A Note on Almon's Endpoint Constraints

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What Do Almon's Endpoint Constraints Constrain?

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In this paper we show that Almon's (1965) endpoint constraints do not constrain the endpoints, as commonly thought. In particular, the endpoints are not constrained to equal zero. Consequently, these constraints have neither a basis in economic theory nor the econometric justification frequently ascribed to them.

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Estimation of dynamic models in which variables adjust slowly over time has always been difficult. Serious problems are encountered when equations require long distributed lags because of both collinearity and limited degrees of freedom. An attractive, apparent solution to these problems was offered by Almon's (1965) polynomial distributed lag (PDL) estimation technique. Subsequent authors have pointed out limitations of this technique, especially the dangers of misspecifying the lag length or polynomial degree.[1/]

One important aspect of PDL estimation has received relatively little attention. This is the role of endpoint constraints. Almon suggested that these should always be imposed; however, Schmidt and Waud (1973) have argued that they represent *ad hoc* restrictions in that there is no *a priori* reason to expect them to be true. Following up on Schmidt and Waud's suggestion, Maddala (1977) and Seaks and Allen (1980) have suggested that the endpoint constraints should be tested and imposed only if they are not rejected by the data. Nevertheless, the endpoint constraints continue to be employed rather routinely in econometric studies—e.g., Edwards (1983), Miller (1980), Mishkin (1982a, 1982b), Spinelli (1983), and Topel (1982)--and in large-scale econometric models.

Perhaps the endpoint constraints continued to be widely used because no one has pointed out the simple and somewhat obvious fact that the endpoint constraints which Almon suggested do not in fact constrain the endpoints of the distributed lag weights to zero. Thus, these constraints are *ad hoc* in a broader sense than previously suggested and do *not* have the interpretation that is frequently ascribed to them. We begin by reviewing briefly the basic PDL model and the conventional implementation of the endpoint constraints.

1. The PDL model

Consider the PDL model

(1) $\underline{Y} = X\underline{\beta} + \underline{u}$,

where $\underline{Y}$ is a $T \times 1$ vector of observations on the dependent variable, X is a $T \times (\ell+1)$ vector of observations on a ℓth-order distributed lag of the independent variable and $\underline{\beta}$ is a $(\ell+1) \times 1$ vector of parameters. The vector of random disturbances is given by $\underline{u}$. The PDL model assumes that the distributed lag weights fall on a polynomial of degree p. That is, the vector $\underline{\beta}$ is related to the $(p+1) \times 1$ vector of polynomial coefficients, $\underline{\alpha}$, as

(2) $\underline{\beta} = H\underline{\alpha}$,

where H is a $(\ell+1) \times (p+1)$ matrix of known coefficients.[2/]

The endpoint constraints suggested by Almon are

(3) $\beta_{-1} = \beta_{\ell+1} = 0$.

These homogeneous restrictions on the distributed lag weights outside of the relevant range of the distributed lag imply homogeneous restrictions on the distributed lag weights inside the relevant range via restrictions on $\underline{\alpha}$. That is, the above restrictions require

$$(4)\quad \begin{aligned} &\alpha_0 + \alpha_1 (-1) + \alpha_2 (-1)^2 + \ldots + \alpha_p (-1)^p = 0 \\ &\alpha_0 + \alpha_1 (\ell+1) + \alpha_2 (\ell+1)^2 + \ldots + \alpha_p (\ell+1)^p = 0, \end{aligned}$$

or

$$(5)\quad R\underline{\alpha} = 0.$$

Since $\underline{\alpha} = H^+\underline{\beta}$, where H^+ is $(H'H)^{-1} H'$,

$$(6)\quad R\underline{\alpha} = RH^+\underline{\beta} = R^*\underline{\beta} = 0.$$

Thus, the endpoint constraints imply a set of homogeneous restrictions, R^*, on $\underline{\beta}$.

2. The Endpoint Constraints Do Not Constrain the Endpoints

The question concerning what the endpoint constraints constrain appears to depend critically on the interpretation of β_{-1} and $\beta_{\ell+1}$. Schmidt and Waud argue that since $\beta_0, \beta_1, \ldots, \beta_\ell$ are coefficients on $x_t, x_{t-1}, \ldots, x_{t-\ell}$, by analogy β_{-1} and $\beta_{\ell+1}$ could be interpreted as coefficients on x_{t+1} and $x_{t-\ell-1}$, respectively. They argue that since x_{t+1} does not influence y_t, it is reasonable to conclude that $\beta_{-1}=0$. They observe, however, that it would be just as reasonable to conclude that $\beta_{-2}=\beta_{-3}=\ldots=0$, but note that this is impossible since no more than p of the βs can equal zero unless they all do. They conclude that one should only be concerned with the coefficients $\beta_0, \beta_1, \ldots, \beta_{\ell+1}$. The behavior outside this range is simply irrelevant.

Actually, the behavior of the parameters outside the range which the researcher specifies may or may not be important: the crucial point is that restrictions on parameters which lie inside this range imply nothing about the parameters outside of this range. If β_{-1} and $\beta_{\ell+1}$ are interpreted as coefficients on x_{t+1} and $x_{t-\ell-1}$, however, this is what the endpoint constraints seem to imply.

To see this point, consider the following time-series representation introduced by Sims (1972)

$$(7)\quad y_t = \sum_{i=-L}^{K} \beta_i x_{t-i} + u_t \qquad t = 1, 2, 3, \ldots, T,$$

where the infinite order of the theoretical model is replaced by the finite order L+K+1.[3/] If the coefficients are assumed to fall on a polynomial of degree p, the polynomial can be at most of degree L+K+1. Thus, the homogeneous restrictions imposed to reduce the theoretically infinite order to a finite one likewise restrict the range of p.

In the case where L=1 and K=p+1, imposing the endpoint constraints β_{-1} and $\beta_{p+1}=0$ is tantamount to reducing (7) to (1). Thus, if these parameters are interpreted as coefficients on x_{t+1} and $x_{t-\ell-1}$, then imposing the endpoint constraints as additional restrictions on (1), as is commonly done, is absurd: they were already imposed when the model was specified as (1). While Schmidt and Waud have suggested that one can argue that β_{-1} and $\beta_{\ell+1}$ are coefficients on x_{t+1} and $x_{t-\ell-1}$ (and others have interpreted them in this way), it is clear that the endpoint constraints have no such interpretation since they do not constrain these coefficients when applied to (1).[4/] Hence, there is no meaningful sense in which the endpoint constraints can be seen as restricting the coefficients on x_{t+1} and $x_{t-\ell-1}$, as is frequently suggested.

If endpoint constraints cannot be interpreted in this way, how can they be interpreted? The answer is that they are simply a name given to the particular set of homogeneous restrictions on the polynomial coefficients, $\underline{\alpha}$, given by (4). They have no particular relevance in economic theory nor meaning in econometric theory. They are completely ad hoc, and there is no reason a priori to prefer them over any other set of two homogeneous restrictions on $\underline{\alpha}$.

3. Conclusions

We have shown that the Almon's endpoint constraints are not true endpoint constraints, since the true endpoint constraints were effectively imposed in specifying the model. Consequently, the endpoint constraints are completely ad hoc. They have neither a basis in economic theory nor the econometric justification often ascribed to them.

FOOTNOTES

1/ See Judge, et. al. pp. 641-48.

2/ The H matrix is of the form

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & 2^3 & \cdots & 2^p \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ 1 & \ell & \ell^2 & \ell^3 & \cdots & \ell^p \end{bmatrix}$$

See Judge, et. al. (1980), p. 641.

3/ The Sims model is used here for illustrative purposes because it is not clear in the context of this model that the coefficients on β_{-1}, β_{-2}, etc., are necessarily zero.

4/ There appears to be some difference of opinion as to whether theoretical and applied econometricians actually interpret the endpoint constraints in this way. Certainly, while Schmidt and Waud offered this as an interpretation by analogy, they stopped short of using it. It is difficult to find examples of this interpretation in the literature since econometric texts often do not discuss the endpoint constraints and applied researchers simply impose them without comment. Nevertheless, here are a few examples: Maddala (1977, p. 358, footnote one), Kmenta (1971, p. 494, figure 11-14), Seaks and Allen (1980, p. 824 figure 1) and Andersen and Jordan (1968).

Actually, Kelejian and Oates (1974, pp. 176-78) interpret the endpoint constraints in this way, but represent them as $\beta_0=\beta_\ell=0$, instead of the conventional way. They note that imposing these constraints is equivalent to setting $\beta_0=\beta_\ell=0$ in (1). They fail to note, however, that if $p < \ell$, imposing these constraints on $\underline{\alpha}$ can at best approximate the true constraints on $\underline{\beta}$, i.e., if x_{t+1} and $x_{t-\ell-1}$ were included in (1) and if $p<\ell+2$, then the constraints applied on $\underline{\alpha}$ could only approximate the true constraints on $\underline{\beta}$ because H is not full rank. For example, if $\ell=10$ and $p=4$, then

$$R^* = \begin{bmatrix} .98 & .06 & -.08 & -.01 & .04 & .02 & -.02 & -.03 & -.01 & .03 & .02 & -.03 & .01 \\ .01 & -.03 & .02 & .03 & -.01 & -.03 & -.02 & .02 & .04 & -.01 & -.08 & .06 & .98 \end{bmatrix},$$

not

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

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