of firms. This justifies why the cutoff is high in the first place. Since only high productivity firms are operating, TFP is high. Conversely, in a steady state where capital is low and lower productivity firms are operating (i.e., the cutoff is low), the wage is low. Since the wage

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<sup>1</sup>Galí (1995) obtains multiple equilibria and poverty traps in a model where increasing returns stem from endogenous markups. However, the empirical evidence summarized in Section 4.2 below suggests that the degree of increasing returns is much smaller than the level required to obtain poverty traps in Galí's model.

is low, lower profits are sufficient to cover the operating cost. That is, the low operating cost sullies the pool of producers, leading to lower TFP and capital. In a good equilibrium high productivity firms produce more than in a bad equilibrium, despite facing a higher wage and the same interest rate. This is optimal because firms face a higher demand for their goods, which offsets the contractionary pressure of higher factor prices.

The role of endogenous firms entry as a powerful amplifying mechanism to increasing returns to scale is not limited to the study of poverty traps. Barseghyan and DiCecio (2008) analyze indeterminacy and sunspot fluctuations in an endogenous TFP model. Jaimovich (2007) analyzes a model where endogenous net business formation is procyclical: This implies endogenous countercyclical variations in markup and indeterminacy. Endogenizing entry (see Bilbiie, Ghironi, and Melitz, 2007; Jaimovich and Floetotto, 2008) can also provide a powerful propagation mechanism for real business cycle models.

Empirical motivation for our work stems from the studies of the determinants of cross-country income differences of Klenow and Rodriguez-Clare (1997), Hall and Jones (1999), and Caselli (2005). These authors find that income differences can be attributed, at least in part, to differences in TFP. Previous studies of poverty trap models with endogenous TFP pointed to the failure of adopting the most productive technology as the cause of low TFP and income in poor countries.<sup>2</sup> However, there is evidence pointing to the fact that differences in TFP across economies are related to the lowest level of firms' productivity. Comin and Hobijn (2004) take a comprehensive look at the uses of various technologies as determinants of TFP and find that the key is not when new, better technologies are adopted, but when old, obsolete ones are relinquished. Also, the empirical evidence on the importance of international knowledge spillovers summarized in Klenow and Rodriguez-Clare (2005) suggests that all countries can easily access frontier technologies. Banerjee and Duflo (2005) cite the McKinsey Global Institute (2001) report on India, which finds that while larger production units (firms) use relatively new technologies, smaller (in home) production units have low productivity. Finally, Mokyr (1990, 2001) argues that the Industrial Revolution was characterized by a shift from less productive forms of production

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<sup>2</sup>See, for example, Murphy, Shleifer, and Vishny (1989) and Ciccone and Matsuyama (1996). For comprehensive reviews of the literature on poverty traps, see Matsuyama (2005) and Azariadis and Stachurski (2005).

(workshops) to more productive ones (factories).

A successful model of cross-country income and productivity differences should also provide a plausible story of how a “growth miracle” can occur, i.e., it should be consistent with the transition of a country from low to high output and productivity. In our model economy, a growth miracle is a transition from a bad equilibrium (low productivity cutoff) to a good one (high productivity cutoff). Such a take-off can be triggered by technological progress that makes the highest productivity firms even more productive or by a decline in the entry cost. In the first case, the increase in productivity of the best firms makes them more competitive, raising factor prices and driving low productivity firms out of business. In the second case, a decline in the entry cost brings about more competition from entering firms, driving out of the market low productivity firms. In both cases, along the transition path, the economy’s TFP, output, capital, and firms’ average productivity (and size) rise. An increase in the average firm size, caused by a massive shift of employment from small to large establishments, is a defining feature of the Industrial Revolution. A similar increase is recorded in the case of Japan’s growth miracle. Between 1957 and 1969, the employment share of Japan’s smallest establishments declined from 41 to 31.5 percent.

A calibrated version of our model is able to generate sizeable differences across two stable steady states. For example, with 25 percent increasing returns and a 65 percent capital share<sup>3</sup> TFP in the high steady states is 45 percent larger than in the low steady state. The corresponding difference in levels of output across steady states is larger than 600 percent.

Finally, our model’s implications are consistent with the literature which explores the effects that various barriers have on productivity: e.g., Parente and Prescott (1994, 2000), Restuccia and Rogerson (2003), Erosa and Hidalgo Cabrillana (2005), Herrendorf and Teixeira (2005a,b), and Barseghyan (2008).

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 studies its steady state, dynamics properties, and some extensions. Section 4 provides an interpretation of growth miracles that arise naturally in the model and discusses the quantitative implications of a calibrated version of the model. We conclude in Section 5. We provide proofs in an appendix.

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<sup>3</sup>A capital share this high is meant to capture the fact that our model does not include explicitly other forms of capital besides physical capital.

## 2 The Model

Our model is a variant of the neoclassical growth model. The model departs from the standard framework by having a richer structure of the production side of the economy. We model firms following Lucas (1978), Jovanovic (1982), and Hopenhayn (1992). Firms are heterogenous: each firm has monopoly power over the good it produces, and firms have different productivity levels. Two features of the production side of the economy are crucial for the results of the paper:

1. a sunk entry cost;
2. an operating cost: in addition to capital and labor used directly in production, firms pay for a fixed amount of overhead labor.

A part of the entry costs stems from satisfying different official regulatory requirements (see Djankov, La Porta, Lopez-de-Silanes, and Shleifer, 2002). In addition, in some countries, entry requires significant side payments to local officials.<sup>4</sup> Entry cost may also include expenses related to acquisition of firm-specific capital,<sup>5</sup> acquisition of appropriate technology,<sup>6</sup> and market research.

The operating cost typically refers to overhead labor and expenses that are lumpy in nature (e.g., renting a physical location). According to the findings of Domowitz, Hubbard, and Petersen (1988), in U.S. manufacturing plants, the overhead labor accounts for 31 percent of total labor. Ramey (1991) suggests that overhead labor is about 20 percent. The preferred estimate of overhead inputs in Basu (1996) is 28 percent.

We also assume that firms learn their productivity only after a sunk entry cost is paid. This assumption reflects very high uncertainty faced by entering firms. This is routinely found in the data and documented, for example, by Klette and Kortum (2004) as a stylized fact.

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<sup>4</sup>In the case of Peru, this is documented by De Soto (1989).

<sup>5</sup>Ramey and Shapiro (2001) show that in some instances the specificity of firm capital is so extreme that the sale price of such capital after a firm has been dissolved is only a small fraction of the original cost.

<sup>6</sup>See, for example, Atkeson and Kehoe (2005).

## 2.1 Households

There is a continuum of households that supply a fixed amount of labor, consume, invest, and own all firms in the economy. The problem of the representative household is given by

$$\begin{aligned} \max \sum_{t=0}^{\infty} \beta^t U(C_t), \quad \beta \in (0, 1) \\ \text{s.t. } C_t + I_t = r_t K_t + w_t + \Pi_t + T_t, \\ I_t = K_{t+1} - (1 - \delta) K_t, \end{aligned} \tag{1}$$

where  $C_t$  denotes consumption,  $I_t$  is investment,  $K_t$  denotes the total household capital,  $r_t$  is the rental rate on capital, and  $w_t$  is the wage.<sup>7</sup>  $\Pi_t$  is the firms' profits, and  $T_t$  is a lump-sum transfer from the government;  $\beta$  and  $\delta \in (0, 1)$  are the discount rate and depreciation rate, respectively. We assume a constant elasticity of substitution utility function with elasticity  $\sigma > 0$ .

## 2.2 Firms

### 2.2.1 Final Good Producers

The final consumption good in this economy is produced by perfectly competitive firms, according to the following production function:

$$Y_t = \left[ \int_0^{\mu_t} [y_t(i)]^{\frac{1}{\lambda}} di \right]^{\lambda},$$

where  $\mu_t$  is the number of intermediate goods produced in the economy,  $\lambda$  is a constant which is greater than 1, and  $y_t(i)$  is the quantity of the intermediate good  $i$ . Let  $p_t(i)$  be the price of the  $i^{\text{th}}$  intermediate good relative to the final good. Then, the maximization problem of the final good producer can be written as

$$\Pi_t^{FF} = \max \left[ \int_0^{\mu_t} [y_t(i)]^{\frac{1}{\lambda}} di \right]^{\lambda} - \int_0^{\mu_t} p_t(i) y_t(i) di, \tag{2}$$

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<sup>7</sup>We assume that the household inelastically supplies one unit of labor.

and the first-order optimality condition implies that the demand function for the  $i^{\text{th}}$  intermediate good is given by

$$p_t(i) = \left[ \frac{y_t(i)}{Y_t} \right]^{-\frac{\lambda-1}{\lambda}}.$$

### 2.2.2 Intermediate Goods Producers

A firm in the intermediate goods sector lives for one period and is profit maximizing. All firms are ex-ante identical. There is a sunk entry cost,  $\kappa$ .<sup>8</sup> Once the entry cost is paid, a firm gains the ability to produce an intermediate good. The firm has monopoly power over the good it produces. Next, the firm draws a productivity parameter  $A(j)$ , where  $j$  is drawn from an i.i.d. uniform distribution over  $[0, 1]$ . The production function for the good  $j$  is given by

$$[A(j)]^{1-\gamma} [k(j)^\alpha n(j)^{1-\alpha}]^\gamma,$$

where  $k(j)$  and  $n(j)$  denote capital and labor, respectively. The productivity parameter differs among the firms. A firm with a higher index has a higher productivity parameter, i.e.,  $A(j) > A(i)$  for  $j > i$ . In addition, function  $A(j)$  is assumed to be continuous, and  $A(0) = 0$ . The parameter  $\gamma \in (0, \lambda)$  determines the degree of returns to scale in variable inputs.<sup>9</sup> The parameter  $\alpha$  is between zero and 1.

If a firm decides to produce, it must incur an operating cost in terms of wages paid to  $\phi$  units of overhead labor. Consider the decision of a firm born in time  $t$  with a draw  $j$ . If it decides to produce, its profits are

$$\begin{aligned} \pi_t^P(j) &= \max_{k_t(j), n_t(j)} p_t(j) y_t(j) - r_t k_t(j) - w_t [n_t(j) + \phi] \\ \text{s.t. } y_t(j) &= [A(j)]^{1-\gamma} [k_t^\alpha(j) n_t^{1-\alpha}(j)]^\gamma, \quad p_t(j) = \left[ \frac{y_t(j)}{Y_t} \right]^{\frac{1-\lambda}{\lambda}}. \end{aligned} \quad (3)$$

The decision to produce or not depends on whether  $\pi_t^P(j)$  is positive. Therefore, the  $j^{\text{th}}$  firm's profits,  $\pi_t^F(j)$ , are given by

$$\pi_t^F(j) = \max\{\pi_t^P(j), 0\}. \quad (4)$$

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<sup>8</sup>We assume that  $\kappa$  is denominated in consumption units and that all entry-cost payments are rebated to the households in a lump-sum fashion. Alternatively, one can model  $\kappa$  as a sunk investment, i.e., in units of capital. Such a formulation would not change any of our results, but it would make the exposition more cumbersome.

<sup>9</sup>This is what Lucas (1978) calls managers' span of control.

Free entry implies that, in equilibrium, firms' expected profits must be equal to the entry cost,  $\kappa$ :

$$\int_0^1 \pi_t^F(j) dj = \kappa. \quad (5)$$

### 2.2.3 Firms' average productivity

We derive the equilibrium relationship between the firms' average productivity and the operating cost. First, we determine the lowest productivity level necessary for a firm to decide to produce. The existence of economy-wide competitive factor markets implies that in equilibrium, the gross profits, capital, and labor ratios of any two firms are equal to their (scaled) productivity ratio:

$$\frac{p_t(j)y_t(j)}{p_t(i)y_t(i)} = \frac{k_t(j)}{k_t(i)} = \frac{n_t(j)}{n_t(i)} = \frac{a(j)}{a(i)}, \quad \forall i, j, \quad (6)$$

where  $a(j) \equiv A(j)^{\frac{1-\gamma}{\lambda-\gamma}}$ . The first-order conditions of problem (3) imply that profits from producing are equal to the firm's share of the gross profits  $(1 - \frac{\gamma}{\lambda})$  minus the operating cost<sup>10</sup>:

$$\pi_t^P(j) = (1 - \frac{\gamma}{\lambda})p_t(j)y_t(j) - \phi w_t.$$

Clearly  $\pi_t^P(j)$  is increasing in  $j$  and, since  $a(j) = 0$ , there exists a cutoff firm,  $J_t$ , which is indifferent between producing or not:

$$(1 - \frac{\gamma}{\lambda})p_t(J_t)y_t(J_t) = \phi w_t. \quad (7)$$

Firms with indices higher than  $J_t$  will produce, and those with lower indices will not. Thus, firms' zero profit condition in (5) can be written as

$$\kappa = \phi w_t \int_{J_t}^1 \left[ \frac{a(j)}{a(J_t)} - 1 \right] dj. \quad (8)$$

The previous equation defines the cutoff  $J_t$  as a function of the operating cost  $\phi w_t$ . An increase in the cutoff  $J_t$  has two effects: Profits of every firm decline, and the number of producing firms as a fraction of entering firms declines. Therefore, the right-hand side of (8) is decreasing in  $J_t$  and increasing in the

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<sup>10</sup>Later on, with some abuse of terminology, we will refer to  $(1 - \frac{\gamma}{\lambda})p_t(j)y_t(j)$  as firms' gross profits.

fixed cost  $(\phi w_t)$ . Hence, the cutoff is increasing in the operating cost. Therefore, firms' average productivity,  $\bar{a}(J_t) = \frac{\int_{J_t}^1 a(j) dj}{1 - J_t}$ , is an increasing function of the operating cost.

#### 2.2.4 Entry and the number of operating firms

Entry in this model refers to the number of firms that pay the entry cost,  $\kappa$ . The number of entering firms differs from the number of operating firms because only a fraction of entrants will have productivity high enough to operate: the pool of producers consists only of firms which have an index higher than  $J_t$ . In particular, let  $\nu_t$  denote the number of entering firms and  $\mu_t$  the number of operating firms. Then

$$\mu_t = \nu_t \int_{J_t}^1 dj. \quad (9)$$

### 2.3 Aggregate Output and TFP

Aggregate capital, labor, and output can be expressed as

$$K_t = \nu_t \int_{J_t}^1 k_t(j) dj, \quad (10)$$

$$N_t = \nu_t \int_{J_t}^1 [n_t(j) + \phi] dj, \quad (11)$$

$$Y_t = \left[ (\mu_t \bar{a}(J_t))^{(\lambda - \gamma)} u_t^{(1 - \alpha)\gamma} \right] K_t^{\alpha\gamma} (N_t)^{(1 - \alpha)\gamma}, \quad (12)$$

where  $u_t$  is the fraction of labor used in production. Finally, the rental rate on capital, the wage rate, and the equation determining the cutoff  $J_t$  can be written as

$$r_t = \alpha \frac{\gamma}{\lambda} \frac{Y_t}{K_t}, \quad (13)$$

$$w_t = (1 - \alpha) \frac{\gamma}{\lambda} \frac{Y_t}{u_t N_t}, \quad (14)$$

$$\left(1 - \frac{\gamma}{\lambda}\right) \frac{a(J_t)}{\bar{a}(J_t)} \frac{Y_t}{(1 - u_t) N_t} = w_t. \quad (15)$$

## 2.4 Closing the Model

The resource constraint is given by

$$C_t + K_{t+1} = Y_t + (1 - \delta)K_t. \quad (16)$$

The only role the government has in the model is to collect the entry fees  $\nu_t \kappa$  from firms and rebate them lump-sum to the households:

$$T_t = \nu_t \kappa. \quad (17)$$

Profits and the labor market clearing condition are

$$\Pi_t = \Pi_t^{FF}, \quad (18)$$

$$N_t = 1. \quad (19)$$

The definition of equilibrium is standard.

## 3 Steady States, Dynamics, and Some Extensions

In this section we analyze the existence and stability of the steady states, and discuss some extensions to our basic model. The main finding is that there can be multiple stable steady states for an arbitrarily low degree of increasing returns to scale.

Intuitively, if there are multiple steady states, their existence is due to the endogenous productivity mechanism embedded in the model. Equation (8) relates the cutoff  $J$  to the operating cost,  $\phi w_t$ . The integral on the right-hand side of this equation is decreasing in  $J_t$ . Thus, a higher operating cost translates into a higher cutoff and vice versa. In an economy where the operating cost is high, higher (gross) profits are required to cover this cost. Only high productivity firms can generate such profits. Therefore, the lower productivity firms are forced out from the pool of producers. As the operating cost increases, the entry cost relative to operating cost falls, allowing more firms to enter. However, only the ones with higher productivity firms are profitable enough to operate. This relation between the operating cost and the cutoff provides economic intuition for the existence of multiple steady states. If multiple steady states exist, then one steady state has high capital

and only high productivity firms are operating. High capital stock and high productivity imply that the wage rate is high, and so is the operating cost. A high operating cost, in turn, justifies why only high productivity firms are operating. Finally, since productivity is high, a high capital stock is necessary to equate the return on capital to  $1/\beta$ . Conversely, in a “low” steady state, the capital stock and firms’ average productivity are low. This implies a low operating cost which allows lower productivity firms to operate. Since firms’ average productivity is low, the capital stock must be low to have the return on capital equal to  $1/\beta$ . A firm productive enough to be active in different steady states produces more in a good steady state than in a bad one, despite a higher wage and the same interest rate. This is optimal because it faces a higher demand for its goods, which offsets the contractionary pressure of higher factor prices.

### 3.1 Steady States

We present the argument formally in Propositions 1 and 2; we provide proofs in Appendix A. First, note that the number of firms is proportional to the total amount of labor used to cover the fixed cost:

$$\mu_t = \frac{1 - u_t}{\phi} N_t.$$

Therefore, aggregate output is given by

$$Y_t = TFP_t K_t^{\lambda s_k}, \quad (20)$$

where  $s_k = (\alpha\gamma)/\lambda$  denotes the capital share of output, and total factor productivity is

$$TFP_t = \left[ \phi^{\gamma-\lambda} [\bar{a}(J_t)]^{\lambda-\gamma} (1 - u_t)^{\lambda-\gamma} u_t^{(1-\alpha)\gamma} \right]. \quad (21)$$

There are two components of TFP: firms’ average productivity  $(\bar{a}(J_t))^{\lambda-\gamma}$  and the term  $u_t^{(1-\alpha)\gamma} (1 - u_t)^{\lambda-\gamma}$ , which we call the *labor allocation* component. Firms’ average productivity is increasing in  $J_t$ . The labor allocation component is a function of  $J_t$  as well, though not necessarily monotonic. However, the effect of  $J_t$  on average productivity dominates, and  $TFP_t$  is increasing in  $J_t$ .

The following proposition allows us to present the model economy in a more familiar, neoclassical framework.

**Proposition 1** *The aggregate production function in (12) and the total factor productivity in (21) are increasing in the cutoff  $J_t$ . The cutoff  $J_t$ , the wage  $w_t$ , and the aggregate output  $Y_t$  are all increasing functions of capital  $K_t$ . The rate of return on capital  $R_t \equiv (r_t + 1 - \delta)$  is a function of  $K_t$ .*

**Proof.** See Appendix A. ■

The proposition above implies that the dynamics of the economy can be characterized by the following system of difference equations,

$$\begin{aligned} \left(\frac{c_{t+1}}{c_t}\right)^{1/\sigma} &= \beta R(K_{t+1}), \\ C_t + K_{t+1} &= Y(K_t) + (1 - \delta)K_t, \end{aligned} \quad (22)$$

plus a transversality condition. We now turn to the existence and multiplicity of steady states.

**Proposition 2** *The economy characterized by the system in (22) generically has an odd number of steady states. For any  $\lambda > 1$ , there exist a distribution of productivities,  $a(j)$ , and a value of  $\kappa$  such that the system (22) has multiple steady-state equilibria.*

**Proof.** (sketch) Straightforward manipulations of the first order conditions lead to the following relation between the rate of return on capital and the cutoff  $J$  in steady state:

$$r^{\frac{1-\alpha\gamma}{\alpha\gamma}} \kappa = \eta \cdot \Phi(J), \quad (23)$$

where

$$\Phi(J) \equiv \left[ \frac{\bar{a}(J)}{\bar{a}(J) + \frac{\lambda-\gamma}{(1-\alpha)\gamma} a(J)} \right]^{\frac{\lambda-1}{1-\alpha\gamma}} a(J)^{\frac{\lambda-\gamma}{1-\alpha\gamma}} \int_J^1 \left[ \frac{a(j)}{a(J)} - 1 \right] dj \quad (24)$$

and  $\eta$  is a constant. Since  $\Phi(J)$  is continuous and  $\Phi(0) = \infty$ ,  $\Phi(1) = 0$ , there always exists at least one value of  $J \in (0, 1)$  which satisfies equation (23). To have multiple solutions, it is necessary that the function  $\Phi$  has to be non-monotone (see Figure 2). In Appendix A we show that there always exists a function  $a(j)$  such that this is the case. With a non-monotone  $\Phi(J)$ , it is trivial to find a value of  $\kappa$  such that equation (23) has multiple solutions. Note that equation (23) implies that if  $\Phi(J)$  is increasing, so is  $r(K)$ : The necessary condition for the existence of multiple steady states is that for some values of  $K$  the return on capital must be increasing. The properties of the

function  $\Phi$  mimic those of firms' expected profits, i.e., the right-hand side of the zero-profit condition (8). A necessary condition for existence of multiple steady state is that firms' expected profits are increasing in  $J$ . An increase in the cutoff  $J$  has two opposite effects. On one hand the wage rate increases, increasing expected profits. On the other hand, a higher  $J$  implies a lower value of the integral on the right-hand side of (8). For expected profits to rise with  $J$ , the increase in the wage has to dominate the fall in the value of the integral. A sufficient condition for this is that  $\partial \bar{a}(J) / \partial J \gg \partial a(J) / \partial J$ : A relatively high derivative of the average productivity guarantees a strong positive effect on TFP and the wage rate, while a relatively low derivative of the function  $a(j)$  implies a mild negative response of the integral term. A function  $a(j)$  which is sufficiently flat on some interval and increases rapidly for higher values of  $j$  has this property. ■

Given Propositions 1 and 2 it is easy to establish that the “high  $J$ ” economy has higher capital stock, higher output, higher total factor productivity, and higher average productivity for firms.

## 3.2 Dynamics

The following proposition characterizes the behavior of the economy around the steady state(s).<sup>11</sup>

**Proposition 3** *Steady states with an odd index are saddles. Steady states with an even index can be classified as follows:*

1. *source, if  $Y' - \delta > \frac{\sigma CR'}{R}$  and  $[(Y' - \delta) - \frac{\sigma CR'}{R}]^2 > 4 \frac{\sigma CR'}{R}$ ;*
2. *unstable spiral, if  $Y' - \delta > \frac{\sigma CR'}{R}$  and  $[(Y' - \delta) - \frac{\sigma CR'}{R}]^2 < 4 \frac{\sigma CR'}{R}$ ;*
3. *sink, if  $Y' - \delta < \frac{\sigma CR'}{R}$  and  $[(Y' - \delta) - \frac{\sigma CR'}{R}]^2 > 4 \frac{\sigma CR'}{R}$ ;*
4. *stable spiral, if  $Y' - \delta < \frac{\sigma CR'}{R}$  and  $[(Y' - \delta) - \frac{\sigma CR'}{R}]^2 < 4 \frac{\sigma CR'}{R}$ .*

**Proof.** See appendix A. ■

For the parameter values we consider in the rest of the paper, we obtain three steady states, with the odd steady state unstable (cases 1 and 2 in Proposition 3). In comparing output and TFP across steady states we will focus on the two stable steady states.

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<sup>11</sup>An analysis of the global dynamics of our model is beyond the scope of this paper, and we refer the reader to Galí (1995) and Slobodyan (2005).

### 3.3 Extending the Basic Model

In this section we consider two possible extensions of our basic model. First, we consider the implications of endogenizing the entry cost. Then, we analyze infinitely lived firms.

#### 3.3.1 Entry and Operating Costs

The key feature of the model that accounts for the existence multiple steady-state equilibria is the asymmetry between the entry cost and the operating cost. While the operating cost is endogenous, the entry cost is not. This assumption can be relaxed to allow for both entry and operating costs to be endogenous. In this case, multiple steady-state equilibria exist as long as a weaker form of asymmetry is preserved. In particular, the operating cost has to be increasing in capital faster than the entry cost, so that the ratio of the operating cost to the entry cost is increasing in capital. We suggest a simple example, based on Atkeson and Kehoe (2005). Let the entry cost take the form of  $\kappa$  units of entry services, which firms need to purchase to enter. Let the production function of these services be exactly the same as it is for consumption goods, except for a different labor intensity. It can be shown that in a steady state the zero-profit condition in (8) becomes

$$\kappa = \eta w^\nu \int_J^1 \left[ \frac{a(j)}{a(J)} - 1 \right] dj, \quad (25)$$

where  $\eta$  and  $\nu$  are positive constants. When  $\nu$  is 1, it is the same zero-profit condition discussed above. When  $\nu$  is zero, it is the same as in Atkeson and Kehoe (2005). As long as  $\nu$  is positive, the key relation between wage  $w$  and the cutoff  $J$  is preserved and the model admits multiple equilibria.

#### 3.3.2 Infinitely Lived Firms

One unrealistic simplifying assumption in our model is that firms live only for one period. Our results are easily extended to infinitely-lived firms with time-varying productivity. We consider two opposite cases:

1. firms' productivity in every period is given by  $A(j)$ , where  $j$  is the original draw.
2. firms draw a new  $j$  which is independent of past draws.

We also assume that firms die with constant probability  $(1 - p)$ . Consider a period- $t$  decision of a firm born in time  $s$  with a draw  $j$ . The Bellman equation of this firm is

$$V_t(s, j) = \max \left[ \pi_t^P(s, j) + \frac{p}{R_{t+1}} E_t V_{t+1}(s + 1, j') \right], \quad (26)$$

where  $\pi_t^P(s, j)$  is the profits from producing as defined in Section 2.2.2, and  $R_{t+1}$  is the rate of return on capital (i.e., the interest rate).<sup>12</sup> The law of motion of  $j$  is  $j' = j$  in case 1 and  $j'$  is i.i.d. uniform over  $[0, 1]$  in case 2. The free entry condition implies that

$$\kappa = \int_0^1 V_t(t, j) dj. \quad (27)$$

Proposition 2 extends to both cases. If firms' productivity is the same as the original draw, the function  $\Phi$  is unchanged. For the case of i.i.d. draws, the function  $\Phi$  is replaced by the following:

$$\tilde{\Phi}(J) \equiv \frac{\left[ \frac{\bar{a}(J)}{\bar{a}(J) + \frac{(\lambda-\gamma)}{(1-\alpha)\gamma} \left(1 + \frac{p}{R} \int_J^1 \left[\frac{a(j)}{a(J)} - 1\right] dj\right) a(J)} \right]^{\frac{\lambda-1}{1-\alpha\gamma}}}{\left(1 + \frac{p}{R} \int_J^1 \left[\frac{a(j)}{a(J)} - 1\right] dj\right)^{1 - \frac{(\lambda-\gamma)}{1-\alpha\gamma}}} a(J)^{\frac{\lambda-\gamma}{1-\alpha\gamma}} \int_J^1 \left[\frac{a(j)}{a(J)} - 1\right] dj. \quad (28)$$

Notice that when  $p = 0$ , case 2 simplifies to our baseline model in Section 2, i.e.,  $\tilde{\Phi}(J)|_{p=0} = \Phi(J)$ .

## 4 Properties of the Model

In this section we discuss some qualitative and quantitative properties of our model. First we consider a growth miracle in the model, driven by technological progress or a decrease in entry costs. We argue that the decline in the share of small firms is consistent with Japan's post-World War II experience and with the Industrial Revolution. Then we compute differences in output and TFP in a calibrated version of the model, and we perform

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<sup>12</sup>Recall that the firms are owned by the households and there is no aggregate uncertainty in the economies we consider. Therefore,  $1/R_{t+1}$  is the relevant discount factor.

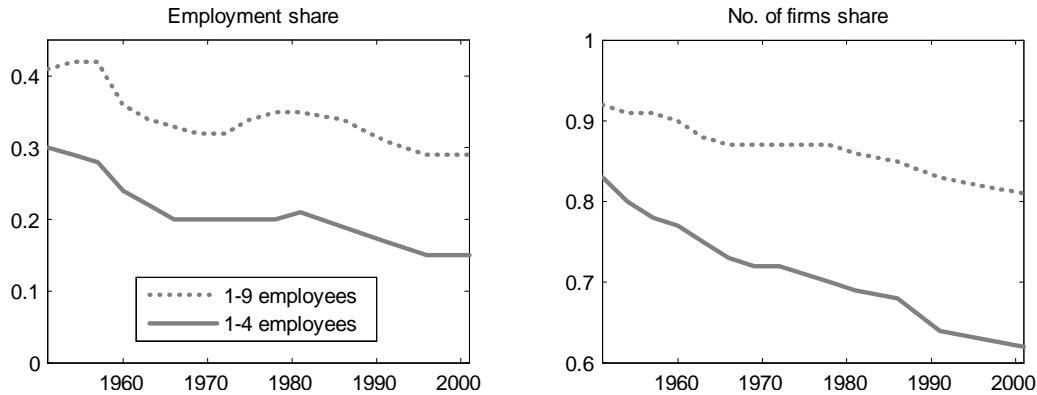


Figure 1: Smallest establishments in Japan: employment and number of firms shares.

sensitivity analysis on the degree of increasing returns to scale and capital share parameters. We conclude that the differences between high and low steady states are sizable.

#### 4.1 Growth Miracles: An Interpretation

A puzzle closely related to cross-country income differences is the question of how and why countries grow and what causes growth miracles. A common view in the literature is that growth miracles are a result of a dramatic shift towards more productive firms and better forms of industrial organization. For example, Mokyr (2001) states that the Industrial Revolution was accompanied by “the ever-growing physical separation of the unit of consumption (household) from the unit of production (plant),...” due to “... concentration of former artisans and domestic workers under one roof (plants), in which workers were more or less continuing what they were doing before, only away from home ...” and “... a more radical change in production technique, with substantial investment in fixed capital combined with strict supervision and rigid discipline.” Thus, plants and factories (i.e., bigger establishments) must have been more productive than “in home” production units (i.e., the smallest establishments), and the Industrial Revolution can be viewed as a shift of resources from smaller, less productive units to larger, more productive ones.

Japan postwar growth miracle is similar in this respect to the Industrial Revolution (see Figure 1): the labor share of the smallest establishments (i.e., establishments with nine employees or fewer) fell by 9 percentage points between 1957 and 1969.<sup>13</sup> The period from 1957 to 1969 was a period of remarkable economic growth, which Parente and Prescott (2005) classify as a period of a growth miracle. Such a shift in our model’s framework depends on the properties of the function  $a(j)$ . If the corresponding probability density function of productivities is one that implies the existence of multiple steady states (i.e., it has a high density somewhere at the lower tail), then a shift from small to the large establishments occurs when the economy moves away from a “low  $J$ ” steady state to a “high  $J$ ” steady state.

There are two reasons that can cause such a shift. The first one, is a decline in entry barriers, i.e., a decline in the entry cost,  $\kappa$ .

To illustrate this point, it is useful to start with Figure 2. For larger values of  $\kappa$ , there is a unique, low-cutoff steady state, and for lower  $\kappa$ ’s there is a unique steady state, with large  $J$ . For intermediate values of  $\kappa$  there can be two steady states. A small change in the value of  $\kappa$  can lead to large differences in  $J$  and the corresponding values of capital and output. In our model economy, the best technologies available are used regardless of the magnitude of the entry cost. The usage of worse technologies, on the other hand, depends on the entry cost. A reduction in the entry cost can cleanse the economy of lower productivity firms, increasing firms’ average productivity and TFP. This mechanism of growth miracles shares a common driving force, reduction of barriers, with the one of Parente and Prescott (2000). However, the effect of the reduction of the barriers is different. In their model new, better technologies are not being used because of the barriers. Here, the entry barriers determine not the highest, but the lowest level of technology that is being used in the economy.

The second reason for a growth miracle is technological progress. A natural way to introduce this into our model is to consider a one-time permanent increase in the function  $a(j)$  for values of  $j$  close to 1.<sup>14</sup> That is, the best

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<sup>13</sup>The data is from various issues of the *Japan Statistical Yearbook*, edited by the Statistical Training Institute and published by the Statistics Bureau, both under the Ministry of Internal Affairs and Communications. Data is available every three years for the period 1951-1981 and every five years subsequently. The unit of observation is the establishment, i.e., a single physical location where the business is conducted or where services or industrial operations are performed.

<sup>14</sup>A better model to address the effect of productivity improvements would be one where

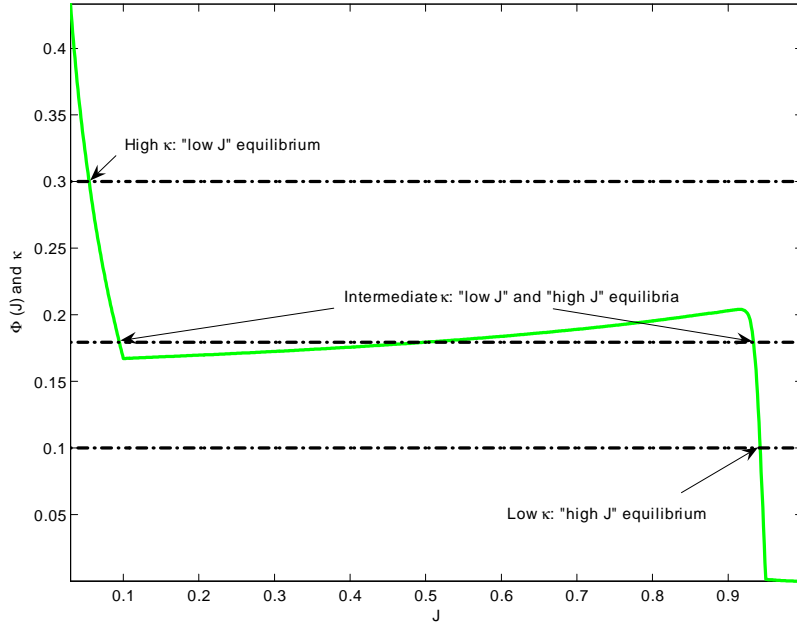


Figure 2: The role of the entry cost,  $\kappa$ .

technologies become even better. Mathematically, this can be written, for example, as

$$\hat{a}(j) = \begin{cases} a(\hat{J}) \left( \frac{a^{OLD}(j)}{a(\hat{J})} \right)^q, & \text{if } j \geq \hat{J}, \\ a(j), & \text{otherwise;} \end{cases}$$

where  $\hat{J}$  is close to 1, and  $q$  is greater than 1. For any  $J < \hat{J}$ , the change in the function  $a$  causes  $\Phi(J)$  to rise. If such a rise is sufficiently large, the “low  $J$ ” steady state will disappear (see Figure 3 below), and the economy will start growing toward a “high  $J$ ” steady state.

## 4.2 Model Calibration

Our model contains seven parameters  $(\beta, \delta, \lambda, \gamma, \alpha, \phi, \kappa)$ , plus any additional parameters determining the function  $a(j)$ . The model’s implications are robust to the choice of  $\beta$  and  $\delta$  for the commonly used values of  $\beta \in (0.94, 0.99)$

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the highest level of technology that is available in the economy grows over time. Building and examining such a model is left for future research.

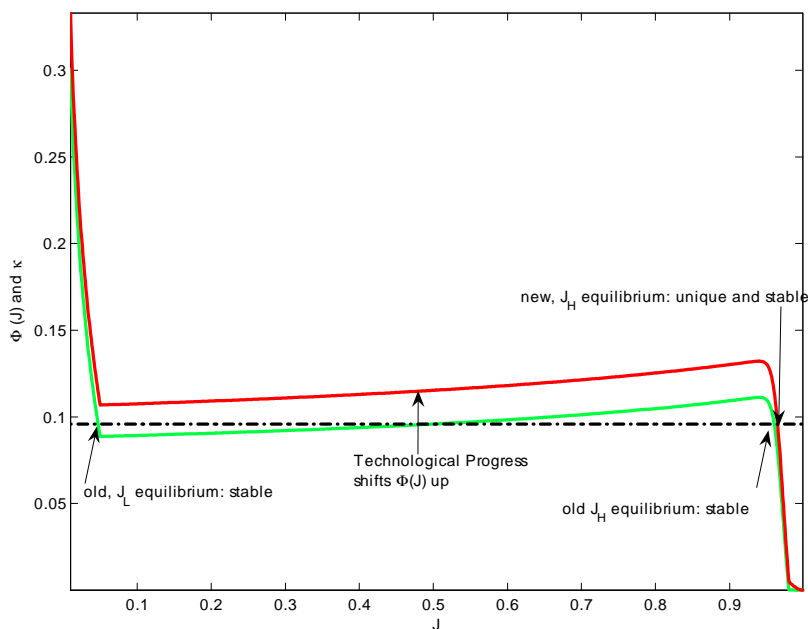


Figure 3: A growth miracle driven by technological progress.

and  $\delta \in (0.08, 0.12)$ . Therefore, we set  $\beta = 0.95$  and  $\delta = 0.10$ . The parameters  $\lambda$ ,  $\gamma$ , and  $\alpha$  deserve more consideration.

The first parameter,  $\lambda$ , governs the degree of increasing returns to scale in the economy. There has been a large debate in the recent literature on the magnitude of increasing returns in the economy. While earlier researchers (most notably, Hall, 1988) suggested that there are large increasing returns to scale in the economy, subsequent work has shown that the returns to scale can be best described as constant or at most moderately increasing. The latest estimates of  $\lambda$  are probably those constructed by Laitner and Stolyarov (2004). Their preferred point estimate is  $\lambda = 1.1$ , with confidence interval (1.03, 1.2). These figures are close to the estimates of Bartelsman, Caballero, and Lyons (1994), Burnside (1996), Burnside, Eichenbaum, and Rebelo (1995), Basu (1996), Basu and Fernald (1997), and Harrison (2003). Hence, we calibrate our model with  $\lambda = 1.1$ .

The next parameter,  $\gamma$ , represents the share of output that goes to capital and labor used directly in production, for a given value of  $\lambda$ . Note that in the model there is a difference between aggregate returns to scale and firm level

returns to scale. While at the aggregate level there are increasing returns to scale, at the firm level, as long as  $\gamma < 1$ , the returns to scale in *variable* inputs are decreasing. In our model, heterogenous productivity leads to a heterogenous degree of returns to scale in *all* inputs. For firms with higher productivity, the decreasing returns to scale in variable inputs dominate the increasing returns to scale effect of the fixed cost; for the firms with lower productivity, it is the opposite. These observations are broadly consistent with empirical findings of Basu (1996), and Basu and Fernald (1997).<sup>15</sup> As a benchmark, we consider  $\gamma = 0.85\lambda$ , which is the preferred value of Atkeson and Kehoe (2005) and is very close to the estimated value of 0.84 in Basu (1996).

The choice of the next parameter,  $\alpha$ , depends on the interpretation of  $s_k$ . Interpreted literally, this is the capital share of output. However, if a part of firms' (entrepreneurs') share of output, i.e.,  $(1 - \gamma/\lambda)$ , is interpreted as capital income, then  $s_k$  is less than the capital share of output. With this interpretation, one needs to take a stand on how the firm's share of output is divided between capital and labor. A commonly used rule is to split this share so that the capital share of output is  $\alpha$ . As a starting point, we set  $s_k$  to 0.4. This implies that when  $\gamma$  is set to  $(0.85\lambda)$ ,  $\alpha$  is equal to 0.47.

We have shown that for some functions  $a(j)$  there will be multiple stable steady states. The key property of the function  $a(j)$  that generates multiplicity of equilibria is that  $\bar{a}_J$  strongly dominates  $a_J$  for some  $J$ .<sup>16</sup> A function that has this property is one that is sufficiently flat on some interval  $(J_1, J_2)$ . The larger this interval is, the farther apart the stable steady states are from each other. In terms of firms' productivity distribution, this translates into the lower steady state having a large number of firms with nearly the same low productivity. Hence, we parameterize the function  $a(j)$  as follows:

$$a(j) = \begin{cases} j, & \text{if } j \leq J_1 \\ J_1 + b \left( \frac{j-J_1}{J_2-J_1} \right)^{N_1}, & \text{if } J_1 < j \leq J_2 \\ (J_1 + b) \left( \frac{j}{J_2} \right)^{N_2}, & \text{if } j > J_2. \end{cases} \quad (29)$$

We normalize  $\phi$  to 1,<sup>17</sup> and we choose  $\kappa$  and the five parameters pinning

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<sup>15</sup>See Kim (2004) for a detailed discussion on different ways of modeling increasing returns to scale.

<sup>16</sup>See proof of Proposition 2 in Appendix A.

<sup>17</sup>Notice that for our results only  $\kappa/\phi^{\frac{\lambda-\gamma}{1-\alpha\gamma}}$  matters (see equations (23) and (37) in

| $J_1$    | $J_2$ | $b$  | $N_1$ | $N_2$ | $\kappa$ |
|----------|-------|------|-------|-------|----------|
| $1e - 6$ | 0.97  | 2.13 | 280.2 | 104.2 | 0.50     |

Table 1: Parameter Values: function  $a(j)$  and fixed cost

down the productivity distribution ( $J_1, J_2, N_1, N_2, b$ ) so that the distribution of firms by size implied by our model in the two stable steady state is as close as possible to the distributions of firms by size in the average Least Developed Country (LDC) and in the U.S. (see Tybout, 2000, Table 1). Notice how in the average LDC the distribution of firms by size is characterized by a much higher share of small firms than in the U.S.

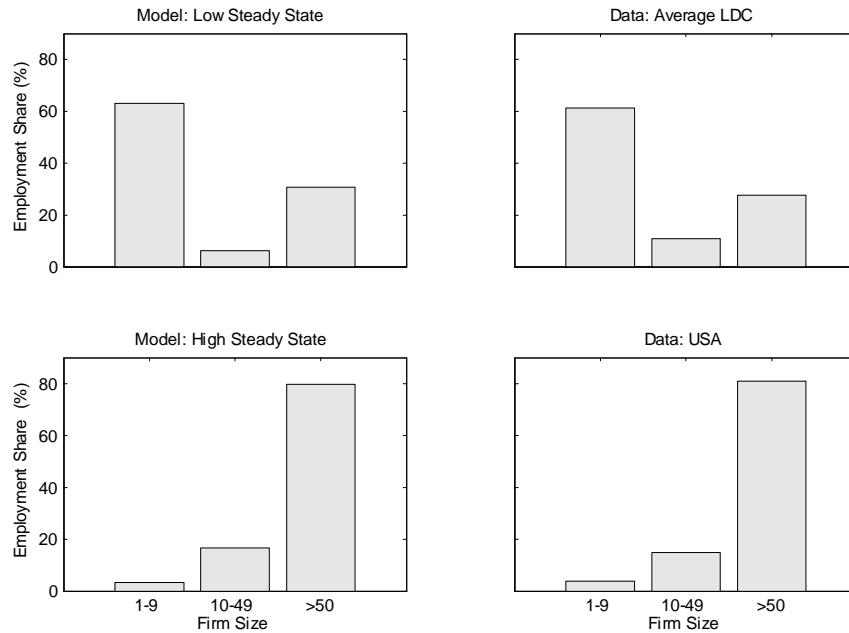


Figure 4: Employment Shares: model versus data.

Figure 4 portrays the distributions of firms by size for the U.S. and the average LDC (right column), together with the distributions for the high and low steady states of our model (left column).

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Appendix A).

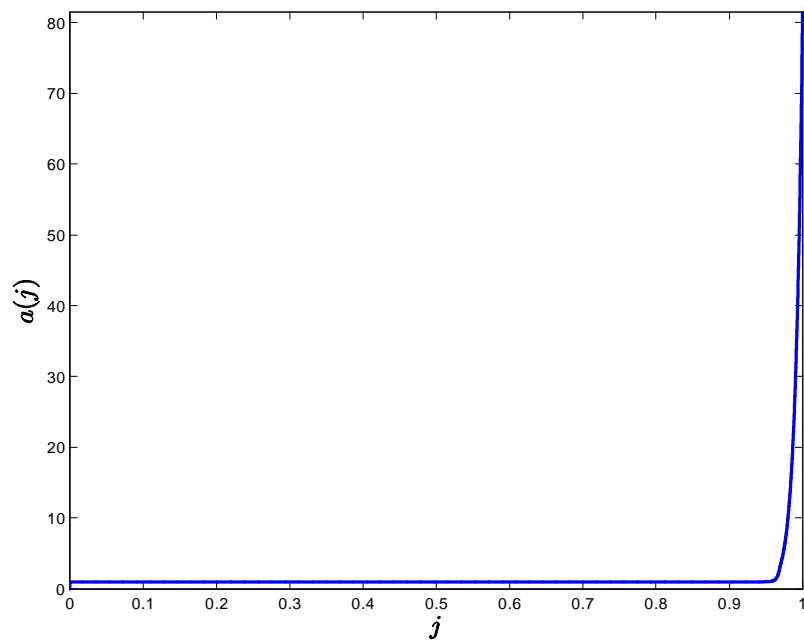


Figure 5: Calibrated function  $a(j)$ .

Figure 5 reports the function  $a(j)$ , which minimizes the distance between the model distributions and their empirical counterparts.

### 4.3 Sensitivity Analysis

In this section we conduct sensitivity analysis of the baseline calibration by analyzing how varying the degree of increasing returns to scale and the capital share maps into differences across the high and the low steady states of our model.

Tables 2-5 present the ratios of values of output and TFP levels for the two stable steady states for different parameter values. In the first column of Tables 2-5 we report the worst-case scenario of no increasing returns to scale, together with the most favorable function  $a(j)$ . In the remaining columns we maintain  $a(j)$  fixed to the calibrated function discussed above, and we analyze the effect of varying the capital share and the ratio of  $\gamma$  to  $\lambda$ .

|          |                                 | $\lambda$ |             |             |                         |             |            |             |
|----------|---------------------------------|-----------|-------------|-------------|-------------------------|-------------|------------|-------------|
|          |                                 | $1^a$     | <b>1.01</b> | <b>1.05</b> | <b>1.1</b>              | <b>1.15</b> | <b>1.2</b> | <b>1.25</b> |
| $\gamma$ | <b>0.95<math>\lambda</math></b> | 1.05      | 1.03        | 1.03        | 1.03                    | 1.03        | 1.03       | 1.04        |
|          | <b>0.9<math>\lambda</math></b>  | 1.12      | 1.06        | 1.06        | 1.07                    | 1.07        | 1.08       | 1.08        |
|          | <b>0.85<math>\lambda</math></b> | 1.19      | 1.10        | 1.10        | <b>1.11<sup>b</sup></b> | 1.12        | 1.13       | 1.14        |
|          | <b>0.8<math>\lambda</math></b>  | 1.28      | 1.14        | 1.16        | 1.17                    | 1.18        | 1.20       | 1.22        |

<sup>a</sup> Theoretical upper bound for  $\lambda \rightarrow 1$ . <sup>b</sup> Benchmark calibration.

Table 2: Relative TFP and returns to scale ( $s_k = 0.4$ )

|          |                                 | $\lambda$ |             |             |                         |             |            |             |
|----------|---------------------------------|-----------|-------------|-------------|-------------------------|-------------|------------|-------------|
|          |                                 | $1^a$     | <b>1.01</b> | <b>1.05</b> | <b>1.1</b>              | <b>1.15</b> | <b>1.2</b> | <b>1.25</b> |
| $\gamma$ | <b>0.95<math>\lambda</math></b> | 1.09      | 1.05        | 1.05        | 1.06                    | 1.06        | 1.07       | 1.07        |
|          | <b>0.9<math>\lambda</math></b>  | 1.20      | 1.10        | 1.11        | 1.12                    | 1.14        | 1.15       | 1.17        |
|          | <b>0.85<math>\lambda</math></b> | 1.33      | 1.17        | 1.19        | <b>1.21<sup>b</sup></b> | 1.23        | 1.26       | 1.30        |
|          | <b>0.8<math>\lambda</math></b>  | 1.50      | 1.25        | 1.28        | 1.32                    | 1.37        | 1.42       | 1.48        |

<sup>a</sup> Theoretical upper bound for  $\lambda \rightarrow 1$ . <sup>b</sup> Benchmark calibration.

Table 3: Relative output and returns to scale ( $s_k = 0.4$ )

The limiting case of  $\lambda$  equal to 1 has the least favorable implications for the existence of multiple steady states, because the model essentially

|       |             | $\lambda$ |             |             |                         |             |            |             |
|-------|-------------|-----------|-------------|-------------|-------------------------|-------------|------------|-------------|
|       |             | $1^a$     | <b>1.01</b> | <b>1.05</b> | <b>1.1</b>              | <b>1.15</b> | <b>1.2</b> | <b>1.25</b> |
| $s_k$ | <b>0.36</b> | 1.19      | 1.10        | 1.10        | 1.11                    | 1.12        | 1.13       | 1.13        |
|       | <b>0.4</b>  | 1.19      | 1.10        | 1.10        | <b>1.11<sup>b</sup></b> | 1.12        | 1.13       | 1.14        |
|       | <b>0.45</b> | 1.19      | 1.10        | 1.11        | 1.12                    | 1.13        | 1.14       | 1.15        |
|       | <b>0.55</b> | 1.20      | 1.10        | 1.11        | 1.12                    | 1.13        | 1.14       | 1.16        |
|       | <b>0.6</b>  | 1.20      | 1.11        | 1.12        | 1.13                    | 1.14        | 1.16       | 1.18        |
|       | <b>0.65</b> | 1.22      | 1.12        | 1.13        | 1.15                    | 1.19        | 1.26       | 1.45        |

<sup>a</sup> Theoretical upper bound for  $\lambda \rightarrow 1$ . <sup>b</sup> Benchmark calibration.

Table 4: Relative TFP and capital share ( $\gamma = 0.85$ )

|       |             | $\lambda$ |             |             |                         |             |            |             |
|-------|-------------|-----------|-------------|-------------|-------------------------|-------------|------------|-------------|
|       |             | $1^a$     | <b>1.01</b> | <b>1.05</b> | <b>1.1</b>              | <b>1.15</b> | <b>1.2</b> | <b>1.25</b> |
| $s_k$ | <b>0.36</b> | 1.31      | 1.16        | 1.17        | 1.19                    | 1.21        | 1.23       | 1.26        |
|       | <b>0.4</b>  | 1.33      | 1.17        | 1.19        | <b>1.21<sup>b</sup></b> | 1.23        | 1.26       | 1.30        |
|       | <b>0.45</b> | 1.38      | 1.19        | 1.21        | 1.24                    | 1.28        | 1.32       | 1.37        |
|       | <b>0.55</b> | 1.43      | 1.22        | 1.25        | 1.29                    | 1.34        | 1.40       | 1.48        |
|       | <b>0.6</b>  | 1.50      | 1.26        | 1.30        | 1.35                    | 1.43        | 1.54       | 1.70        |
|       | <b>0.65</b> | 1.75      | 1.39        | 1.48        | 1.66                    | 1.99        | 2.87       | 7.27        |

<sup>a</sup> Theoretical upper bound for  $\lambda \rightarrow 1$ . <sup>b</sup> Benchmark calibration.

Table 5: Relative output and capital share ( $\gamma = 0.85$ )

collapses to the standard neoclassical model. It is important to see how large the steady state differences can be for  $\lambda$  arbitrarily close to 1. The condition for the existence of multiple steady states translates to  $a(J)$  being (almost) a constant over some interval. In this case, the extremes of this interval correspond to the two steady-state values of  $J$ . This implies that the ratio of total factor productivity, capital, and output levels in the two stable steady states can be at most:

$$\frac{TFP^H}{TFP^L} \leq \left( \frac{1 - s_k}{\gamma - s_k} \right)^{1-s_k}, \quad (30)$$

$$\frac{K^H}{K^L} \leq \frac{1 - s_k}{\gamma - s_k}, \quad (31)$$

$$\frac{Y^H}{Y^L} \leq \frac{1 - s_k}{\gamma - s_k}. \quad (32)$$

When our economy approaches constant returns to scale, the endogenous TFP mechanism alone is quite powerful and it can generate differences in TFP and output across steady states of up to 28 and 50 percent, respectively.

In the studies of the long-run behavior of an economy, using the proper measure of capital share of output is of crucial importance. For example, for the unified theory of Parente and Prescott (2005) to be successful, the capital share of output should be between 0.55 and 0.65. The magnitude of this share depends on the definition of investment (capital). In the context of this paper it is proper to define investment as “any allocation of resources that is designed to increase future productivity” (see Parente and Prescott, 2000). That is, investment should include maintenance and repair, research and development, software, investment in organizational capital, and investment in human capital. Parente and Prescott (2000) find that including these items in investment implies that the capital share of output is larger than 1/2 and can reach as high as 2/3.<sup>18</sup>

The capital share is important for two reasons. First, there is a standard neoclassical effect: The higher the capital share is, the higher the effect of TFP is on the economy. For two economies differing only in their TFP, the steady state capital ratio relates to the TFP ratio as follows:  $(K^H/K^L) = (TFP^H/TFP^L)^{\frac{1}{1-s_k\lambda}}$ . The higher the share of capital is, the higher the difference in steady state capital is between the two economies.

Second, the capital share directly impacts TFP, because it enters into the definition of TFP in (21) and into the definition of the function  $\Phi(J)$  in (24). Because of the highly non-linear nature of TFP and  $\Phi$  as functions of the cutoff  $J$ , it is not possible to derive analytically the effect of an increase in the capital share on the resulting TFP differences across the steady states. However, when  $\lambda$  tends to 1 the theoretical upper bound on these differences gets larger as the capital share grows (see equation (31) above). For all numerical experiments (Table 4) the increase in the capital share of output increases the TFP differences. Combined with the “neoclassical effect” described above, this leads to even larger differences in output and in capital across the steady states (Table 5).

For  $s_k = 0.4$  and  $\lambda = 1.1$ , our baseline calibration, TFP and output differ

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<sup>18</sup>For details and references see the original paper. A large portion of the unmeasured capital is organization capital. Findings of Atkeson and Kehoe (2005) imply that the value of organizational capital in the US manufacturing sector is *larger* than the value of physical capital.

across steady states by a factor of 1.1 and 1.21, respectively. Differences across steady states increase in  $\lambda$  and  $s_k$ . When both  $s_k$  and  $\lambda$  are high, the resulting differences in output are large, reaching as much as 627 percent.

## 5 Conclusions

Recent empirical studies attribute a sizable fraction of cross-country income differences to differences in TFP. These differences reflect, in part, the fact that the fraction of low productivity firms in less developed countries is much higher than in industrialized countries. We introduce heterogeneity in productivity across firms in an otherwise standard model. In our model differences in TFP arise endogenously, and we obtain multiple steady-state equilibria for an arbitrarily small degree of increasing returns to scale. If an economy is in a good steady state, only the most productive firms operate, leading to high TFP, capital, and output. In an economy locked in a poverty trap the pool of producers is sullied by low productivity firms, with low TFP, capital and output.

We analyze the qualitative properties of our model by studying a growth miracle. A growth miracle can be induced by technological progress or by a decline in entry barriers and it is accompanied by a shift of employment from small to large firms. This is consistent with the Industrial Revolution and postwar Japan growth experiences. We calibrate our model using standard parameter values and a distribution of productivity across firms which matches the distribution of firms by size across developed and LDC countries. Our calibrated economy displays large differences in TFP across stable steady states and even larger differences in output levels.

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# A Proofs of Propositions

## A.1 Proof of Proposition 1

Equations (14) and (15) imply that the fraction of labor used in production  $u_t$  is a function only of the cutoff  $J_t$  :

$$u_t = \frac{\bar{a}(J_t)}{\bar{a}(J_t) + \frac{\lambda-\gamma}{(1-\alpha)\gamma}a(J_t)}.$$

Substituting this expression of  $u_t$  into equation (21), we obtain:

$$\begin{aligned} TFP(J_t) &= \phi^{\gamma-\lambda} \left[ \frac{\frac{\lambda-\gamma}{(1-\alpha)\gamma}a(J_t)}{\bar{a}(J_t) + \frac{\lambda-\gamma}{(1-\alpha)\gamma}a(J_t)} \right]^{\lambda-\gamma} \times \\ &\times \left[ \frac{\bar{a}(J_t)}{\bar{a}(J_t) + \frac{\lambda-\gamma}{(1-\alpha)\gamma}a(J_t)} \right]^{(1-\alpha)\gamma} [\bar{a}(J_t)]^{\lambda-\gamma} \end{aligned} \quad (33)$$

Differentiating the previous expression,

$$\text{signum}(TFP_J) = \text{signum} \left[ \begin{aligned} &\frac{(\lambda-\gamma)(1-\alpha)\gamma}{(1-\alpha)\gamma\bar{a} + (\lambda-\gamma)a} \frac{a_J}{a} (\bar{a} - a) + \\ &\bar{a}_J \left( \frac{1}{\bar{a}} - \frac{1}{\bar{a} + \frac{\lambda-\gamma}{(1-\alpha)\gamma}a} \right) \end{aligned} \right], \quad (34)$$

where

$$TFP_J = \frac{\partial TFP(J)}{\partial J}, \quad a_J = \frac{\partial a(J)}{\partial J}, \quad \bar{a}_J = \frac{\partial \bar{a}(J)}{\partial J}.$$

The terms in parenthesis in (34) are positive and they are multiplied by positive terms. Hence,  $TFP_J > 0$ .

Using the firms' first-order condition in (14) and the zero profit condition in (8) we get that the following relation between the cutoff  $J_t$  and capital  $K_t$  :

$$\begin{aligned} \kappa &= (1-\alpha) \frac{\gamma}{\lambda} \left[ \frac{\lambda-\gamma}{(1-\alpha)\gamma} \right]^{\lambda-\gamma} \left[ \frac{\bar{a}(J_t)}{\bar{a}(J_t) + \frac{\lambda-\gamma}{(1-\alpha)\gamma}a(J_t)} \right]^{(1-\alpha)\gamma + (\lambda-\gamma) - 1} \\ &\quad [a(J_t)]^{\lambda-\gamma} K_t^{\alpha\gamma} \left[ \frac{1}{a(J_t)} \int_{J_t}^1 a(j) dj - (1 - J_t) \right] \end{aligned}$$

For a given  $K_t$  the left-hand side of this equation varies with  $J_t$  from  $+\infty$  to zero. Moreover, one can easily show that the left-hand side is decreasing in  $J_t$ . Thus, there exists a unique  $J_t$  which solves the equation. In addition, it is increasing in  $K_t$ . Because  $J_t$  is increasing in  $K_t$ , so is output  $Y_t$  and wage  $w_t$ . In addition, since, for a given  $K_t$ , output  $Y_t$  is uniquely determined, so is the  $R_t$ ; i.e.,  $R_t$  is a function of  $K_t$ . ■

## A.2 Proof of Proposition 2

We use equations (13) and (20) to express  $K_t$  as a function of  $r_t$  and  $J_t$ . By substituting this expression of  $K_t$  into equation (33) and by using equation (14), we get

$$r_t^{\frac{1-\alpha\gamma}{\alpha\gamma}} \kappa = \eta \cdot \Phi(J_t) \quad (35)$$

where

$$\Phi(J) \equiv \left[ \frac{\bar{a}(J)}{\bar{a}(J) + \frac{\lambda-\gamma}{(1-\alpha)\gamma} a(J)} \right]^{\frac{\lambda-1}{1-\alpha\gamma}} a(J)^{\frac{\lambda-\gamma}{1-\alpha\gamma}} \int_J^1 \left[ \frac{a(j)}{a(J)} - 1 \right] dj, \quad (36)$$

and  $\eta$  is a constant:

$$\eta = \phi^{\frac{\gamma-\lambda}{1-\alpha\gamma}} \left(1 - \frac{\gamma}{\lambda}\right) \left(\frac{\alpha\gamma}{\lambda}\right)^{\frac{\alpha\gamma}{1-\alpha\gamma}} \left[ \frac{\lambda-\gamma}{(1-\alpha)\gamma} \right]^{\frac{\lambda-\gamma}{1-\alpha\gamma}-1}. \quad (37)$$

Since  $\Phi(J)$  is continuous and  $\Phi(0) = \infty$ ,  $\Phi(1) = 0$ , there always exists a  $J^*$  that satisfies the equation below:

$$[1/\beta - (1-\delta)]^{\frac{1-\alpha\gamma}{\alpha\gamma}} \kappa = \eta \cdot \Phi(J^*). \quad (38)$$

We have to show that for any  $J^*$  satisfying equation (38) there exists a pair  $(c^*, K^*)$ , both positive, such that  $R(K^*) = 1/\beta$  and  $c^* = Y(K^*) - \delta K^*$ . This is an immediate consequence of proposition 1.

If there is more than one  $J^*$  satisfying equation (38), then there will be multiple steady states. Note that for given parameters  $\lambda$ ,  $\gamma$ , and  $\alpha$ , the shape of the function  $\Phi(J)$  is entirely determined by the shape of function  $a(j)$ : If  $a(j)$  is such that  $\Phi_J > 0$  then (38) can have multiple solutions. To conclude the proof, we must show that there exists a function  $a(j)$  such that  $\Phi_J > 0$ .

The sign of  $\Phi_J$  can be checked as follows:

$$\text{signum}(\Phi_J) = \text{signum} \left\{ \begin{aligned} & \frac{\lambda-1}{1-\alpha\gamma} \left[ \frac{\bar{a}_J}{\bar{a}} - \frac{\bar{a}_J + \frac{\lambda-\gamma}{(1-\alpha)\gamma} a_J}{\bar{a} + \frac{\lambda-\gamma}{(1-\alpha)\gamma} a} \right] \\ & + \left[ \frac{\lambda-\gamma}{1-\alpha\gamma} - \frac{1-J}{\int_J^1 (a(j)-a(J)) dj} \right] \frac{a_J}{\bar{a}} \end{aligned} \right\}.$$

Consider a function

$$a(j) = \begin{cases} j, & \text{if } j \leq J_1 \\ J_1 + b_1 \left( \frac{j-J_1}{J_2-J_1} \right)^{N_2}, & \text{if } J_1 < j \leq J_2 \\ b_2 \left( \frac{j}{J_2} \right)^{N_2}, & \text{if } j > J_2. \end{cases} \quad (39)$$

where  $b_i, N_i > 0$ . The constant and  $b_2$  must satisfy the following restrictions to guarantee continuity:

$$b_2 = (J_1 + b_1).$$

Taking limits for  $N_1 \rightarrow \infty$ :

$$\begin{aligned} \lim_{N_1 \rightarrow \infty} a(J) &= J_1 \\ \lim_{N_1 \rightarrow \infty} a_J(J) &= 0 \\ \lim_{N_1 \rightarrow \infty} \bar{a}(J) &= \frac{1}{1-J} \left[ (J_2 - J) J_1 + \frac{J_1 + b_1}{1 + N_2} \left( \frac{1}{(J_2)^{N_2}} - J_2 \right) \right] > J_1 > 0 \\ \lim_{N_1 \rightarrow \infty} \bar{a}_J(J) &= \frac{\lim_{N_1 \rightarrow \infty} \bar{a}(J) - J_1}{(1-J)} > 0 \end{aligned}$$

Therefore, as long as  $\lambda > 1$ ,  $\lim_{N_1 \rightarrow \infty} \Phi_J > 0$ . It follows that there exists a finite  $N_1$  for which  $\Phi_J > 0$ . If  $\Phi_J > 0$ , then  $\Phi$  will have at least one local minimum and one local maximum, say  $\underline{\Phi}$  and  $\bar{\Phi}$ . For any  $\kappa \in \left( [1/\beta - (1-\delta)]^{\frac{\alpha\gamma-1}{\alpha\gamma}} \eta \underline{\Phi}, [1/\beta - (1-\delta)]^{\frac{\alpha\gamma-1}{\alpha\gamma}} \eta \bar{\Phi} \right)$  our model has multiple steady-state equilibria. ■

### A.3 Proof of Proposition 3

Linearizing (22) about a steady state:

$$\begin{bmatrix} \hat{K}_{t+1} \\ \hat{C}_{t+1} \end{bmatrix} = \begin{bmatrix} Y' + 1 - \delta & -1 \\ \frac{\sigma R'}{R} C (Y' + 1 - \delta) & 1 - \frac{\sigma R'}{R} C \end{bmatrix} \begin{bmatrix} \hat{K}_t \\ \hat{C}_t \end{bmatrix}$$

| Condition  | Steady state stability |
|--|------------------------|
| $R' < 0$   | saddle                 |
| $Y' - \delta > \frac{\sigma CR'}{R} > 0$<br>$[(Y' - \delta) - \frac{\sigma CR'}{R}]^2 > 4\frac{\sigma CR'}{R}$ | source                 |
| $Y' - \delta > \frac{\sigma CR'}{R} > 0$<br>$[(Y' - \delta) - \frac{\sigma CR'}{R}]^2 < 4\frac{\sigma CR'}{R}$ | unstable spiral        |
| $\frac{\sigma CR'}{R} > Y' - \delta > 0$<br>$[(Y' - \delta) - \frac{\sigma CR'}{R}]^2 > 4\frac{\sigma CR'}{R}$ | sink                   |
| $\frac{\sigma CR'}{R} > Y' - \delta > 0$<br>$[(Y' - \delta) - \frac{\sigma CR'}{R}]^2 < 4\frac{\sigma CR'}{R}$ | stable spiral          |

Table A.1: Steady-state stability for different parameters configurations

The eigenvalues of the transition matrix are given by:

$$\xi_{1,2} = 1 + \frac{(Y' - \delta) - \frac{\sigma CR'}{R} \pm \sqrt{[(Y' - \delta) - \frac{\sigma CR'}{R}]^2 - 4\frac{\sigma CR'}{R}}}{2}.$$

If  $R' < 0$  (odd steady states) both eigenvalues are real and  $\xi_1 < 1 < \xi_2$ . If  $R' > 0$  (even steady states) there are four possible cases:

1.  $Y' - \delta > \frac{\sigma CR'}{R} \wedge [(Y' - \delta) - \frac{\sigma CR'}{R}]^2 > 4\frac{\sigma CR'}{R} \Rightarrow \xi_{1,2} \in \mathbb{R}, \|\xi_{1,2}\| > 1;$
2.  $Y' - \delta > \frac{\sigma CR'}{R} \wedge [(Y' - \delta) - \frac{\sigma CR'}{R}]^2 < 4\frac{\sigma CR'}{R} \Rightarrow \xi_{1,2} \in \mathbb{C}, \|\xi_{1,2}\| > 1;$
3.  $Y' - \delta < \frac{\sigma CR'}{R} \wedge [(Y' - \delta) - \frac{\sigma CR'}{R}]^2 > 4\frac{\sigma CR'}{R} \Rightarrow \xi_{1,2} \in \mathbb{R}, \|\xi_{1,2}\| < 1;$
4.  $Y' - \delta < \frac{\sigma CR'}{R} \wedge [(Y' - \delta) - \frac{\sigma CR'}{R}]^2 < 4\frac{\sigma CR'}{R} \Rightarrow \xi_{1,2} \in \mathbb{C}, \|\xi_{1,2}\| < 1.$

■