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Collapsing Exchange Rate Regimes:
A Reinterpretation

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COLLAPSING EXCHANGE RATE REGIMES: A REINTERPRETATION

ABSTRACT

Microfoundations are provided for a common model of balance of payments crises by showing formal equivalence to a standard overlapping generations model. Certain domestic policy rules are inconsistent with fixed exchange rate equilibrium, which is in agreement with previous results. However, when policy *is* consistent with a fixed exchange rate equilibrium (1) there are multiple stationary equilibria, and the fixed exchange rate equilibrium is generally unstable under rational expectations; (2) the fixed exchange rate equilibrium is the attractor when agents learn over time; (3) complicated exchange rate paths cannot be ruled out by utility maximization and rational expectations alone.

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1. Introduction

This paper develops some microfoundations for a commonly employed simple model of the collapse of a managed exchange rate regime.¹ Once the microfoundations are provided, the interpretation of the model is altered in important ways relative to the standard view, and balance of payments crises can occur for reasons quite different from those previously postulated. The paper therefore provides both a justification for research on this model, via the microfoundations, as well as a new characterization of the equilibrium outcomes.

Much of the literature has focused on domestic policy rules inconsistent with a fixed exchange rate equilibrium. The model used in this paper encompasses the key previous result in the following sense: supposing the system is in a stationary fixed exchange rate equilibrium, certain policies imply a permanent drain on the home country's foreign reserves, so that such an equilibrium is ruled out. This fact has been used as the basis for rational explanations of balance of payments crises, and applies equally well to the present research. However, the primary emphasis in this paper is that rational crises can occur even when policy is exactly consistent with the fixed exchange rate goal.² Heuristically, the problem is that there are two stationary equilibria in the model, and the one where the exchange rate is fixed is unstable.

The microfoundations are provided by showing a formal equivalence between the conventional model and a standard version of the overlapping generations model. A dynamic analysis indicates that under rational expectations, the system typically converges

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¹The literature derives from the notion of rational speculative attack developed by Salant and Henderson (1978) and Salant (1983), and applied to balance of payments crises by, among others, Krugman (1979), Flood and Garber (1984), Obstfeld (1984, 1986ab), Connolly and Taylor (1984), Grilli (1986), Buiter (1987), Calvo (1987), and Cumby and Van Wijnbergen (1989). Rational speculative attack has also been discussed in connection with target zone models by Krugman (1988), Flood and Garber (1989), and Buiter and Grilli (1989).

²This idea is in the same spirit as Obstfeld (1986b).

to a crawling peg stationary equilibrium instead of the fixed exchange rate stationary equilibrium. In addition, under a strict overlapping generations interpretation, the existence of a stationary equilibrium where the exchange rate is actually fixed is an exceptional case. In general, the monetary authority requires assistance from the fiscal authority in the model in order to engineer a fixed rate.

Considering the dynamics of the model under rational expectations, the domestic government might have difficulty achieving a stationary fixed exchange rate. However, many authors might argue that agents learn over time and that it therefore might be reasonable to base analyses on the fixed exchange rate equilibrium. Using techniques recently developed by Marcet and Sargent (1989), this issue is investigated by introducing least squares learning into the simple model, replacing the rational expectations assumption. The fixed exchange rate equilibrium is locally stable under this learning scheme provided a certain condition holds. When the condition does not hold, a crisis can occur.

In general, rational expectations and utility maximization alone are insufficient to rule out complicated dynamics in this model. The linear approximation for money demand which is commonly employed in this literature amounts to a gross substitutes assumption in the overlapping generations framework. Relaxing this assumption would allow complicated dynamics and chaotic equilibrium paths for the exchange rate.

The next section introduces the model and illustrates the usual logic used to explain a balance of payments crisis. Section three considers a policy that might be expected to fix the exchange rate. Section four presents an overlapping generations interpretation of this model. Section five describes the dynamics of the model under rational expectations, while section six describes the dynamics under least squares learning. The final sections offer some interpretations.

2. A nonstochastic model

This section presents the model and briefly illustrates a central finding in previous research. The model is a version of those studied in Flood and Garber (1984), Obstfeld (1986b), Buiter (1987), Cumby and Van Wijnbergen (1989), and elsewhere. It is simple and deterministic in order to focus attention on the issues involved in the sharpest possible way. The overlapping generations version of the same model will be presented in section 4.

There is perfect international capital mobility for a small country the residents of which produce and consume a single homogeneous good. The model consists of six equations and determines domestic money market equilibrium. An asterisk indicates "foreign." Money demand is given by

$$(1) \quad H_t/P_t = \alpha - \gamma i_t$$

where $\alpha, \gamma > 0$, H_t is money, P_t is the price level, and i_t is the nominal interest rate. The supply of money is

$$(2) \quad H_t = R_t + D_t$$

where the initial conditions are $D_0 > 0$ and $R_0 \geq \underline{R}$ given for some $\underline{R} \leq 0$, where R_t and D_t represent the domestic central bank's foreign and domestic assets, respectively. A negative value of R_t implies borrowed reserves, allowed up to the limit $\underline{R}$. The domestic government finances a fixed real deficit ξ via

$$(3) \quad D_t = \theta D_{t-1} + \xi P_t$$

where $\theta, \xi \geq 0$. Absolute purchasing power parity is expressed as

$$(4) \quad P_t = P_t^* s_t$$

and the world price level evolves exogenously according to

$$(5) \quad P_t^* = \delta P_{t-1}^*$$

with $P_0^* = 1$ given and $\delta \geq 1$. The parameter δ is the gross world inflation rate.

Uncovered interest parity constitutes the final equation:

$$(6) \quad i_t = i_t^* + F_t \left[(s_{t+1}/s_t) - 1 \right]$$

where s_t is the spot exchange rate and F_t indicates a forecast based on the information available at time t . The model can be closed by describing how expectations are formed, that is, by naming a process for $F_t s_{t+1}$. Later in the paper the model will be closed once under perfect foresight and then again under least squares learning.

Assume that the foreign interest rate is constant and set it equal to $i_t^* = \delta - 1$. If the world price level is constant, the foreign interest rate will be constant and equal to zero, as in Obstfeld (1986b). Equations (4) and (5) can be combined to imply $P_t = \delta^t s_t$. The model can be written as a two equation system in money and exchange rates

$$(7a) \quad \frac{D_t + R_t}{\delta^t s_t} = [\alpha + \gamma(2 - \delta)] - \gamma F_t s_{t+1} / s_t$$

$$(7b) \quad D_t = \theta D_{t-1} + \xi \delta^t s_t.$$

Most of the analysis in the paper concerns the solution and interpretation of this system.

When the domestic government does not run a fixed real deficit ($\xi = 0$) and the exogenous world price level is constant ($\delta = 1$), the model is a version of those analyzed by, among others, Flood and Garber (1984), Buiter (1987), and Obstfeld (1986b). Purely for purposes of comparison, this will be labeled the "benchmark" case. None of the qualitative conclusions of the paper depend on whether the benchmark or the general case is being analyzed, but at times the parameter restrictions of the benchmark case simplify the discussion and permit more intuitive explanations. In the benchmark version, the D_t variable is labeled domestic credit, and the value of θ in equation (3) describes domestic credit policy (since $\xi = 0$). This convention will be followed here even though, as will be shown, θ has a nonpolicy interpretation in the overlapping generations version of this system.³

Because the focus of the paper is on the existence and stability of a stationary fixed

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³This turns out to be of little consequence, since values of θ will be reinterpreted as values of ξ given θ ; the ξ parameter represents fiscal policy.

exchange rate equilibrium in this system, the following assumption, while not strictly necessary, will be useful. It guarantees that the demand for money (7a) will be positive when the exchange rate is fixed:

Assumption 1. Low rates of world inflation. Let $\delta \in [1, (\alpha + \gamma)/\gamma)$.

The assumption will be invoked as needed below.

The system described in (7ab) is generally not well defined without further restrictions. If R_t is constant, however, the system determines positive sequences $\{s_t, D_t\}_{t=0}^{\infty}$ when closed with a definition for $F_t s_{t+1}$. An important result from previous research, which can be derived without placing more restrictions on the model, is that when the domestic government attempts an inconsistent policy of fixing the exchange rate and simultaneously pursuing domestic credit growth, a fixed exchange rate equilibrium is unsustainable.

Lemma 1. Set $\delta = 1$ and $\xi = 0$. Growing domestic credit ($\theta > 1$) is incompatible with fixed exchange rate equilibrium.

Proof. Suppose that at time $t = 0$, $F_t s_{t+1} = s_t = \bar{s} \ \forall t \geq 0$, where $\bar{s}$ is some fixed value. Then $R_t < \underline{R} \ \forall t > \tau'$, for some $\tau' > 0$, because under the conditions in the lemma, equations (7ab) can be written as

$$R_t = \alpha \bar{s} - \theta^t D_0$$

so that $\lim_{t \rightarrow \infty} R_t = -\infty$. ■

A version of this lemma also holds for the more general model being considered here.

Lemma 2. Domestic credit growing faster than the world inflation rate ($\theta > \delta$) is incompatible with fixed exchange rate equilibrium.

Proof. Suppose that at time $t = 0$, $F_{t+1}s_t = s_t = \bar{s} \ \forall t \geq 0$, where $\bar{s}$ is some fixed value. Then $R_t < \underline{R} \ \forall t > \tau'$, for some $\tau' > 0$, because under the conditions in the lemma, equations (7ab) can be written as

$$R_t = [\alpha + \gamma - \delta\gamma]\delta^t\bar{s} - \theta^t D_0 - \xi\bar{s} \sum_{j=0}^t \theta^{t-j}\delta^j$$

so that $\lim_{t \rightarrow \infty} R_t = -\infty$. ■

This result formalizes the idea that balance of payments crises in this model are the result of inconsistent macroeconomic policies. In various forms, this result has been used as the explanation for how an apparently irrational event like a speculative attack can be reconciled with rational expectations. Most of the literature has focused on the timing and magnitude of the speculative attack.

Lemmas one and two illustrate that policies exist that are inconsistent with the goal of fixing the exchange rate. In the next section, a policy consistent with fixed exchange rate equilibrium is described.⁴

3. A fixed exchange rate policy

A policy regime in this model consists of a sequence $\{R_t\}_{t=0}^{\infty}$, where $R_t > \underline{R} \ \forall t$. Assume that the value of the parameter is θ set by the authorities at time zero. The authorities are assumed to be intent on fixing the exchange rate for all time. Focus is

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⁴Obstfeld (1986b) showed how policies consistent with fixed exchange rate equilibrium could nevertheless lead to a breakdown. In Obstfeld's (1986b) model, the collapse hinged on agents' expectations of policy in the post-collapse regime. This paper follows Obstfeld in analyzing policies that are consistent with fixed exchange rate equilibrium but nevertheless may not produce pegged rates.

placed completely on the following policy:

Constant reserve growth policy. Set $R_t = \theta^t R_0 \quad \forall t$.

If for some $\tau > 0$, $F_t s_{t+1} = s \quad \forall t > \tau$, this policy can achieve a fixed exchange rate equilibrium. In the benchmark case ($\xi = 0$, $\delta = 1$), the policy is one of constant money growth $H_t = \theta H_{t-1}$, and it will produce a fixed exchange rate if $\theta = 1$: from equations (7ab), $s_t = (R_0 + D_0)/\alpha \quad \forall t > \tau$. This is just another version of the result that only constant domestic credit is consistent with fixed exchange rate equilibrium in the benchmark case. Similarly, if $\xi = 0$ but $\delta > 1$, a fixed exchange rate will be achieved if and only if $\theta = \delta$. The fixed exchange rate equilibrium is more problematic in the general case, and so discussion will be delayed until section 5. The next section shows how equations (7ab) can be derived from an overlapping generations model.

4. An overlapping generations model

A standard version of the overlapping generations model with population growth is employed; for a detailed account see Sargent (1987). An infinite horizon economy is populated by agents who live for two periods. The agents are indexed $n = 1 \dots N(t)$, where $N(t)$ indicates the number of agents born at time t . There is no storage and there are no bequests. Agents are endowed with $w_t^n(t)$, $w_t^n(t+1)$; taxed at $\tau_t^n(t)$, $\tau_t^n(t+1)$; and consume $c_t^n(t)$, $c_t^n(t+1)$. The agents maximize utility $U_t^n[c_t^n(t), c_t^n(t+1)]$ where (i) indifference curves are convex; (ii) no satiation; (iii) $U_{t_1}^n/U_{t_2}^n \rightarrow \infty$ as $c_{t_1}^n(t)/c_{t_2}^n(t) \rightarrow 0$ and $U_{t_1}^n/U_{t_2}^n \rightarrow 0$ as $c_{t_1}^n(t)/c_{t_2}^n(t) \rightarrow \infty$; and (iv) $c_t^n(t)$, $c_t^n(t+1)$ are normal goods. First period saving, $w_t^n(t) - \tau_t^n(t) - c_t^n(t)$, is a function of the gross interest rate $r(t)$. The government consumes $G(t) \geq 0$ units of the consumption good, levies lump-sum taxes $\tau_t^n(t)$, $\tau_{t-1}^n(t)$, and lends (borrows) $L^g(t) > 0$ (< 0) by negotiating one period loans at

time t . The loans are repaid the next period in the amount of $\tau(t)L^g(t)$.

The solution to the agent's problem requires that $U_{t2}^n/U_{t1}^n = 1/\tau(t)$. The individual agent's supply of savings $\ell_t^n = f_t^n[w_t^n(t) - \tau_t^n(t), w_t^n(t+1) - c_t^n(t+1), \tau(t)]$ is increasing in the first period after tax endowment, decreasing in the second period after tax endowment, and, generally speaking, ambiguous in the gross rate of interest $\tau(t)$. For now assume that $f_{t3}^n > 0$; the case where $f_{t3}^n < 0$ will be discussed in greater detail in section 7.

Denote the nonnegative price level by $P(t)$, and suppose that at time $t = 1$ the government owns $H(1) > 0$ units of currency. The government budget constraint is

$$G(t) = \tau(t-1)L^g(t-1) - L^g(t) + [H(t) - H(t-1)]/P(t) + \sum_n \tau_{t-1}^n(t) + \sum_n \tau_t^n(t).$$

Arbitrage requires that the rate of return to loans equals the rate of return to holding currency, $\tau(t) = P(t)/P(t+1)$. The government budget constraint can therefore be rewritten

$$G(t) = \tau(t-1)[L^g(t-1) - H(t-1)/P(t-1)] - [L^g(t) - H(t)/P(t)] + \sum_n \tau_{t-1}^n(t) + \sum_n \tau_t^n(t)$$

where $H(0) = 0$ and $L^g(0) = 0$. Loan market equilibrium requires

$$\sum_n f_t^n[\tau(t), w_t^n(t) - \tau_t^n(t), w_t^n(t+1) - \tau_t^n(t+1)] = H(t)/P(t) - L^g(t).$$

Three definitions allow derivation of a system analogous to (7ab). First, denote

$$N(t) = \eta N(t-1),$$

with $N(0)$ given, so that η is the constant gross rate of population growth. Define per capita real government indebtedness as

$$h(t) = \frac{-L^g(t) + H(t)/P(t)}{N(t)},$$

and define the net of interest per capita government deficit as ξ , where

$$\xi = \frac{G(t) - \sum_n \tau_{t-1}^n(t) - \sum_n \tau_t^n(t)}{N(t)}.$$

With these definitions, a system analogous to (7ab) can be developed. Set $L^g(t) =$

$0 \forall t$, and use the arbitrage condition $r(t-1) = P(t-1)/P(t)$ to obtain

$$\frac{H(t)}{P(t)N(t)} = \frac{\sum_n f_t^n[r(t)]}{N(t)} = f[r(t)]$$

where $f[\cdot]$ is the per capita savings function. Equation (7a) can be viewed as involving a linear approximation for $f[\cdot]$ where savings depends positively on the rate of interest, so that

$$\frac{H(t)}{P(t)N(t)} = a_0 - \frac{a_1}{r(t)}$$

with $a_0, a_1 > 0$. To obtain equation (7a), set $P(t) = P_t$, $H(t)/N(t) = H_t$, $a_0 = [\alpha + 2\gamma - \delta\gamma]$, $a_1 = \gamma\delta^{-1}$, $r(t) = P_t/F_t P_{t+1}$, and use the purchasing power parity condition so that

$$(8a) \quad H_t/\delta^t s_t = [\alpha + 2\gamma - \delta\gamma] - \gamma F_t s_{t+1}/s_t$$

where H_t is now interpreted as a per capita variable.

Now suppose the monetary authorities hold foreign assets, so that $H(t) = D(t) + R(t)$, and impose the constant growth reserves policy $R(t) = \theta^t R(0)$. The definitions of $h(t)$ and the per capita real deficit ξ , along with the arbitrage condition $r(t-1) = P(t-1)/P(t)$ and the simplification that $L^g(t) = 0 \forall t$, imply

$$\frac{H(t)}{N(t)} = -\frac{1}{\eta} \frac{H(t-1)}{N(t-1)} + \xi P(t).$$

Employing the reserve policy implies

$$\frac{D(t)}{N(t)} = -\frac{1}{\eta} \frac{D(t-1)}{N(t-1)} + \xi P(t)$$

or

$$(8b) \quad D_t = \theta D_{t-1} + \xi \delta^t s_t$$

where $D_t = D(t)/N(t)$ is now a per capita variable, $\theta = \eta^{-1}$ is the inverse of the population growth rate, ξ is the per capita real deficit, and $P(t) = P_t = \delta^t s_t$ is the aggregate price level in the home country.

While the systems described by (7ab) and (8ab) are analytically equivalent, the

overlapping generations framework implies some alternative interpretations. First, the role of a linear money demand function in equation (1) is brought out explicitly as a linearization of the savings function in the inverse of the interest rate. This aspect of the derivation, which rules out cycles and chaos, will be discussed in more detail in section 7.

Secondly, the parameter θ no longer represents policy and instead is a mere datum. The overlapping generations approach restricts "policy" to specifying a process for reserves and, perhaps, a setting for the real level of the fiscal deficit ξ . In the next section it will be shown that interpreting the model in terms of values for θ set by the authorities can equally well be thought of as setting the level of the fiscal deficit ξ . Since fiscal policy is normally taken as exogenous in the overlapping generations framework, this will amount to specifying some degree of fiscal cooperation needed to fix the exchange rate.

The overlapping generations interpretation amounts to a one country model with some conditions attached that represent the rest of the world. This is a manifestation of the small country assumption. It should be noted that the implications of this model may not be anything like the implications of a true two country overlapping generations model of exchange rates, such as those based on Karekin and Wallace (1981).

One advantage of the overlapping generations derivation of (8ab) is that it shows that the system is one about which quite a lot is known. There are, in general, two stationary equilibria in this model and a maximum level of the fixed real deficit that can be financed by seigniorage. The dynamics of the model are explored in detail in the next section.

5. Dynamics under rational expectations

The model can be closed with an assumption regarding expectations formation. In this section, rational expectations (perfect foresight) is imposed using the equations

$$(9a) \quad F_t s_{t+1} = \beta_t s_t$$

$$(9b) \quad \beta_t = s_{t+1}/s_t$$

where the variable β_t is the expected gross rate of exchange rate change between t and $t+1$. Using equations (9ab), the model can be rewritten as (see appendix 1)

$$(10) \quad \beta_{t+1} = \gamma^{-1}[\alpha + 2\gamma - \delta\gamma - \xi + \theta\gamma\delta^{-1}] - \theta\delta^{-1}\gamma^{-1}(\alpha + 2\gamma - \delta\gamma)\beta_t^{-1}.$$

As an equation in β_t , this function is upward sloping with a negative second derivative provided assumption 1, "low world inflation," holds.

In working with equation (10), it is convenient to make the following substitution for a certain constant:

$$k \equiv (\alpha + 2\gamma - \delta\gamma)\gamma^{-1}.$$

The system has two stationary values, one where the rate of exchange rate change is low, and another where the rate is high. These values are the roots of (10), given by

$$\beta_1^*, \beta_2^* = \frac{(k - \gamma^{-1}\xi + \theta\delta^{-1}) \pm \sqrt{(k - \gamma^{-1}\xi + \theta\delta^{-1})^2 - 4\theta\delta^{-1}k}}{2}.$$

These roots will be real provided the discriminant is positive. This fact provides an upper limit on the per capita deficit ξ , given by

$$\xi_{\max} = \gamma \left[k + \theta\delta^{-1} - \sqrt{4\theta\delta^{-1}k} \right].$$

An equilibrium is defined as:

Definition. A positive sequence $\{\beta_t\}_{t=0}^{\infty}$ that satisfies the difference equation (10) is a rational expectations equilibrium for $\{s_t, D_t\}$.

Consider first the benchmark case with no population growth, so that $\xi = 0$ and $\theta = \delta = 1$. Equation (10) simplifies to:

$$\beta_{t+1} = 1 + k(1 - \beta_t^{-1}),$$

where $k > 1$.⁵ The stationary equilibria of equation (10) are the fixed exchange rate stationary state $\beta_1^* = 1$ and the crawling peg stationary state $\beta_2^* = k$. At the crawling peg stationary state, money demand falls to zero. Under perfect foresight, there is a continuum of equilibria indexed by $\beta_0 \in [\beta_1^*, k)$. The hyperinflation stationary equilibrium is the attractor for all permissible initial conditions $\beta_0 > \beta_1^*$. The fixed exchange rate is the asymptotic outcome only if $\beta_0 = \beta_1^*$. If $\beta_0 \in (0, \beta_1^*)$, no equilibrium sequence exists under perfect foresight. These facts are summarized by the qualitative graph given in figure 1.

A second special case, a variation on the first, is also instructive. Again set the fiscal deficit to zero, but now let world inflation be positive ($\delta > 1$), and allow for domestic population growth or decline. The stationary equilibria are now given by $\beta_1^* = \theta/\delta$ and $\beta_2^* = k$. A fixed exchange rate equilibrium exists if and only if $\theta = \delta$. Such a case would be exceptional since there is no reason why, in general, the inverse of the population growth rate would equal a world inflation rate. Thus, the interpretation of θ would appear to be crucial: if θ is controlled by policymakers, a fixed exchange rate stationary state can be made to exist in this case. However, the interpretation of θ does not turn out to be so important, because once seigniorage is incorporated into the model, variations in the fiscal deficit can accomplish the fixed exchange rate objective.

To see this, consider the general case where $\xi > 0$ and $\delta > 1$. Both steady states now involve positive money holdings. In general, the equilibrium with the low rate of exchange rate change is not a fixed exchange rate equilibrium. In order to get the rate to be fixed, one must relax the assumption that the fiscal deficit is exogenously given. Assume instead that the fiscal authority chooses a value of ξ at time zero. This assumption can be thought of as one of fiscal policy coordination. The required value for ξ can be found by

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⁵The constant k simplifies to $(\alpha + \gamma)/\gamma$ in this case.

setting the lower root equal to one and solving for ξ :

$$\xi^* = \gamma(1 - k)(\theta/\delta - 1).$$

This value of ξ sets $\beta_1^* = 1$ in the general case. If $\theta = \delta$, $\xi^* = 0$, which is the special case discussed above. Since $(1 - k) < 0$, ξ^* is nonnegative provided $\theta \leq \delta$. Because θ is the inverse of the gross rate of domestic population growth and δ is the gross world inflation rate, the condition should hold generally. Other possibilities (i.e., $\theta > \delta$) will be set aside. The following assumption provides a formalization:

Assumption 2. The gross domestic population growth rate is greater than or equal to one. The gross world inflation rate is greater than or equal to one.

The mechanics of the existence of a stationary fixed exchange rate equilibrium are illustrated in figure 2. When $\xi = 0$, the steady states occur at θ/δ and k . Assumption 2 implies that $\theta < \delta$. When $\xi = \xi_{\max}$, there is a single stationary equilibrium at $\beta_1^* = \beta_2^*$. A decrease in ξ from $\xi_{\max}$ has the effect of shifting the curve (the qualitative graph of equation (10)) upward in the diagram. As $\xi \rightarrow \xi^*$, $\beta_1^* \rightarrow 1$. When $\xi = \xi^*$, a stationary fixed exchange rate equilibrium exists.

A rational balance of payments crisis in this model amounts to a domestic hyperinflation caused by the fact that the stationary equilibrium with the high rate of exchange rate change is the attractor under perfect foresight. The fixed exchange rate stationary equilibrium is unstable. Asymptotic outcomes can be characterized for all permissible initial conditions.

6. Dynamics under least squares learning

The results under perfect foresight suggest that the domestic government will have difficulty fixing its exchange rate. This section shows how a learning argument can be invoked to essentially reverse the stability properties of the stationary equilibria and thus suggest that fixed exchange rates might not be so difficult for a small country to achieve. It will be shown that, nevertheless, balance of payments crises can occur under learning. The learning argument gives the following character to the model: balance of payments crises occur sometimes, but not all the time. In particular, crises can occur under learning when the deficit nears its maximum value.

These results make sense if rational expectations is viewed as an approximation to some actual underlying learning scheme used by agents. Generally, this is not a problem since the rational expectations assumption is usually justified by supposing learning is complete: agents have observed their environment for some time and understand how it works. Lucas (1987), for instance, suggests that macroeconomists study rational expectations equilibria because learning on the part of agents makes these equilibria plausible. But if the notion that learning underlies the rational expectations assumption is often noted, the method that agents use to eliminate the forecast errors over time is the subject of dispute. Nevertheless, one forecasting technique that comes immediately to mind and for which some rigorous analytical techniques now exist (see Marcet and Sargent, 1989) is least squares regression. In particular, a good starting point might be to consider agents that run autoregressions in order to forecast the future spot exchange rate.

Marcet and Sargent (1989) provide an analysis of a model similar to this one under least squares learning. Their findings will be briefly summarized.⁶

The model (1)–(6) is closed using the assumption that expectations are formed

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⁶See also the work of Bruno and Fischer (1990).

using the least squares formula

$$(11a) \quad F_t s_{t+1} = \beta_t s_t$$

$$(11b) \quad \beta_t = \left[\sum_{j=1}^{t-1} s_{j-1}^2 \right]^{-1} \left[\sum_{j=1}^{t-1} s_j s_{j-1} \right]$$

where the expected rate of exchange rate change is now a coefficient in a regression of s_t on s_{t-1} . The least squares formula (11b) can be rewritten in the recursive form⁷

$$(12a) \quad \beta_t = \beta_{t-1} + C_{t-1}^{-1} s_{t-2} \left[s_{t-1} - s_{t-2} \beta_{t-1} \right]$$

$$(12b) \quad C_{t-1} = \sum_{j=1}^{t-1} s_{j-1}^2.$$

The model can now be solved again, this time where β_t is described by (12ab). Equations (1)–(6) and (12a) imply (see appendix 2)

$$(13a) \quad \beta_t = \beta_{t-1} - g_{t-1} \left[\frac{[k - \beta_{t-2}]}{[k - \beta_{t-1}]} S(\beta_{t-1}) - \beta_{t-1} \right]$$

where

$$(13b) \quad g_{t-1} = s_{t-2}^2 \left[\sum_{j=1}^{t-1} s_{j-1}^2 \right]^{-1}$$

and

$$(13c) \quad S(\beta_{t-1}) \equiv \frac{\theta(k - \beta_{t-1})}{\delta(k - \beta_{t-1} - \xi \gamma^{-1})}.$$

The function $S(\beta)$ has been shown by Marcet and Sargent (1989) to be of key importance in determining the dynamics of (13). The steady states can be found by setting $\beta_t = \beta_{t-1} = \beta_{t-2}$ and finding the zeros; this amounts to finding the zeros of the equation $S(\beta) - \beta = 0$. The roots are the same as those for equation (10). Marcet and Sargent (1989) conclude that the limit points of the system are unaltered by the introduction of learning. Equilibrium is described by the following:

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⁷See Harvey (1981, p. 54).

Definition. A set of positive sequences $\{\beta_t, D_t, s_t\}_{t=0}^{\infty}$ satisfying (1)–(6) and (13abc) is an equilibrium under least squares learning.

Equation (13) describes the evolution of expectations for this model under least squares learning. While it is a complicated difference equation, Marcet and Sargent (1989) offer some intuition in the following way. Rewrite (13) as

$$\beta_t = (1 - g_{t-1})\beta_{t-1} + g_{t-1} \frac{[k - \beta_{t-2}]}{[k - \beta_{t-1}]} S(\beta_{t-1}).$$

Since $g_{t-1} \in (0,1]$, one interpretation of (13) is that the current expected exchange rate change is a convex combination of last period's expected exchange rate change and a certain term involving β_{t-1} and β_{t-2} . If one imagines a neighborhood of a stationary state, so that $\beta_{t-1} \approx \beta_{t-2}$, equation (13) can be thought of as very nearly a convex combination of β_{t-1} and $S(\beta_{t-1})$. Agents are making forecasts by adjusting their prediction for the previous period by a small amount. Figure 3 illustrates the qualitative graph of $S(\beta_{t-1})$. A convex combination is a point between the 45 degree line and the curve; the dynamics of $\beta_t = S(\beta_{t-1})$ therefore suggest that the lower fixed point will be the attractor.

Marcet and Sargent (1989) suggest methods for analyzing (13). To make progress, first find the limiting value of g_{t-1} . Then use a linear approximation at a stationary equilibrium replacing g_{t-1} by its limiting value. Local stability of a steady state can then be deduced via an eigenvalue condition. The limit of g_{t-1} is given by the following lemma.

Lemma 1. If $\beta_t \rightarrow \bar{\beta} \geq 1$, $\liminf_{t \rightarrow \infty} g_t = 1 - S(\bar{\beta})^2$.

Proof. See Marcet and Sargent (1989).

The stationary equilibria of the model are greater than or equal to one when ξ is sufficiently large (i.e., $\xi \geq \xi^*$). For the purposes of the remainder of this section, we will simply look at this particular case, that is, the case where both roots are greater than or equal to one. Use the result in Lemma 1 to obtain the following.

Theorem 1. Let $k > \beta_2^* > \beta_1^* \geq 1$. Let $K = [S(\beta_1^*) - S(\beta_1^*)^{-1}]/[k - \beta_1^*]$. Under least squares learning, the stationary equilibrium with a low rate of exchange rate change is locally stable if $K \leq 1$. That is, for β_0 near 1 and g_0 near $1 - S(\beta_1^*)^{-2}$, $\lim_{t \rightarrow \infty} \beta_t = \beta_1^*$.

Proof. Consider (13) as the following system in $\mathbb{R}^3$:

$$(14a) \quad \beta_t = \beta_{t-1} + g_{t-1} \left[\frac{\theta(k - \beta_{t-2})}{\delta(k - \beta_{t-1} - \xi\delta^{-1})} - \beta_{t-1} \right]$$

$$(14b) \quad \beta_{t-1} = \beta_{t-1}$$

$$(14c) \quad g_t = \left[g_{t-1}^{-1} \left[\frac{\alpha + \gamma(1 - \beta_{t-2})}{\alpha + \gamma(1 - \beta_{t-1})} \right]^{-2} + 1 \right]^{-1}.$$

To obtain this last equation, divide the numerator and the denominator of g_t by s_{t-1}^2 and use equation (B7) in appendix 2.

Determine the local stability properties of the fixed exchange rate stationary equilibrium. Let $z_t = [\beta_t, \beta_{t-1}, g_t]'$, so that $z_t = G(z_{t-1})$, where $G(\cdot)$ is defined by the right hand side of equation (14). Let $z^* = [\beta_1^*, \beta_1^*, 1 - S(\beta_1^*)^{-2}]'$, where $1 - S(\beta_1^*)^{-2}$ is the limiting value of g_t . Then $z^* = H(z^*)$. The Jacobian is given by

$$\left. \frac{\partial G}{\partial z} \right|_{z=z^*} = J = \begin{bmatrix} J_{11} & J_{12} & 0 \\ 1 & 0 & 0 \\ J_{31} & J_{32} & J_{33} \end{bmatrix},$$

where

$$J_{11} = S(\beta_1^*)^{-2} + \frac{[S(\beta_1^*)^{-1} - S(\beta_1^*)]}{k - \beta_1^* - \xi\gamma^{-1}},$$

$$J_{12} = \frac{S(\beta_1^*)^{-1} - S(\beta_1^*)}{k - \beta_1^*} = -K$$

$$J_{33} = S(\beta_1^*)^{-1}$$

where J_{31} and J_{32} are unevaluated derivatives. The associated characteristic equation is given by

$$[S(\beta_1^*)^{-1} - \rho_3]\{\rho^2 - J_{11}\rho + K\} = 0.$$

The root $\rho_3 < 1$ whenever $\beta_1^* > 1$. The remaining roots are also less than unity provided $K < 1$; see Lemma 2 of Marcet and Sargent (1989). For the case where $\beta_1^* = 1$, the Jacobian is given by

$$\left. \frac{\partial G}{\partial z} \right|_{z=z^*} = J = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ J_{31} & J_{32} & 1 \end{bmatrix}$$

This case can be handled by Theorem 7.1 of Ruelle (1989, p. 32). Conclude that the fixed exchange rate equilibrium is locally stable under least squares learning. ■

Consider the contrast between the results under least squares learning as opposed to rational expectations. Under rational expectations, the fixed exchange rate stationary equilibrium was unstable, and the system was generally attracted to a hyperinflationary stationary state. Under least squares learning, the fixed exchange rate stationary equilibrium is locally stable. A government attempting to fix the exchange rate in an economy with least squares learners may succeed.

Nevertheless, balance of payments crises can still occur under least squares learning. Suppose that world inflation approaches its maximum value under assumption 1, that is, δ

$\rightarrow (\alpha + \gamma)/\gamma$. Then $\xi^* \rightarrow \xi_{\max}$, that is, the value of the deficit required to create a stationary fixed exchange rate equilibrium approaches the maximum allowable deficit. Marcet and Sargent (1989) have shown how nonconvergence possibilities can be realized in this situation under least squares learning. Heuristically, the equilibria are pushed close together as $\xi \rightarrow \xi_{\max}$ (in fact, $\beta_1^* = \beta_2^*$ when $\xi = \xi_{\max}$). Since the least squares algorithm can be thought of as a convex combination of β_{t-1} and $\left[[k - \beta_{t-2}] / [k - \beta_{t-1}] \right] S(\beta_{t-1})$, and since this latter term can be greater than β_2^* , situations exist where the updated value of β_t is greater than β_2^* . Figure 3 suggests that this is a region where no equilibrium sequences exist under least squares learning.

The next section returns to the linearization of the savings function in the derivation of equations (8ab) via the overlapping generations framework.

7. Effects of linearization of money demand

So far, this paper has emphasized an overlapping generations interpretation of a simple model commonly used in the balance of payments crises literature. Comparability was obtained by imposing a certain linearization on the savings function. The linearization has a relatively simple interpretation and far reaching consequences for the dynamics of the model.

The standard assumptions on utility given by (i)–(iv) in section 4 are in general insufficient to require savings to depend positively on the interest rate in the overlapping generations model. An auxiliary assumption that savings does depend positively on the interest rate amounts to an assumption that $c_t^n(t)$ and $c_t^n(t+1)$ are gross substitutes (see, for instance, Sargent 1987). Linearization of the savings function is more than sufficient to impose the gross substitutes assumption in this model. Aggregate savings is aggregate excess demand, and the work of Sonnenschein (1973) and others has shown that any continuous function could be an excess demand function. Full generality in this regard

would allow complicated dynamics under rational expectations, and possibly even chaos, as has been shown by Grandmont (1985). Therefore, the simple one (small) country overlapping generations model of exchange rates has, as a logical possibility, complicated and chaotic equilibrium exchange rate paths based on perfect foresight and utility maximization.

8. Summary

Calvo (1987, p. 21) has commented that the balance of payments crisis literature is "a clear instance where, rather than continuing the search for ad hoc models that yield the desired results, our understanding might be better served by examining a simple monetary model with solid microeconomic foundations." This paper suggests that the standard ad hoc model may not have been so bad after all. Instead, it can be viewed as describing a small overlapping generations economy facing a large and exogenous rest of the world. Interpretations, however, do differ once the microfoundations are provided.

In general, there are two stationary equilibria in the model, one where the rate of exchange rate change is relatively slow, and another where it is rapid. The hyperinflationary equilibrium is the attractor when agents possess perfect foresight. This provides the explanation of balance of payments crises in the model. Thus, the explanation is different from the usual approach in the literature, which relies on the domestic government following policies that are inconsistent with fixed exchange rate equilibrium. The idea that balance of payments crises may occur even when the domestic government pursues the right policy to fix the exchange rate is in the same spirit as Obstfeld (1986b).

The existence of a fixed exchange rate equilibrium is problematic in the model. The stationary equilibrium where the rate of exchange rate change is slow will be a fixed exchange rate equilibrium, generally speaking, only in an exceptional case. However, if one views the per capita real deficit as a policy parameter, a fixed exchange rate exists given

the right amount of fiscal coordination. In particular, there is, generally speaking, a value for the deficit that will create a stationary fixed exchange rate equilibrium.

Given that the fixed rate equilibrium exists, the dynamics under rational expectations suggest it will rarely be achieved. However, if one views rational expectations as an approximation to an underlying learning process, the stability properties of the stationary equilibria can be reversed. In particular, if rational expectations is replaced with a least squares learning algorithm as analyzed by Marcet and Sargent (1989), the fixed exchange rate equilibrium is locally stable under certain conditions. When the conditions are not met, a balance of payments crisis could occur.

The derivation of the model in the overlapping generations framework entails a linearization of the savings function. The linearization amounts to a method of imposing a gross substitutes assumption on the model, and rules out cyclical and chaotic equilibrium exchange rate paths under perfect foresight.

This paper illustrates a theme in some of the recent literature on learning, namely that stability properties are reversed under learning when compared to the same properties under rational expectations (see Grandmont, 1985, and Grandmont and Laroque, 1990). In the present case, introducing learning induced the "good" equilibrium, the fixed exchange rate stationary state. It is important to note, however, that there is no particular association of learning with "good" outcomes. It may well be that, in some other model, rational expectations dynamics lead to a good outcome while learning leads to a bad outcome.⁸

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⁸Nothing has been said in this paper concerning the adequacy of describing actual learning behavior by least squares as opposed to perfect foresight. However, some recent evidence is available on this point. In particular, Marimon and Sunder (1990) consider an experiment based on an overlapping generations model with two stationary equilibria, a "good" low inflation stationary state and a "bad" high inflation stationary state. The dynamics under rational expectations converge to the high inflation stationary state, while the dynamics under least squares learning converge to the low inflation stationary equilibrium. They conduct an experiment based on participation in the model by human

Appendix 1 (Not For Publication)

Algebra for equation (10).

Rewrite (8ab) as

$$(A1) \quad s_t = \frac{\gamma}{(\alpha + \gamma(2 - \delta))} F_t s_{t+1} + \frac{(D_t + R_t)}{(\alpha + 2\gamma - \delta\gamma)\delta^t}$$

$$(A2) \quad D_t = \theta D_{t-1} + \xi \delta^t s_t.$$

Impose perfect foresight via

$$F_t s_{t+1} = \beta_t s_t$$

$$\beta_t = s_{t+1}/s_t$$

so that (A1) becomes

$$(A3) \quad s_t = \frac{\gamma}{(\alpha + \gamma(2 - \delta))} \beta_t s_t + \frac{(D_t + R_t)}{(\alpha + 2\gamma - \delta\gamma)\delta^t}$$

$$(A4) \quad s_{t-1} = \frac{\gamma}{(\alpha + \gamma(2 - \delta))} \beta_{t-1} s_{t-1} + \frac{(D_{t-1} + R_{t-1})}{(\alpha + 2\gamma - \delta\gamma)\delta^{t-1}}$$

or

$$(A5) \quad D_{t-1} = (\alpha + 2\gamma - \delta\gamma)\delta^{t-1} s_{t-1} - \gamma\delta^{t-1} \beta_{t-1} s_{t-1} - R_{t-1}.$$

Substitute (A2) in (A1)

$$(A6) \quad s_t = \frac{\gamma}{(\alpha + \gamma(2 - \delta))} \beta_t s_t + \frac{[(\theta D_{t-1} + \xi \delta^t s_t) + R_t]}{(\alpha + 2\gamma - \delta\gamma)\delta^t}.$$

Substitute (A5) in (A6) and simplify to obtain

$$(A7) \quad \gamma\beta_t = [\alpha + 2\gamma - \delta\gamma - \xi + (\theta\gamma/\delta)] - \frac{\theta}{\delta}(\alpha + 2\gamma - \delta\gamma)\beta_{t-1}^{-1} - \frac{(R_t - \theta R_{t-1})}{\delta^t s_t}$$

Apply the constant reserve growth policy $R_t = \theta^t R_0$.

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subjects. The primary qualitative result is that forecasts showed no tendency to converge to the high inflation stationary equilibrium values, and in fact tend to cluster around the low inflation values. Marimon and Sunder (1990) conclude that least squares provides a good description of actual dynamics in the model, which is very similar to the one used in this paper.

$$(A8) \quad \beta_{t+1} = \gamma^{-1}[\alpha + 2\gamma - \delta\gamma - \xi + \theta\gamma\delta^{-1}] - \theta\delta^{-1}\gamma^{-1}(\alpha + 2\gamma - \delta\gamma)\beta_t^{-1}$$

which is the equation analyzed in the text.

Appendix 2 (Not For Publication)

Algebra for equation (13).

We work with

$$(B1) \quad \beta_t = \beta_{t-1} + C_{t-1}^{-1} s_{t-2} [s_{t-1} - s_{t-2} \beta_{t-1}]$$

and the two equation system

$$(B2) \quad \frac{D_t + R_t}{\delta^t s_t} = [\alpha + \gamma(2 - \delta)] - \gamma F_t s_{t+1} / s_t$$

$$(B3) \quad D_t = \theta D_{t-1} + \xi \delta^t s_t$$

From (B2) we obtain

$$(B4) \quad D_t = \delta^t s_t [\alpha + 2\gamma - \delta\gamma - \gamma\beta_t] - R_t$$

$$(B5) \quad D_{t-1} = \delta^{t-1} s_{t-1} [\alpha + 2\gamma - \delta\gamma - \gamma\beta_{t-1}] - R_{t-1}$$

so that substitution into (B3) and applying the policy $R_t = \theta^t R_0$ implies

$$(B6) \quad s_t = \frac{\theta [\alpha + 2\gamma - \delta\gamma - \gamma\beta_{t-1}]}{\delta [\alpha + 2\gamma - \delta\gamma - \gamma\beta_t - \xi]} s_{t-1}$$

and

$$(B7) \quad s_{t-1} = \frac{\theta [\alpha + 2\gamma - \delta\gamma - \gamma\beta_{t-2}]}{\delta [\alpha + 2\gamma - \delta\gamma - \gamma\beta_{t-1} - \xi]} s_{t-2}$$

Substituting (B7) into (B1) gives

$$(B8) \quad \beta_t = \beta_{t-1} + C_{t-1}^{-1} s_{t-2} \left[\frac{\theta [\alpha + 2\gamma - \delta\gamma - \gamma\beta_{t-2}]}{\delta [\alpha + 2\gamma - \delta\gamma - \gamma\beta_{t-1} - \xi]} s_{t-2} - \beta_{t-1} \right].$$

The equation in the text can be obtained by multiplying the first term in the brackets by $[\gamma^{-1}(k - \beta_{t-1})]/[\gamma^{-1}(k - \beta_{t-1})]$.

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FIGURE ONE.

Graph of $\beta_{t+1} = 1 + k(1 - \beta_t^{-1})$

FIGURE TWO

Mechanics of fixed exchange rate existence.

FIGURE THREE

Qualitative graph of $s(\beta)$.