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MARKOV SWITCHING IN GARCH PROCESSES AND MEAN REVERTING STOCK MARKET VOLATILITY

ABSTRACT

This paper introduces four models of conditional heteroscedasticity that contain markov switching parameters to examine their multi-period stock-market volatility forecasts as predictions of options-implied volatilities. The volatility model that best predicts the behavior of the options-implied volatilities allows the student-*t* degrees-of-freedom parameter to switch such that the conditional variance and kurtosis are subject to discrete shifts. The half-life of the most leptokurtic state is estimated to be weak, so expected market volatility reverts to near-normal levels fairly quickly following a spike.

KEYWORDS: Conditional Heteroscedasticity, Asset Price Volatility, Kurtosis, Markov Switching

JEL CLASSIFICATION: C22, G12

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1. Introduction

Volatility clustering is a well-documented feature of financial rates of return: Price changes that are large in magnitude tend to occur in bunches rather than with equal spacing. A natural question is how long financial markets will remain volatile, because volatility forecasts are central to calculating optimal hedging ratios and options prices. Indeed we can study the behavior of options-implied stock-market volatilities to find stylized facts that parametric volatility models should aim to capture. Two stylized facts that conventional volatility models, notably generalized autoregressive conditional heteroskedasticity [GARCH, Bollerslev (1986)], find hard to reconcile are: 1) conditional volatility can increase substantially in a short amount of time at the onset of a turbulent period; 2) the rate of mean reversion in stock-market volatility appears to vary positively and nonlinearly with the level of volatility. In other words, stock-market volatility does not remain persistently two to three times above its normal level the same way it can persist at 30-40 percent above normal.

Hamilton and Susmel (1994) and Lamoureux and Lastrapes (1993) highlight the forecasting difficulties of conventional GARCH models by showing that they can provide worse multi-period volatility forecasts than constant-variance models. In particular, multi-period GARCH forecasts of the volatility are too high in a period of above-normal volatility. Friedman and Laibson (1989) address the forecasting issue by not allowing the conditional variance in a GARCH model to respond proportionately to 'large' and 'small' shocks. In this way, the conditional variance is restrained from increasing to a level from which volatility forecasts would be undesirably high. One drawback of this approach is that in such a model the conditional volatility might understate the true variance by not responding sufficiently to large shocks and thereby never be pressed to display much mean reversion. Thus, such "threshold" models do not necessarily address the two stylized facts listed above: sharp upward jumps in volatility, followed by fairly rapid reversion to near-normal levels. This article endeavors to craft a volatility model that can address these two stylized facts from within the class of GARCH models with markov-switching parameters. Markov-switching parameters ought to enable the volatility to experience discrete shifts and discrete changes in the persistence parameters.

Partly in response to Lamoureux and Lastrapes (1990), who observed that structural breaks in the variance could account for the high persistence in the estimated conditional variance, Hamilton and Susmel (1994) and Cai (1994) introduced markov-switching parameters to ARCH models and I extend the approach to GARCH models, since the latter are more flexible and widely used. The next section presents tractable methods of estimating GARCH models with markov-switching parameters. The third section describes four specifications that are estimated and provides in-sample and out-of-sample goodness-of-fit test results. The fourth section uses the estimated models to generate multi-period forecasts of stock-market volatility and compares the forecasts with options-implied volatilities to see which of the GARCH/Markov switching models best explains the two stylized facts described above.

2. GARCH/Markov switching volatility models

Each of the volatility model specifications will assume a student-t error distribution with n_t degrees of freedom in the dependent variable y:

$$\begin{aligned} y_t &= \mu_t + \epsilon_t \\ &\epsilon_t \sim \text{student-}t(\text{mean} = 0, n_t, h_t) \\ &n_t > 2 \end{aligned} \tag{1}$$

In all of the models, the conditional mean, μ_t, is allowed to switch according to a markov process governed by a state variable, S_t:

$$\begin{aligned} \mu_t &= \mu_l S_t + \mu_h(1 - S_t) \\ &S_t \in \{0, 1\} \qquad \forall t \\ \text{Prob.}(S_t = 0 \mid S_{t-1} = 0) &= p \\ \text{Prob.}(S_t = 1 \mid S_{t-1} = 1) &= q \end{aligned} \tag{2}$$

The unconditional probability of $S_t = 0$ equals $(1 - q)/(2 - p - q)$. The variance of ϵ_t is denoted σ_t^2 and is a function of n_t and h_t in all of the models considered such that $\sigma_t^2 = f(n_t, h_t)$, where the specific function f varies across the models. In all cases, however, h is assumed to be GARCH(1,1) process with markov-switching parameters also governed by S, so that a general form for h is

$$h_t(S_t, S_{t-1}, ..., S_0) \;=\; \gamma(S_t) + \alpha(S_{t-1})\epsilon_{t-1}^2 + \beta(S_{t-1})h_{t-1}(S_{t-1}, ..., S_0) \tag{3}$$

Note that the presence of lagged h on the right side of (3) causes the GARCH variable to be a function of the entire history of the state variable. If h were an ARCH(p) process, then h would depend only on the p most recent values of the state variable, as in Cai (1994) and Hamilton and Susmel (1994). Here I discuss how methods described in Kim (1994) can be applied to make estimation feasible for GARCH processes subject to markov switching.

Clearly it is not practical to examine all of the possible sequences of past values of the state variable when evaluating the likelihood function for a sample of more than a thousand observations, as the number of cases to consider exceeds 1000 by the time $t = 10$. Kim (1994) addresses this problem by introducing a collapsing procedure that greatly facilitates evaluation of the likelihood function at the cost of introducing a degree of approximation that does not appear to distort the calculated likelihood by much. The collapsing procedure, when applied to a GARCH process, calls for treating the conditional dispersion, h_t, as a function of at most the most recent M values of the state variable S. For the filtering to be accurate, Kim notes that when h is p-order autoregressive, then M should be at least $p+1$. In the GARCH(1,1) case $p = 1$, so we would have to keep track of M^2 or four cases, based on the two most recent values of a binary state variable. Thus, h_t is treated as a function of only S_t and S_{t-1}: $h_t^{(i,j)} = h_t(S_t = i, S_{t-1} = j)$.

Denoting φ_t as the information available through time t, we keep the number of cases to four by integrating out S_{t-1} before plugging lagged h into the GARCH equation:

$$h_t^{(i)} = \sum_{j=0}^{1} \text{Prob.}(S_{t-1} = j \mid S_t = i, \varphi_t) h_t^{(i,j)} \tag{4}$$

This method of collapsing of $h_t^{(i,j)}$ onto $h_t^{(i)}$ at every observation gives us a tractable GARCH formula which is approximately equal to the exact GARCH equation from equation (3):

$$h_t^{(i,j)} = \gamma(S_t = i) + \alpha(S_{t-1} = j)\left(\epsilon_{t-1}^{(j)}\right)^2 + \beta(S_{t-1} = j) h_{t-1}^{(j)}, \tag{5}$$

Note that the collapsing procedure integrates out the first lag of the state variable, S_{t-1}, from the GARCH function, h_t, at the right point in the filtering process to prevent the conditional density from becoming a function of a growing number of past values of the state variable.

From this general framework, we choose specifications that differ according to the parameters that switch and the relationship between the GARCH process, h, and the variance σ^2. In several specifications, the GARCH processes are functions of lagged values of the state variable, but not the contemporaneous value, S_t. For these, we treat h_t as a function of only S_{t-1}, so we only need to keep track of two cases: $h^{(j)} = h(S_{t-1} = j)$. Furthermore, after integrating out S_{t-1}, we are left with a scalar in the collapsing process:

$$\hat{h}_t = \text{Prob.}(S_{t-1} = 0 \mid \varphi_t) h_t^{(0)} + \tag{6}$$

$$\text{Prob.}(S_{t-1} = 1 \mid \varphi_t) h_t^{(1)}$$

A tractable GARCH equation is then an even simpler version of equation (5):

$$h_t^{(j)} = \gamma + \alpha(S_{t-1} = j) \left(\epsilon_{t-1}^{(j)}\right)^2 + \beta(S_{t-1} = j) \hat{h}_{t-1}, \tag{7}$$

Another feature of this GARCH/Markov switching framework is that the state variable implies a connection between the mean stock return and the variance and possibly kurtosis. If the mean stock return is lower in the high-volatility state, then the model can explain negatively skewed distributions, both unconditional and conditional on available information. The student-t distributions have zero skewness only when conditional on particular values of the state variables, which are unobservable.

3. Four specifications and estimation results

The first specification is a GARCH analogue to Cai's (1994) ARCH model with markov switching in γ. The variance is assumed to follow a GARCH process so that $\sigma_t^2 = h_t$ and the only parameter in h_t subject to markov switching is γ. This type of switching is tantamount to allowing shifts in the unconditional variance, since the unconditional variance of the ordinary, constant-parameter GARCH(1,1) process is $\gamma/(1 - \alpha - \beta)$. For this model, the GARCH variance takes the form

$$h_t^{(i,j)} = \gamma(S_t = i) + \alpha\left(\epsilon_{t-1}^{(j)}\right)^2 + \beta h_{t-1}^{(j)}, \tag{8}$$

with constant α and β. We denote this model as the GARCH-UV model for GARCH with switching in the unconditional variance. In practice, we parameterize $\gamma(S_t)$ as $g(S_t)\gamma$, where $g(S = 1)$ is normalized to unity.

The second specification is a GARCH analogue to Hamilton and Susmel's (1994) ARCH model with markov switching in a normalization factor g, where the variance $\sigma_t^2 = g_t h_t$. In this case, the GARCH equation (5) takes the form

$$h_t^{(j)} = \gamma + \frac{\alpha}{g(S_{t-1} = j)}\left(\epsilon_{t-1}^{(j)}\right)^2 + \beta \hat{h}_{t-1}, \tag{9}$$

where γ and β are constant and $g(S = 1)$ is normalized to unity. We denote this model as the GARCH-NF model for GARCH with switching in the normalization factor, g. Note that in the GARCH-NF model the GARCH process in equation (9) is not a function of S_t, so estimation is somewhat simplified.

The third specification is a markov-switching analogue to Hansen (1994), where the variance follows a GARCH process ($\sigma_t^2 = h_t$) and the student-t degrees-of-freedom parameter is allowed to switch. Hansen (1994) introduces a model in which the student-t degrees-of-freedom parameter, n_t is allowed to vary over time as a probit-type function of variables dated up to time $t-1$. Because Hansen's (1994) specification is not conducive to multi-period forecasting, however, we chose to make n_t follow a markov process governed by S_t:

$$n_t = n_l S_t + n_h(1 - S_t).$$

Although n_t does not enter the GARCH equation (7) in this specification, the GARCH process is still a function of the state variable, because state-switching in the mean implies that ϵ is a function of the state variable:

$$h_t^{(j)} = \gamma + \alpha \left(\epsilon_{t-1}^{(j)}\right)^2 + \beta \hat{h}_{t-1}. \tag{10}$$

Because the kurtosis of a student-t random variable equals $3(n_t - 2)/(n_t - 4)$ and is uniquely determined by n_t, we call this the GARCH-K model for GARCH with switching in the conditional kurtosis.

The fourth specification is similar to the GARCH-K model except the variance is assumed to be

$$\sigma_t^2 = h_t n_t/(n_t - 2), \tag{11}$$

rather than $\sigma_t^2 = h_t$. In this model, the GARCH process h_t scales the variance of ϵ_t for a given value of the shape parameter n_t. Here it is convenient to define $v_t = \frac{1}{n_t}$, so that

$(1 - 2v_t) = \left(\frac{n_t - 2}{n_t}\right)$ and the GARCH equation (7) becomes

$$h_t^{(j)} = \gamma + \alpha(1 - 2v_{t-1}^{(j)}) \left(\epsilon_{t-1}^{(j)}\right)^2 + \beta \hat{h}_{t-1}. \tag{12}$$

We denote this specification as the GARCH-DF model for GARCH with switching in the degrees-of-freedom parameter. As in the GARCH-NF and GARCH-K models, h is a function of S_{t-1}, but not S_t, in the GARCH-DF model. The GARCH-DF model shares two features with the GARCH-NF model: the variance is subject to discrete shifts and the lagged squared residuals are endogenously downweighted in states where $\frac{\sigma^2}{h}$ is large. With the GARCH-K model, the GARCH-DF model shares the feature of time-varying conditional kurtosis, so that conditional fourth moments are not assumed to be constant.

We also report results on the usual GARCH(1,1) model with markov switching in the mean and a model of switching ARCH with a leverage effect (SWARCH-L), as in Hamilton and Susmel (1994). The SWARCH-L model has switching in a normalizing factor in the variance: $\sigma_t^2 = g_t h_t$, where h_t follows an ARCH(2) process with a leverage effect:

$$h_t^{(j,k)} = \gamma + \frac{(\alpha_1 + \delta D_{t-1}^{(j)})}{g(S_{t-1} = j)} \left(\epsilon_{t-1}^{(j)}\right)^2 + \frac{\alpha_2}{g(S_{t-2} = k)} \left(\epsilon_{t-2}^{(k)}\right)^2, \tag{13}$$

where $D_{t-1}^{(j)}$ is dummy variable that equals one when $\epsilon(S_{t-1} = j)_{t-1} < 0$. The leverage effect posits that negative stock returns increase debt-to-equity ratios, making firms riskier initially. Hence the leverage-effect parameter δ is expected to have a positive sign.

The log-likelihood function for the GARCH-DF model, for example, is

$$\begin{aligned} lnL_t^{(i,j)} &= ln\Gamma(.5(n_t^{(i)}+1)) - ln\Gamma(.5n_t^{(i)}) - .5ln(\pi n_t^{(i)} h_t^{(j)}) \\ &\quad -.5(n_t^{(i)}+1)ln\left(1+\frac{\left(\epsilon_t^{(i)}\right)^2}{h_t^{(j)} n_t^{(i)}}\right) \end{aligned} \tag{14}$$

where $i \in \{0,1\}$ corresponds with $S_t \in \{0,1\}$, $j \in \{0,1\}$ corresponds with $S_{t-1} \in \{0,1\}$ and Γ is the gamma function. The function maximized is the log of the expected likelihood or

$$\sum_{t=1}^{T} ln\left(\sum_{i=0}^{1}\sum_{j=0}^{1}\text{Prob.}(S_t=i, S_{t-1}=j \mid \varphi_{t-1})L_t^{(i,j)}\right) \tag{15}$$

as in Hamilton (1990).

Estimation results

The four GARCH/Markov switching volatility models, the usual GARCH(1,1) model, and the SWARCH-L model are applied to daily percentage changes in the S&P 500 index from January 6, 1982 to December 31, 1991. Observations from the post-1991 period are reserved for evaluation of the out-of-sample fit.

Some interpretation of the parameter estimates in Table 1 follows. The GARCH-DF model shows switching in the student-t degrees-of-freedom parameter between the values of 2.64 and 8.28. This implies that conditional fourth moments do not exist in one state, whereas the conditional kurtosis is $3(n-2)/(n-4) = 4.4$ in the other state. The weight given to lagged squared residuals in the GARCH process is shown to be $\alpha(1-2v(S_{t-1}))$ in equation (12), and this weight shifts with the state variable between .009 and .027. In this way, shocks drawn from the low degree-of-freedom state do not affect the persistent GARCH dispersion process proportionately. Most importantly, shifts in the degrees-of-freedom parameter bring potentially large discrete shifts in the variance. A shift out of the low degree-of-freedom state causes the variance to decrease by about 68 percent, holding the dispersion constant:

$$\left(\frac{\frac{n_h}{n_h-2}}{\frac{n_l}{n_l-2}}\right) = .32$$

The unconditional probability of being in the low degree-of-freedom state is about 10 percent with a half-life of 5 trading days. The unconditional value for the degrees-of-freedom parameter is about 6.8. The GARCH-DF model also suggests that stock returns are negatively skewed, because the mean stock return is below normal in the high-volatility state when $S_t = 0$. In fact, all of the models find negative skewness except the conventional GARCH model.

The GARCH-NF model finds an estimate of the variance inflation factor $g(S=0) = 12.59$ with a large standard error. The effective sample from which to estimate this parameter is small, because the unconditional probability of $S_t = 0$ is only about 1 percent.

The factor $g(S=0)$ raises γ by a significant multiple of 5.7 in the GARCH-UV model, but the unconditional probability of being in that state is only 11 percent. The state with

$g(S = 1)$ is extremely persistent with $q = .995$.

The GARCH-K model estimates that the degrees-of-freedom parameter switches between 10.7 and 4.2, with an unconditional value of about 6. Both states are highly persistent with nearly identical transition probabilties. Two states for the mean stock return are better defined in the GARCH-K model than in the conventional GARCH model with switching in the mean. Table 1 shows that in the usual GARCH(1,1) model, the mean stock return, μ, is virtually identical in both states. Hence the two states are not well-identified and the calculation of standard errors for the transition probabilities failed.

Using daily data, the weights attached to lagged squared residuals are not significant in the SWARCH-L model, with the borderline exception of the leverage-effect parameter, δ. The normalizing factor, g, is estimated to raise the variance by a multiple of 3.78 in the high-volatility state, which has unconditional probability 0.13. The high degree of persistence of both states suggests that low and high volatility states constitute regimes, as opposed to short-lasting episodes. The GARCH-DF model, on the other hand, finds relatively short-lasting low-degree-of-freedom states.

If we were certain that significant state switching occurred in the mean, then likelihood ratio tests of state switching in the degrees-of-freedom parameters and g would be appropriate. But, the GARCH model suggests that switching in the mean cannot be taken for granted, so likelihood ratio tests cannot assume that the transition probabilities are identified under the null of no state switching in v or g. Hansen (1992) has discussed simulation methods to derive critical values for such likelihood ratio tests with nonstandard distributions. The critical values are computationally burdensome to calculate, however, so we do not pursue that strategy here. Instead, we follow Vlaar and Palm (1993) by using a goodness-of-fit test that is valid for data that are not identically distributed. We perform the test over the in-sample period (1982-91) and an out-of-sample period (1992-September 1994). We divide the observations into 100 groups based on the probability of observing a value smaller than the actual residual. If the model's time-varying density function fits the data well, these probabilities should be uniformly distributed between zero and one. Following Vlaar and Palm (1993),

$$
n_i = \sum_{t=1}^{T} I_{it} \qquad \text{where} \qquad \begin{aligned} I_{it} &= 1 \qquad \text{if} \quad \frac{(i-1)}{100} < EF(\epsilon_t, \hat{\theta}) \leq \frac{i}{100} \\ &= 0, \qquad \text{otherwise.} \end{aligned} \tag{16}
$$

The expected value of the cumulative density function, F, is taken across the states that might have held at each time. The goodness-of-fit test statistic equals $100/T \sum_{i=1}^{100}(n_i - T/100)^2$ and is distributed χ^2_{99} under the null.

Table 2 provides results from the goodness-of-fit tests. Only the GARCH-DF model is not rejected on an in-sample basis, with a .57 probability value. All six models are rejected out of sample, however.

To examine the source of failure in models other than GARCH-DF in the goodness-of-fit test, Figures 1a and 1b plot the distribution of the in-sample observations across the 100 groups. Figure 1a shows that the GARCH-DF observations are roughly uniformly distributed across the groups, whereas the GARCH-NF observations have a hump-shaped distribution in Figure 1b. Too many GARCH-NF residuals are near the center of the cumulative density, which implies that the model's conditional densities are overly peaked, i.e., are too leptokurtic. By not allowing the conditional kurtosis to change, the GARCH-NF model apparently fits a constant conditional kurtosis that is too high. If time-varying kurtosis is an important feature of stock returns, then it worth studying the distribution of the observations in the GARCH-K model also. Figure 1c shows that the GARCH-K model also provides conditional densities that are too leptokurtic on average, despite its provision for time-varying kurtosis. The reason might be that the GARCH-K model has a very persistent state in which fourth moments do not exist, because $q = .9986$. It is possible that the GARCH-K model overstates the persistence of periods of fat-tailed stock returns distributions: they might be better described as episodes than regimes, as the GARCH-DF model suggests.

4. Predicting Options-Implied Volatilities

As an economic test of the GARCH/Markov switching models, we use them to predict the next day's opening level of the VIX market volatility index compiled by the Chicago Board Options Exchange. The VIX index is derived from an options-pricing model is not a direct observation of market expectations. Nevertheless, many financial market participants are interested in options-implied volatilities in their own right. The VIX index attempts to represent, as closely as possible, the implied volatility on a hypothetical at-the-money option on the S&P100 with 30 calendar days (22 trading days) to expiration. Details on the construction of the VIX from near-the-money options prices are in Whaley (1993). The implied volatility on an option reflects beliefs about average volatility over the life of the option. Thus, the constant 22 day horizon of the VIX implies that we must use the GARCH/Markov switching volatility models to create multi-period forecasts of volatility for all periods between one and 22 days ahead. In other words, to predict the VIX index well, the GARCH/Markov switching models need to provide good multi-period forecasts for a full range from one to 22 trading days ahead.

Daily data on the VIX index were available from 1986-1992. Because the VIX data are based on the S&P100 and the stock-market data are S&P500 returns, the mean of the VIX index is slightly higher than the average volatility forecast from the GARCH models. The broader S&P500 index is somewhat less volatile than the S&P100. For this reason, I normalize each volatility measure with its 1986-1992 sample mean. Hence a value of 1.5 means that volatility is expected to be one-and-a-half times its normal level in the coming month. Details on the construction of multi-period forecasts from the GARCH/Markov switching models are in the appendix.

I use a minimum forecast error variance criterion to measure the closeness of the model-implied and options-implied monthly volatilities. If we denote the options-implied volatility as VIX and the monthly average of the model-predicted volatilities as $\overline{\sigma}$, then the criterion is

$$\frac{1}{T}\sum_{t=1}^{T}(\overline{\sigma}_t - VIX_t)^2.$$

Note that $\overline{\sigma}_t$ for a Wednesday, for example, is calculated using information available through Tuesday, whereas VIX_t is the data from Wednesday's opening quotes. In this sense, we are using the GARCH/Markov switching models to predict the options-implied volatilities.

Table 3 shows that only the GARCH-DF and GARCH-K models predict the options-implied volatility index better than the conventional GARCH model, and the GARCH-DF model achieves a notable 14 percent reduction in the forecast error variance.

Figures 2a-c depict the 22-day average volatility forecasts for all the models and the VIX volatility in the aftermath of the October 1987 stock market crash. As described in Schwert (1987), for several days after October 19, 1987 options markets became very thin and the options written contained extremely large risk premia, i.e., implied volatilities. Figures 2a-c show that the VIX index reached about eight times its normal level immediately following the crash, but returned to less than two times normal by the end of October 1987. The GARCH-DF model best predicts the VIX index throughout November and early December 1987. The switch to $n(S_t = 0) = 2.64$ led to a downweighting, from .027 to .009, of the lagged squared residuals in the persistent GARCH process. Furthermore, the conditional variance temporarily shifted discretely upward for as long as $n(S_t = 0)$ was expected to persist.

The volatility implied by the GARCH-K model in Figure 2a, in contrast, overpredicts the VIX index for about six weeks, beginning at the end of October 1987. The variance in the GARCH-K model is a GARCH process, so it displays the same over-persistence that characterizes the conventional GARCH model, shown in Figure 2c. In fact, the forecasts from the conventional GARCH model and the GARCH-K model look very similar. The GARCH-NF model in Figure 1a, on the other hand, underpredicts volatility following the crash. The GARCH-NF model quickly switched to the state where $g(S_t = 1) = 12.59$, so the squared residuals were given little weight in the GARCH process and h did not increase much. The variance, $\sigma_t^2 = g_t h_t$ did increase with $g = 12.59$, but the increase was never projected to last long with $p = .75$. Consequently the forecasted average volatility for the month never increased to more than three times the normal level in the GARCH-NF model. In Figure 2b, the GARCH-UV model badly underpredicts the VIX index in late October 1987, but does fairly well in November and December 1987. The GARCH-UV model estimates a constant and relatively low weight, α, on the lagged squared residuals in the GARCH process, so the conditional variance never increases to more than three times normal, in contrast to the spike in the VIX index. In this sense, the GARCH-UV does not necessarily describe the rate of mean reversion in stock-market volatility well, since it does not capture the initial volatility spike. In Figure 2c, the SWARCH-L model shows a good deal of persistence, but does not put enough weight on lagged squared residuals to lift the conditional variance to the levels necessary to match the spike in the VIX index either.

Figures 3 focuses on a milder volatility spike in October 1989, when the VIX index peaked at about 2.5 times its normal level. Again, the GARCH-DF tends to split through the middle of oscillations of the VIX index better than the other model-implied volatilities, although the improvement is less marked than in Figure 2. The same general patterns hold in Figure 3, as in Figure 2, with the GARCH-K and GARCH models tending to overpredict volatility and the GARCH-UV and SWARCH-L models showing persistence, but failing to yield sufficiently dramatic initial increases in volatility.

5. Conclusions

This paper introduces a tractable framework for adding markov-switching parameters to conditional variance models. Four different specifications of markov-switching volatility models are estimated, and the addition of markov-switching parameters is found to have a variety of effects on the behavior of the conditional volatility, relative to the model without switching. The specification found to predict options-implied expectations of stock-market volatility best is the one in which the student-t degrees-of-freedom parameter switches so as to induce substantial discrete shifts in the conditional variance. This model model allows for two sources of mean reversion in the wake of large shocks that are not available in a standard model: A switch out of the fat-tailed state is estimated to induce a 68 percent decrease in volatility for a given level of dispersion; and the weight given to the most recent shock decreases by two-thirds when the fat-tailed state pertains, thereby reducing the influence and persistence of large shocks.

Another novel feature of this model is that it relates stock returns to the degree of leptokurtosis in the conditional returns distribution. Traditional models, in contrast, assume constant conditional kurtosis and relate expected returns to the conditional variance. The point estimates support the hypothesis that stock returns are generally lower in the more fat-tailed state.

We also draw economically relevant comparisons between the behaviors of options-implied volatilities and the conditional variances from the volatility models studied. Since options-implied volatilities serve as useful proxies for market expectations of volatility, it is interesting to observe that the condtional variance from one of the switching-in-the-variance models reverts to normal about as quickly as the options-implied volatility following large shocks, such as the stock market crash of October 1987. The conventional volatility model, in contrast, has a conditional variance that remains above normal with considerably greater persistence. Thus, markov switching in the variance is shown to add a realistic degree of mean reversion to the conditional variance. In addition, the description of time-varying stock return skewness and kurtosis provided by these models could prove useful in analyzing options prices on the S&P500 index.

An interesting extension would be to model the transition probabilities of the markov process as time-varying functions of conditioning variables in order to test whether transitions into and out of fat-tailed states could be better predicted using more information.

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Appendix on multi-period volatility forecasts

Forecasts of the volatility m periods ahead are based on the well-known relationship between GARCH models and autoregressive, moving-average representations of the squared disturbances. A GARCH(1,1) process,

$$h_t = \gamma + \alpha \epsilon_{t-1}^2 + \beta h_{t-1}, \tag{17}$$

implies that the squared residuals obey an ARMA(1,1) process:

$$\epsilon_t^2 = \gamma + (\alpha + \beta)\epsilon_{t-1}^2 - \beta(\epsilon_{t-1}^2 - h_{t-1}) + (\epsilon_t^2 - h_t) \tag{18}$$

where $\epsilon_t^2 - h_t$ is a mean zero error that is uncorrelated with past information. In forecasting the squared residuals m periods ahead with the GARCH-DF model, for example, we define $H_t = \epsilon_t^2(1 - 2v_t)$. In this case H has an ARMA(1,1) representation,

$$H_t = \gamma + (\alpha + \beta)H_{t-1} - \beta(H_{t-1} - \hat{h}_{t-1}) + (H_t - h_t), \tag{19}$$

where

$$E_t[H_{t+1} \mid H_t] = h_{t+1} = \gamma + \alpha H_t + \beta \hat{h}_t \tag{20}$$

Since the sample size is large, longer-range forecasts can be built from the asymptotic forecasting equation for first-order autoregressive processes, so that for $m > 1$

$$E_t[H_{t+m} \mid H_t] = (\alpha+\beta)^{m-1} h_{t+1} + \left[1 - (\alpha+\beta)^{m-1}\right] \frac{\gamma}{1-\alpha-\beta} \tag{21}$$

It remains to integrate out the unobserved states:

$$E_t\epsilon_{t+m}^2 = \sum_{i=0}^{1}\sum_{j=0}^{1} \text{Prob.}(S_{t+m} = i, S_t = j \mid \varphi_t) E_t[H_{t+m} \mid S_t = j] \frac{1}{1 - 2v_{t+m}^{(i)}} \tag{22}$$

where $H_t(S_t = j) = \left(\epsilon_t^{(j)}\right)^2 (1 - 2v_t^{(j)})$. The expected average variance over the next 22 trading days is then taken as

$$\overline{\sigma_t^2} = \frac{1}{22} \sum_{m=1}^{22} E_t\epsilon_{t+m}^2 \tag{23}$$

Similar forecasts are drawn for the other models with H defined such that $H_t = \frac{\epsilon_t^2}{g_t}$ in the GARCH-NF model and $H_t = \epsilon_t^2 - \gamma_t$ in the GARCH-UV model.

For the SWARCH-L model, the multi-period forecasts are derived by recursive substituion as in Hamilton and Susmel (1994).

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Table 1: GARCH/Markov switching models applied to daily percentage changes in S&P 500 index
Note: Standard errors are in parentheses

param.	GARCH-DF	GARCH-NF	GARCH-UV	GARCH-K	GARCH	SWARCH-L
log-lik.	-3294.3	-3294.1	-3292.7	-3295.3	-3301.0	-3311.5
$\mu(S_t = 0)$	.0107	-1.333	-.0971	.0158	.0542	.0366
	(.1431)	(1.487)	(.1287)	(.0219)	(.1287)	(.0884)
$\mu(S_t = 1)$	.0619	.0576	.0636	.0803	.0556	.0585
	(.0207)	(.0164)	(.0169)	(.0224)	(.0890)	(.0168)
$v(S_t = 0)$	.3787	.1478	.1762	.0931	.1860	.1833
	(.0480)	(.0209)	(.0090)	(.0323)	(.0195)	(.0198)
$v(S_t = 1)$	.1208	.1478	.1762	.2393	.1860	δ=.041
	(.0272)	(.0209)	(.0090)	(.0270)	(.0195)	(.025)
γ	.0105	.0109	.0124	.0233	.0228	.6912
	(.0035)	(.0038)	(.0033)	(.0066)	(.0064)	(.0372)
α	.0360	.0334	.0138	.0328	.0344	α_1=1.3E-4
	(.0076)	(.0060)	(.0046)	(.0074)	(.0077)	(.0083)
β	.9466	.9537	.9554	.9307	.9394	α_2=.0192
	(.0102)	(.0082)	(.0090)	(.0323)	(.0121)	(.0139)
$g(S_t = 0)$	n.a.	12.59	5.703	n.a.	n.a.	3.782
		(6.414)	(1.882)			(.4780)
$g(S_t = 1)$	n.a.	1	1	n.a.	n.a.	1
p	.8544	.7479	.9602	.9978	.9144	.9849
	(.0961)	(.1644)	(.0173)	(.0021)		(.0076)
q	.9842	.9980	.9950	.9986	.9420	.9977
	(.0148)	(.0017)	(.0018)	(.0013)		(.0012)

Table 2 Chi-square goodness-of-fit tests for GARCH/Markov switching models in-sample period:1982-1991 out-of-sample: 1992-Sep. 1994 Note: prob. values are in parentheses		
model	*in-sample*	*out-of-sample*
GARCH-DF	96.65 (.577)	152.1 (4.8E-4)
GARCH-NF	193.7 (.000)	208.1 (.000)
GARCH-UV	136.6 (.007)	188.4 (.000)
GARCH-K	235.4 (.000)	294.3 (.000)
GARCH	140.0 (.004)	228.4 (.000)
SWARCH-L	231.0 (.000)	307.9 (.000)

Table 3 **Predicting Options-Implied Volatility Index** **with GARCH/Markov switching models: 1986-92** **Note: Size of forecast error variance** **relative to GARCH model in parentheses**	
model	*Forecast Error Variance*
GARCH-DF	.0365 (.86)
GARCH-NF	.0548 (1.33)
GARCH-UV	.0589 (1.43)
GARCH-K	.0399 (.97)
GARCH	.0413 (1.00)
SWARCH-L	.0956 (2.31)

Figure 1a

Distribution of GARCH-DF Residuals Into 100 Groups Based on Cumulative Density Function

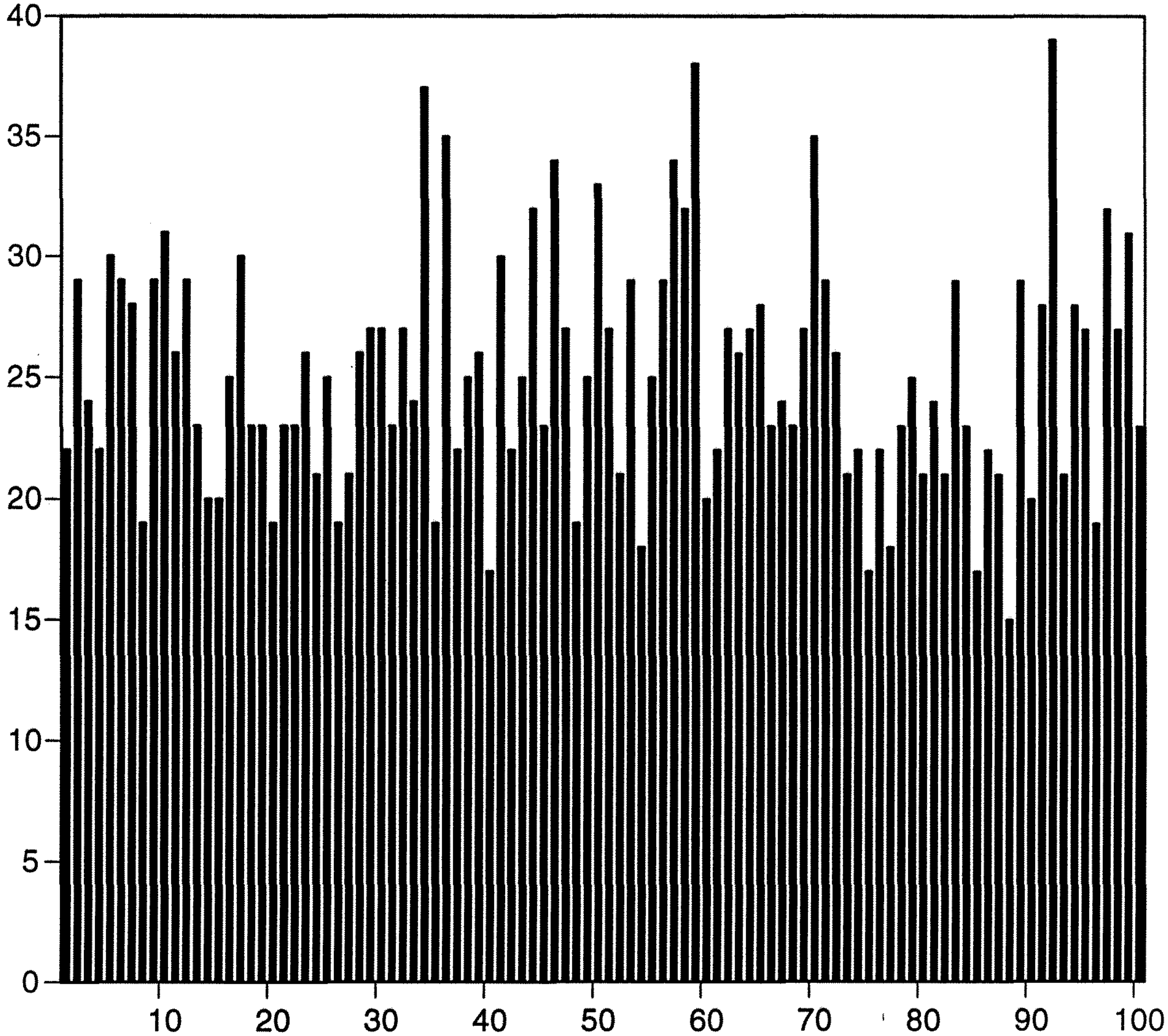


Figure 1b

Distribution of GARCH-NF Residuals Into 100 Groups Based on Cumulative Density Function

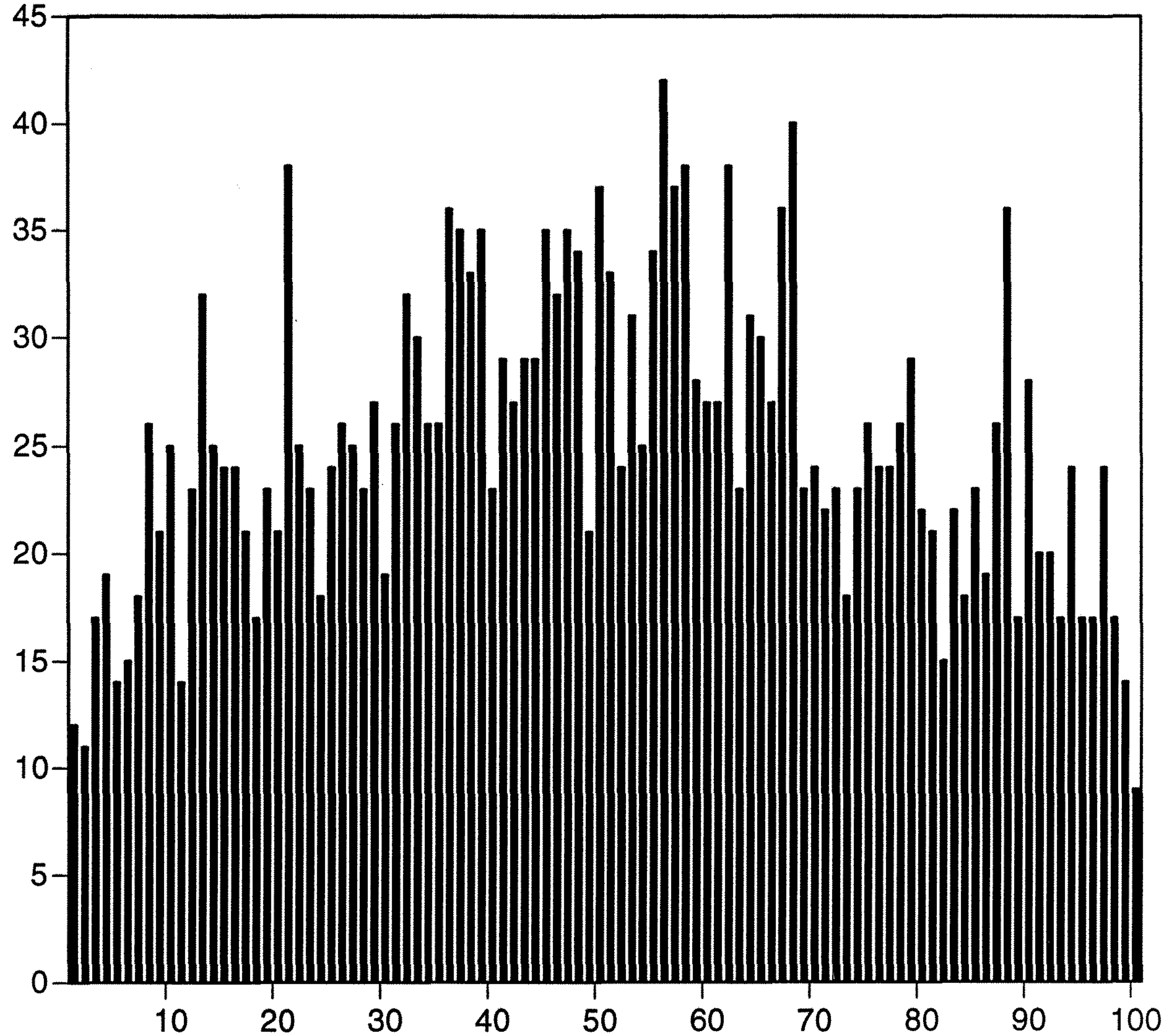


Figure 1c

Distribution of GARCH-K Residuals into 100 Groups Based on Cumulative Density Function

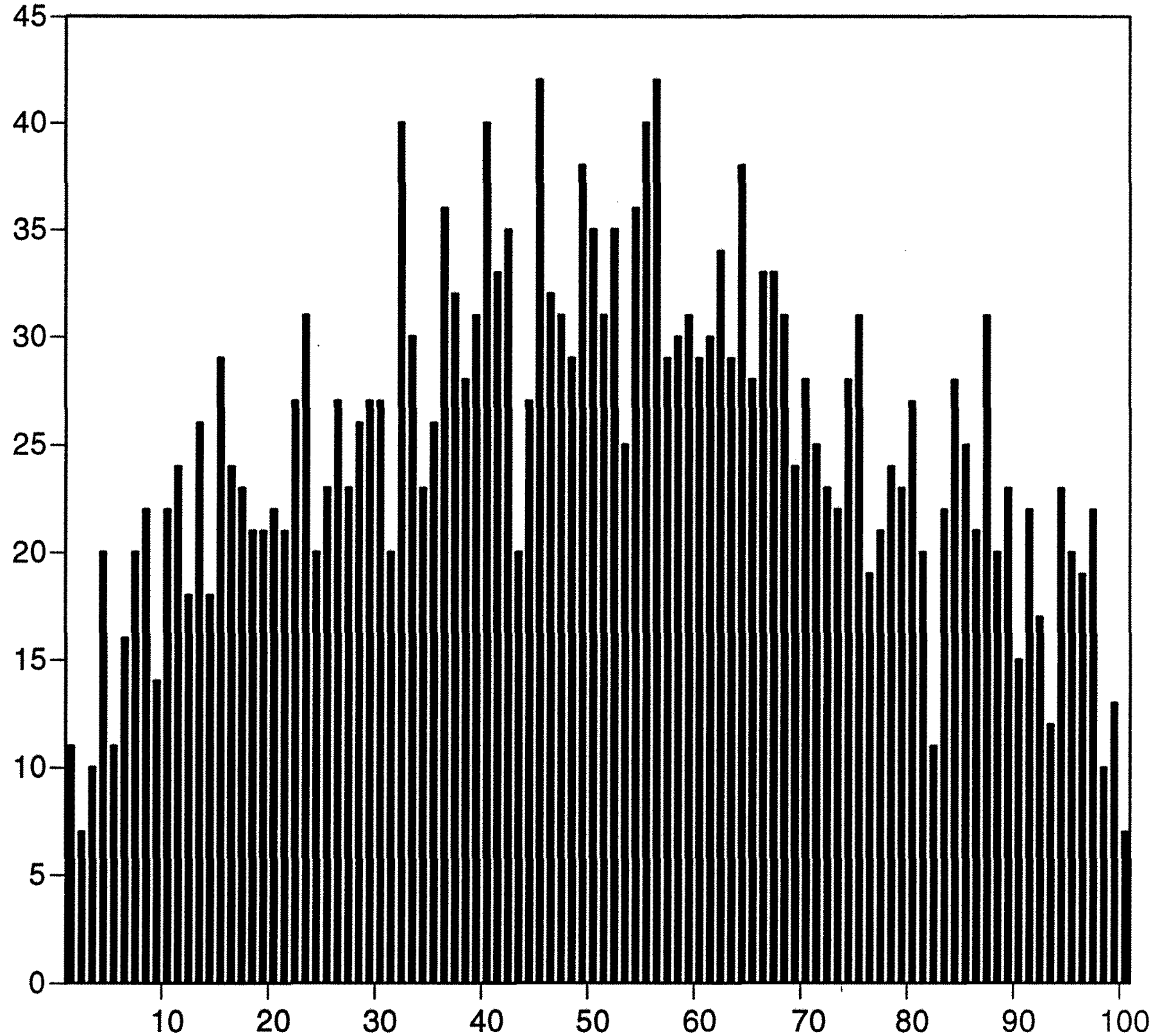


Figure 2a

VIX options-implied and model-implied volatilities

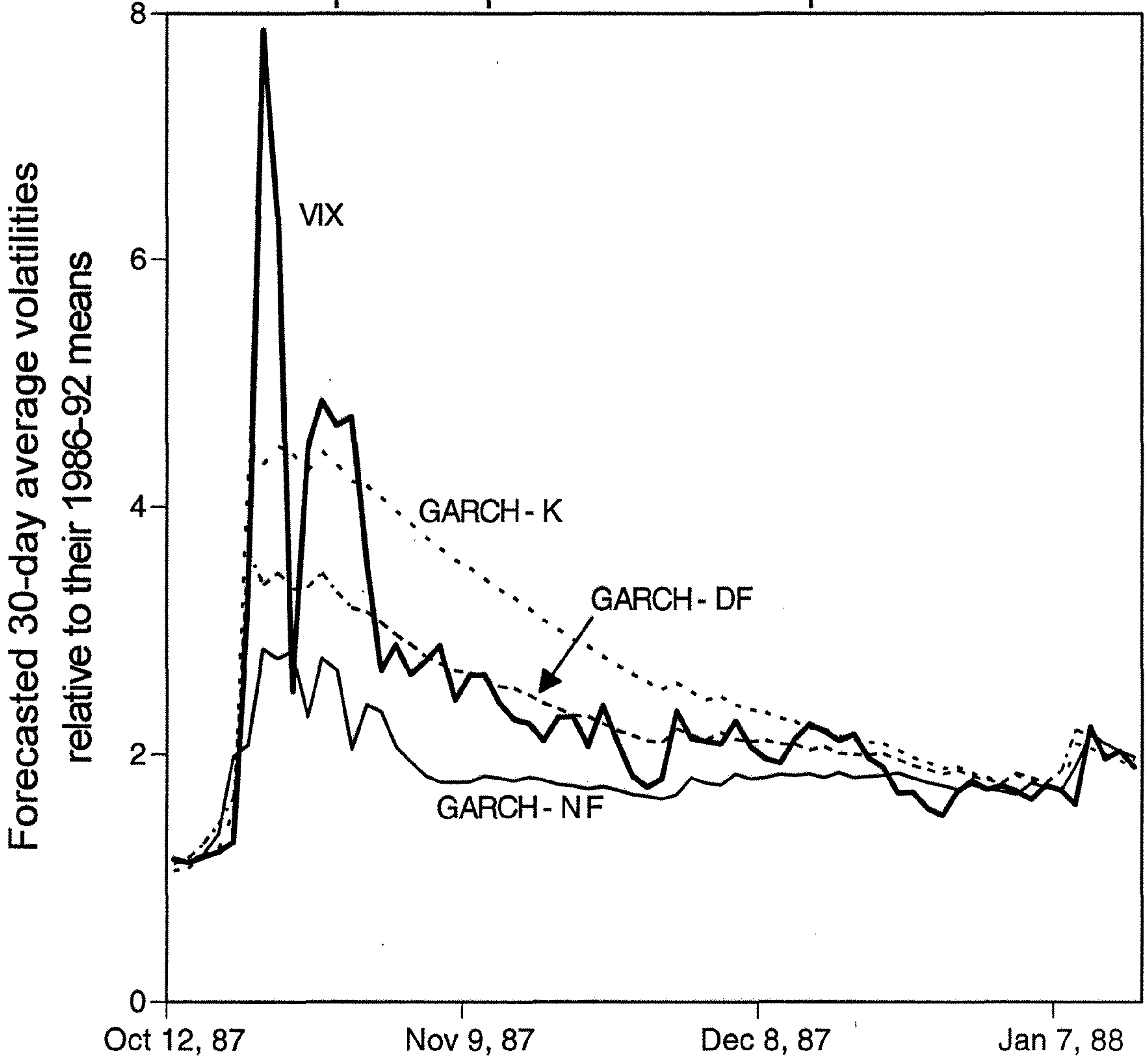


Figure 2b

VIX options-implied and model-implied volatilities

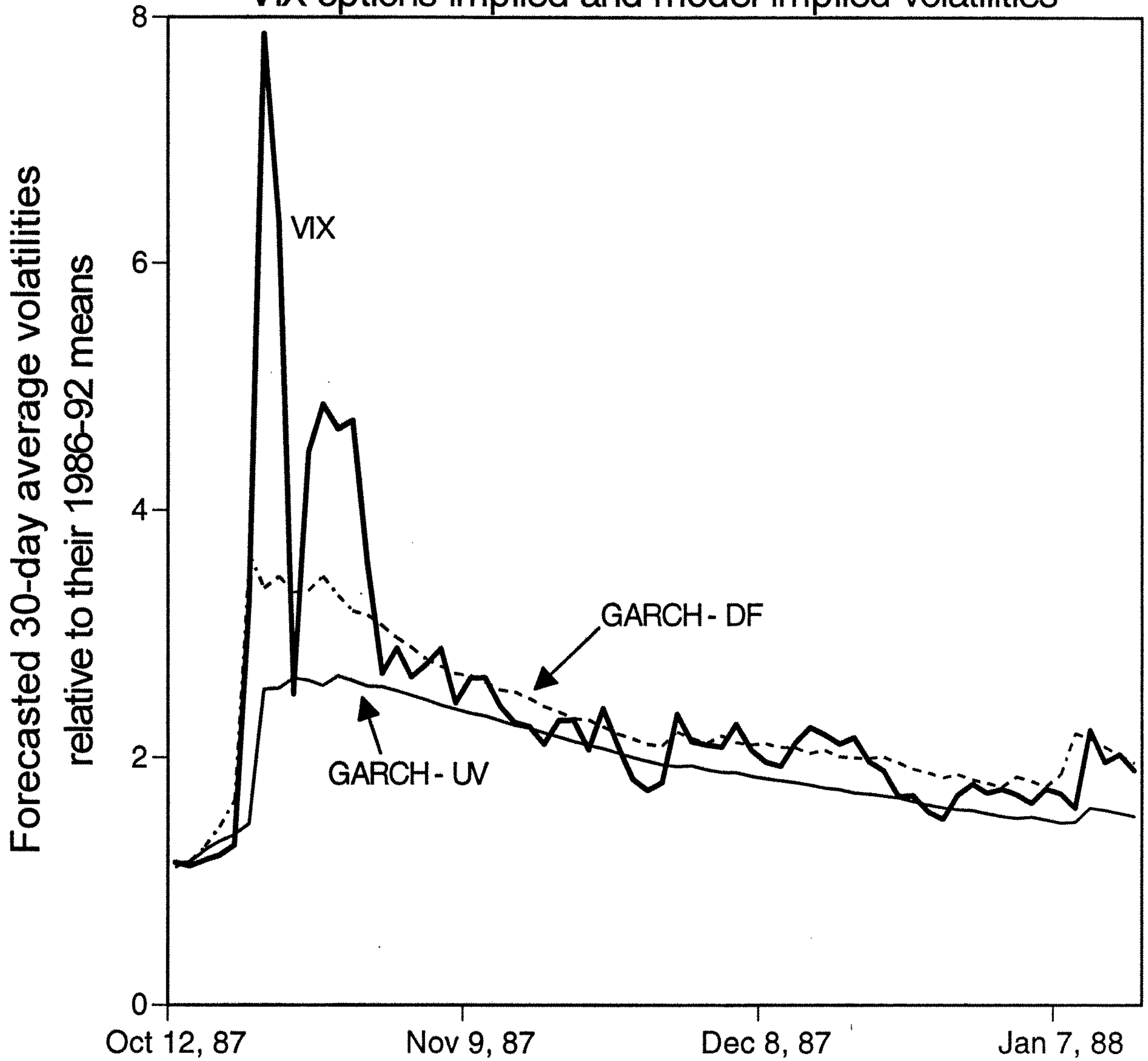
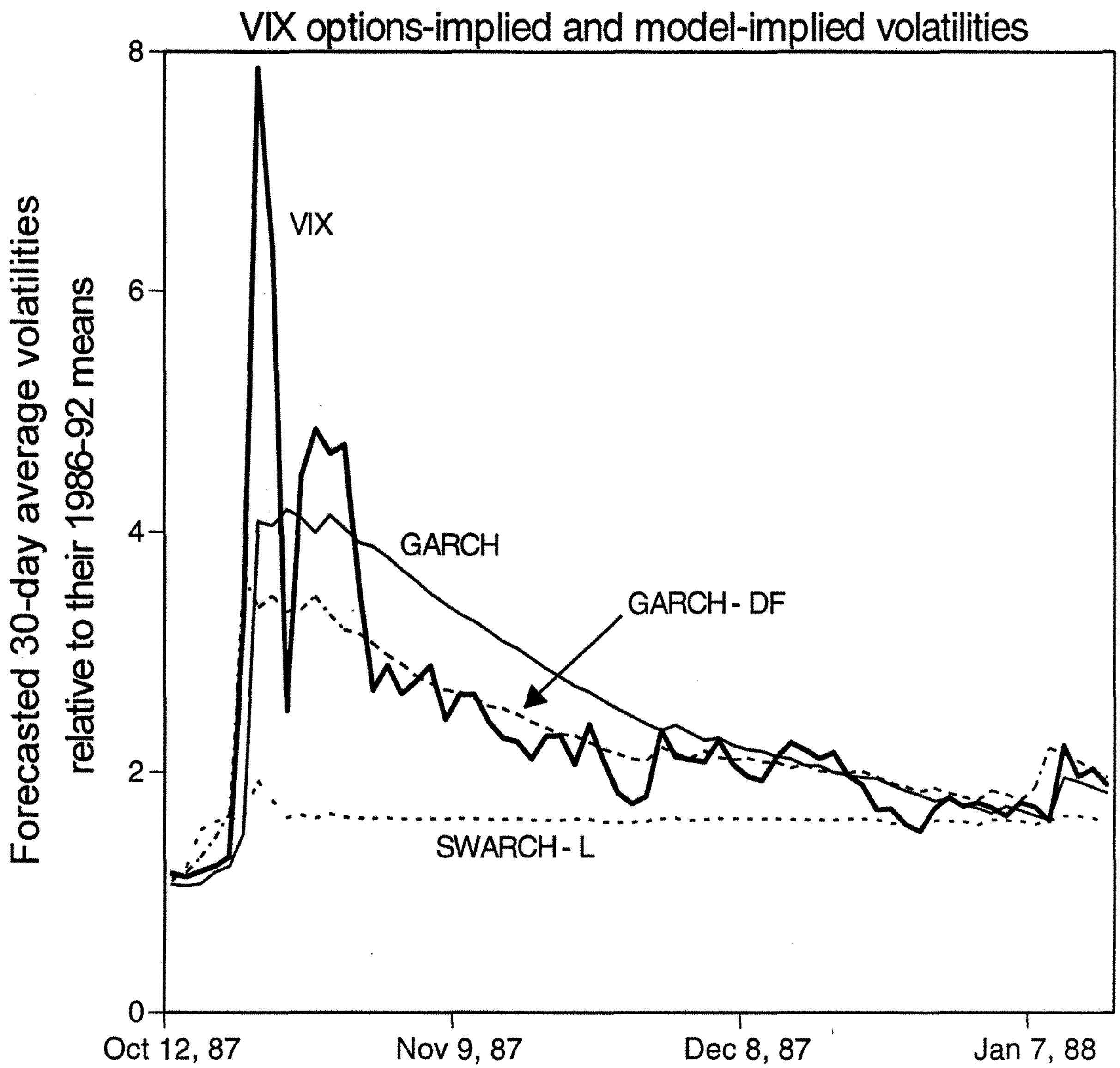


Figure 2c



VIX options-implied and model-implied volatilities
Forecasted 30-day average volatilities relative to their 1986-92 means
8
6
4
2
0
VIX
GARCH
GARCH - DF
SWARCH - L
Oct 12, 87
Nov 9, 87
Dec 8, 87
Jan 7, 88

Figure 3a

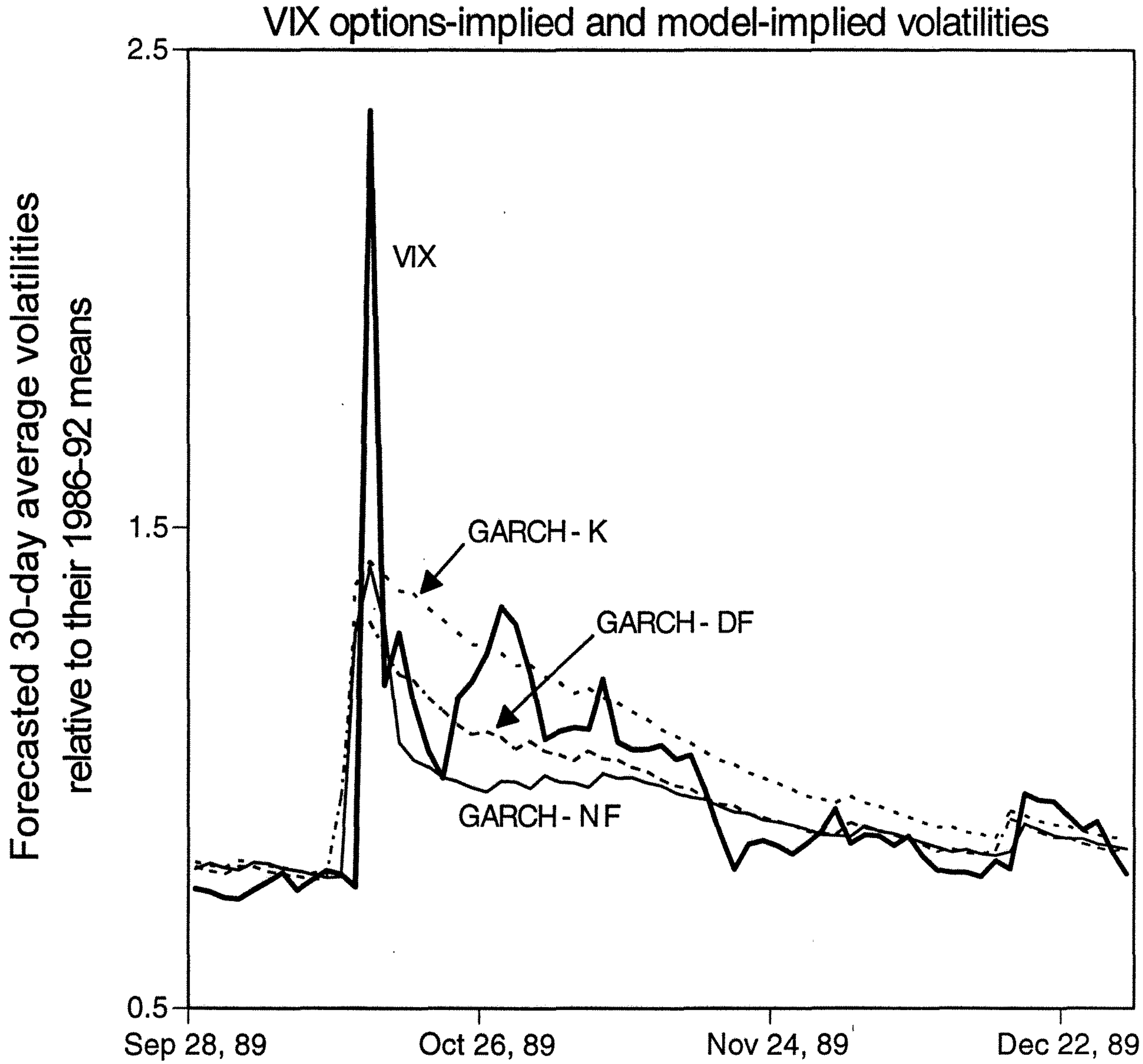

VIX options-implied and model-implied volatilities
Forecasted 30-day average volatilities relative to their 1986-92 means
2.5
1.5
0.5
VIX
GARCH - K
GARCH - DF
GARCH - NF
Sep 28, 89
Oct 26, 89
Nov 24, 89
Dec 22, 89

Figure 3b

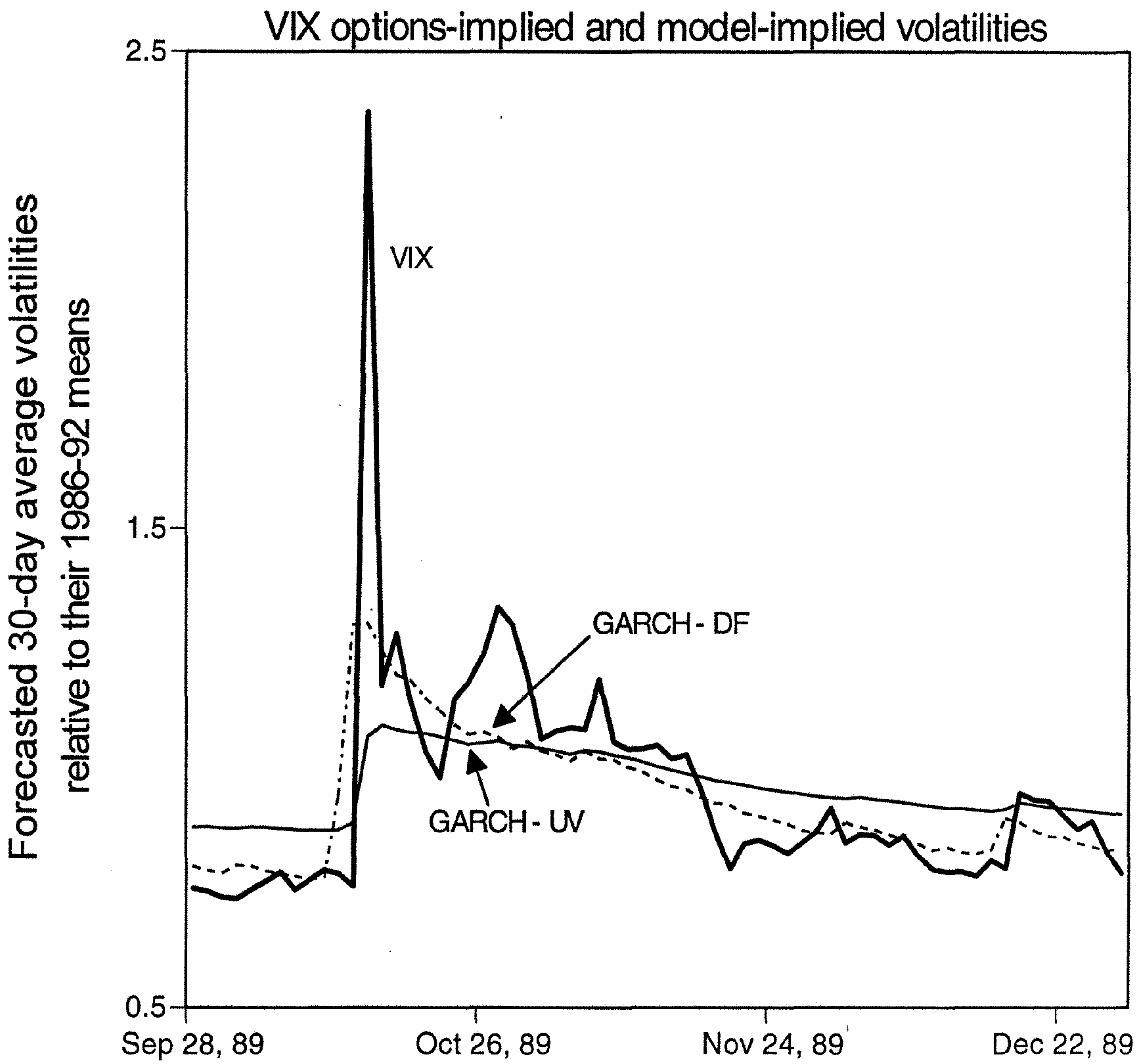
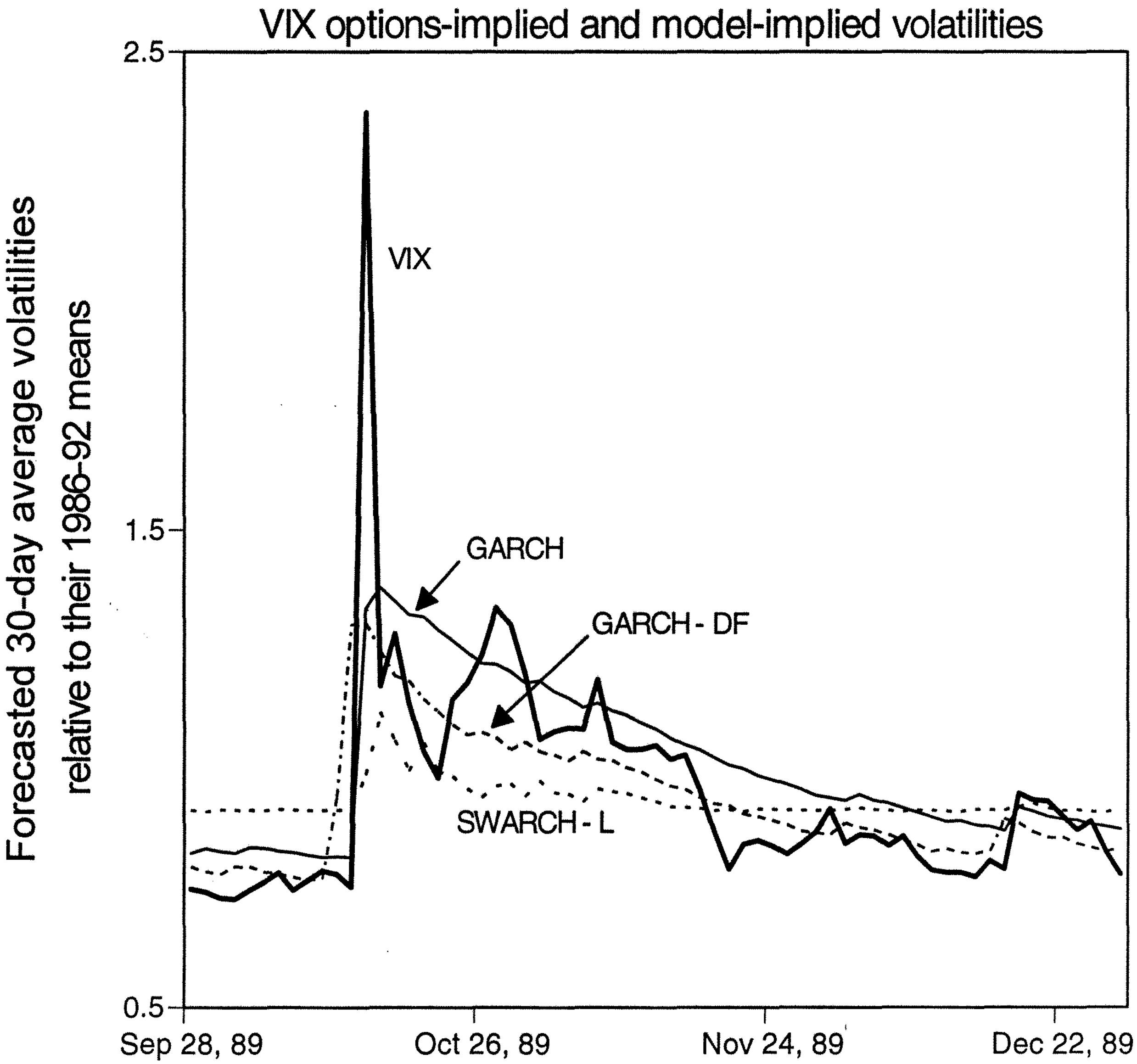

VIX options-implied and model-implied volatilities
Forecasted 30-day average volatilities
relative to their 1986-92 means
2.5
1.5
0.5
VIX
GARCH - DF
GARCH - UV
Sep 28, 89
Oct 26, 89
Nov 24, 89
Dec 22, 89

Figure 3c


VIX options-implied and model-implied volatilities
Forecasted 30-day average volatilities relative to their 1986-92 means
2.5
1.5
0.5
VIX
GARCH
GARCH - DF
SWARCH - L
Sep 28, 89
Oct 26, 89
Nov 24, 89
Dec 22, 89