ON UNIT ROOT TESTS WHEN THE ALTERNATIVE IS A TREND-BREAK STATIONARY PROCESS

Amit Sen*
Department of Economics
University of Missouri–Rolla

December 2000

ABSTRACT

Minimum-t-statistics to test for a unit root are available when the form of break under the alternative evolves according to the Crash, Changing Growth, and Mixed models. Serious power distortions occur if the form of break is misspecified, and so the practitioner should use the Mixed model as the appropriate alternative in empirical applications. The Mixed model may reveal useful information regarding the location and form of break. A new maximal-F-statistic is shown to have greater and less erratic power compared to the minimal-t-statistic. Stronger evidence against the unit root is found for the Nelson-Plosser series and Quarterly Real GNP.

Keywords: Unit Root; Trend Break; Break-Date; Wald Statistic

JEL Classification: C22

*This research was supported by a grant from the University of Missouri Research Board. I would like to thank Richard Bryant, Scott Fullwiler, Greg Gelles, and Alastair Hall for useful discussions. Herman Bierens kindly provided the Nelson-Plosser data set used in this paper. Correspondence: Department of Economics, 1870 Miner Circle, University of Missouri–Rolla, Rolla, MO 65409-1250. Phone: (573) 341-6688, Fax: (573) 341-4866, email: samit@umr.edu

1. INTRODUCTION

Perron (1989) demonstrated that the conventional Dickey and Fuller (1979) t-statistic (t_{DF}) accepts the null hypothesis of a unit root too often when the true data generating process is in fact trend stationary with a break in the intercept and/or the slope of the trend function. Three different characterizations of the trend-break alternative were considered: (a) the Crash model that allows a break in the intercept; (b) the Changing Growth model that allows for a break in the slope with the two segments joined at the break-date; and (c) the Mixed model which allows for a simultaneous break in the intercept and the slope. In order to devise unit root tests that have power against the trend-break stationary alternative, Perron (1989) proposed the following methodology: first specify the location of break-date (T_b), and then estimate a regression that nests the random walk null and the trend-break stationary alternative of choice. Under the null hypothesis, he derives the limiting distribution of the t-statistic on the first lag of the dependent variable, denoted by $t^i_{DF}(T_b)$ where i = A, B, or C corresponds to the Crash, the Changing Growth, or the Mixed model respectively. The limiting distribution of $t^i_{DF}(T_b)$ depends on the location and form of break under the alternative hypothesis.

The assumption that the location of break is known a priori has been criticized by a number of studies, most notably by Christiano (1992).[1] Christiano (1992) argued that

[1] Also see Banerjee, Lumsdaine and Stock (1992), and Zivot and Andrews (1992), and Perron and Vogelsang (1992).

the choice of the break-date is in most cases correlated with the data, for example, the practitioner may determine the location of the break-date by visually inspecting a plot of the time series. And since Perron's (1989) methodology does not account for this 'pretest examination of data,' the unit root null will be rejected too often. A number of studies have proposed extensions for unit root tests that do not require the practitioner to pre-specify the location of break, see Zivot and Andrews (1992), Banerjee, Lumsdiane and Stock (1992), Perron and Vogelsang (1992), Perron (1997), Vogelsang and Perron (1998). The strategy adopted by these studies is to apply Perron's (1989) methodology for each possible break-date in the sample which yields a sequence of t-statistics. Based on this sequence, numerous 'minimum-t-statistics' can be constructed by choosing the t-statistic, based on some algorithm, that maximizes evidence against the null hypothesis. For example, one may simply use the minimum of the sequence of t-statistics, denoted by $t_{DF}^{min}(i)$, i=A,B,C.

With the availability of the minimum-t-statistics, the practitioner no longer needs to pre-specify the location of break. However, one must still specify the form of break under the alternative hypothesis. If one assumes that the location of break is unknown, it is most likely that the form of break will be unknown as well. We argue that selection of the form of break is also correlated with the data and so one must proceed with the break specification according to the most general Mixed model. In addition, one may expect power distortions if the form of break is misspecified, that is, if one imposes the Crash (Changing Growth) model when in fact the Changing Growth (Crash) or the Mixed model is appropriate. Consider, for example, the use of the Crash model specification under the alternative hypothesis when the break occurs according to the Changing Growth or the Mixed model. The Crash model minimum-t-statistic $t_{DF}^{min}(A)$ is designed to maximize evidence against the null hypothesis in favour of this particular alternative. If the true data generating process is given by the Mixed model, we can expect the power of $t_{DF}^{min}(A)$ to be lower than that of $t_{DF}^{min}(C)$, especially if the magnitude of the slope break is relatively large. We may expect even lower power of $t_{DF}^{min}(A)$ if the alternative evolves according to the Changing Growth model. On the other hand, if the Crash hypothesis is the correct specification, then its use will yield superior power compared to the Mixed model statistics. In practice, however, one is uncertain about the form of break. Since one would like to guard against distortion in power owing to misspecification of the form of break, we recommend that the practitioner use the Mixed model specification under the alternative hypothesis.

Our first objective is to assess the performance of minimum-t-statistics when the form of break is misspecified. To this end, we provide simulation evidence: (a) on the performance of tests with the Crash model when the break occurs according to either the Changing Growth or the Mixed model; (b) on the performance of tests with the Changing Growth model when the break occurs according to either the Crash or the Mixed model; and (c) to assess the loss in power of tests with the Mixed model when the break occurs according to either the Crash or the Changing Growth model. We find that the loss in power is quite small if the Mixed model specification is used when in fact the break occurs according to the Crash model. The loss in power is more substantial if the break occurs according to the Changing Growth model. However, serious distortions can occur if the Crash (Changing Growth) model is used when the appropriate model is either the Changing Growth (Crash) model or the Mixed model. Therefore, our results indicate that one should use the form of the break specified under the Mixed model, unless prior information about the nature of the break suggests using either the Crash model or the Changing Growth model. In most cases, we expect that information on the form of break will be accompanied with information on the location of break in which case Perron's (1989) tests should be used. Second, we propose a new statistic for the Mixed model, denoted by F_T^{max}, for the joint null hypothesis that there is a unit root and no break in the intercept and slope of the trend function. We derive the limiting distribution of the new statistic and tabulate the asymptotic and finite sample critical values. Our simulation results show that the power properties of F_T^{max} are less erratic and can be greater than the Mixed model minimum-t-statistics.

Third, we illustrate our arguments within the context of an empirical example that has received much attention in the literature. Numerous studies have used the Nelson and Plosser (1982) data and post-war Quarterly Real GNP to test for the presence of a unit root. Perron (1989) specified the Great Crash of 1929 as the break-date for all Nelson-Plosser series, and the Oil Price Shock of 1973 for Quarterly Real GNP. Conditional on these break-dates, the Crash model was specified for all Nelson-Plosser series, except Common Stock Prices and Real Wages for which the Mixed model was used, and the Changing Growth model was used for Quarterly Real GNP. While subsequent studies by Zivot and Andrews (1992), Perron (1997), and Nunes, Newbold, and Kuan (1997) have allowed for an unknown break-date, they have retained the model specification proposed by Perron (1989). Unlike these studies, we present empirical evidence for the presence of a unit root in all series when the alternative allows for a simultaneous break in the intercept and slope. We find that the unit root null can be rejected for all series with the exception of the GNP Deflator, Consumer Prices, Velocity, and Interest Rate series. Our results are robust to possible misspecification of the form of break and therefore reveal useful information regarding the location and form of break. For instance, the empirical evidence in Zivot and Andrews (1992) indicates that Real Per Capita GNP is characterized as a stationary process with break in the intercept occuring in 1929, and both Money Stock and Quarterly Real GNP contain a unit root. By using the Mixed model as the appropriate alternative, we strengthen the evidence against a unit root in Real Per Capita GNP, and uncovered some evidence against the unit root for Money Stock and Quarterly Real GNP. We estimate the break-date for Real Per Capita GNP in 1938, for Money Stock in 1930, and for Quarterly Real GNP in 1964:IV. Our estimated break-dates for these series are different from the estimated break-dates reported in Zivot and Andrews (1992).

The outline of this paper is as follows. In section 2, we briefly discuss the minimum-t-statistics that have been proposed in the literature. The power properties of these statistics is assessed within the context of a simple simulation experiment when the form of break is misspecified. In section 3, we propose a statistic for the joint null hypothesis of a random walk and no break in the intercept and slope, denoted by F_T^{max}, and present its limiting null distribution and critical values. In section 4, simulation evidence is used to contrast the finite sample size and power properties of F_T^{max} with those of the minimum-t-statistics. Empirical evidence for the Nelson and Plosser (1982) data and U.S. Post-War Quarterly Real GNP is presented in section 5. Concluding comments appear in section 6.

2. BACKGROUND AND MOTIVATION

We begin with a brief discussion of the null and alternative hypotheses of interest, and the class of minimum-t-statistics. Our discussion follows the analysis in Zivot and Andrews (1992). Consider the time series $\{y_t\}_{t=1}^{T}$ where T is the available sample size. The three different characterizations of the alternative hypothesis, originally discussed by Perron (1989), are given by:

$$\text{Model(A)}: \quad y_t = \mu_0 + \mu_1\, DU_t(T_b^c) + \mu_2\, t + \alpha\, y_{t-1} + e_t \tag{1}$$

$$\text{Model(B)}: \quad y_t = \mu_0 + \mu_2\, t + \mu_3\, DT_t(T_b^c) + \alpha\, y_{t-1} + e_t \tag{2}$$

$$\text{Model(C)}: \quad y_t = \mu_0 + \mu_1\, DU_t(T_b^c) + \mu_2\, t + \mu_3\, DT_t(T_b^c) + \alpha\, y_{t-1} + e_t \tag{3}$$

where T_b^c is the correct break-date, $\mathrm{DU_t(T_b^c)} = 1_{(\mathrm{t>T_b^c})}$, $\mathrm{DT_t(T_b^c)} = (\mathrm{t - T_b^c})\, 1_{(\mathrm{t>T_b^c})}$, and $1_{(t>T_b^c)}$ is an indicator function that takes on the value 0 if $\mathrm{t \leq T_b^c}$ and 1 if $\mathrm{t > T_b^c}$. For the asymptotic results, we assume that the break-date is a constant fraction of the sample size, that is, $T_b^c = \lambda^c T$ with the correct break-fraction $\lambda^c \in$ (0,1). We assume that $\mathrm{A(L)\,e_t = B(L)}\,\nu_{\mathrm{t}}$, and ν_t is a sequence of i.i.d. $(0, \sigma^2)$ random variables, A(L) and B(L) are polynomials in the lag operator of order p and q respectively with all roots outside the unit circle. Model (A) is referred to as the Crash model and it allows for a break in the intercept alone, Model (B) is referred to as the Changing Growth model and it allows for a break in the slope with the two segments joined at the break-date, and Model (C) is referred to as the Mixed model and it allows for a simultaneous break in the intercept and the slope of the trend function. Under the null hypothesis, the data generating process is a random walk (possibly with drift), that is,

$$y_t = \mu_0 + y_{t-1} + e_t \tag{4}$$

In order to test the unit root null against the alternatives specified in (1)-(3), the following methodology has been prescribed. Specify the interval $\Lambda = [\lambda_0, 1{-}\lambda_0] \subseteq$ (0,1) that is believed to contain the true break-fraction. For each possible $\lambda \in \Lambda$, estimate the following regression that nests the null and the appropriate alternative:

$$y_t = \hat{\mu}_0^A + \hat{\mu}_1^A\, DU_t([\lambda T]) + \hat{\mu}_2^A\, t + \hat{\alpha}^A\, y_{t-1} + \sum_{j=1}^{k} \hat{c}_j^A\, \Delta y_{t-j} + \hat{e}_t \tag{5}$$

$$y_t = \hat{\mu}_0^B + \hat{\mu}_2^B\, t + \hat{\mu}_3^B\, DT_t([\lambda T]) + \hat{\alpha}^B\, y_{t-1} + \sum_{j=1}^{k} \hat{c}_j^B\, \Delta y_{t-j} + \hat{e}_t \tag{6}$$

$$y_t = \hat{\mu}_0^C + \hat{\mu}_1^C\, DU_t([\lambda T]) + \hat{\mu}_2^C\, t + \hat{\mu}_3^C\, DT_t([\lambda T]) + \hat{\alpha}^C\, y_{t-1} + \sum_{j=1}^{k} \hat{c}_j^C\, \Delta y_{t-j} + \hat{e}_t \tag{7}$$

where [.] is the smallest integer function. The additional 'k' regressors $\{\Delta y_{t-j}\}_{j=1}^{k}$ are included in the regression to eliminate the correlation in the disturbance term. Typically, the value of the lag-truncation parameter (k) is unknown, and so a data-dependent method for choosing the appropriate value of k is used, see Perron and Vogelsang (1992), Hall (1992b,1994), Perron (1997), and Ng and Perron (1995). We use Perron and Vogelsang's (1992) data-dependent method k(t-sig) for selecting the lag-truncation parameter which is described in what follows. Specify an upper bound 'kmax' for the lag-truncation parameter. For each break-date $[\lambda T]$, the chosen value of the lag-truncation parameter (k*) is determined according to the following 'general to specific' procedure: the last lag in an autoregression of order k* is significant, but the last lag in an autoregression of order greater than k* is insignificant. The significance of the coefficient is assessed using the 10% critical values based on a standard normal distribution. Regressions (5)-(7) are estimated for the break-dates $\{[\lambda_0 T], [\lambda_0 T]+1, \ldots, T-[\lambda_0 T]\}$, and the sequence of t-statistics for $H_0 : \alpha = 1$, denoted by $\{t^{\,i}_{DF}(T_b)\}_{T_b=[\lambda_0 T]}^{T-[\lambda_0 T]}$ (i=A,B,C) are calculated. Based on this sequence, a number of minimum-t-statistics can be obtained by using an algorithm to choose the appropriate break-date that maximizes evidence against equation (4), see Perron and Vogelsang (1992), Banerjee, Lumsdaine and Stock (1992), Perron (1997) and Vogelsang and Perron (1998).

We consider two particular algorithms for choosing the break-date. The first statistic, originally proposed by Perron and Vogelsang (1992) and Zivot and Andrews (1992), is obtained by choosing the break-date that maximizes evidence against the unit root null, that is,

$$t_{DF}^{min}(i) = Min_{\, T_b \in \{[\lambda_0 T],[\lambda_0 T]+1,\ldots,T-[\lambda_0 T]\}} \; t_{DF}^{i}(T_b) \tag{8}$$

for i=A,B,C. Perron (1997) demonstrated that $t_{DF}^{min}(i)$ can be calculated with Λ=(0,1) so that no trimming of the sample is necessary. The second statistic proposed by Banerjee, Lumsdaine and Stock (1992) and Vogelsang and Perron (1998) is defined as:

$$\hat{t}_{DF}(i) = t_{DF}(\hat{\lambda}_i) \tag{9}$$

where $\hat{\lambda}_i$ (i=A,B,C) is the break-date that maximizes the Wald statistic, $F_T^i([\lambda T])$, corresponding to $H_0 : \mu_1 = 0$ for the Crash model (i=A), $H_0 : \mu_3 = 0$ for the Changing Growth model (i=B), and $H_0 : \mu_1 = \mu_3 = 0$ for the Mixed model (i=C).

The minimum-t-statistics, $t_{DF}^{min}(i)$ and $\hat{t}_{DF}(i)$ (i=A,B,C), do not require specification of the location of break. However, the practitioner does need to specify the form of break under the alternative hypothesis. Once the break-date is treated as unknown, the practitioner will in most cases be unaware of the form of break. Since the Crash and Changing Growth models are special cases of the Mixed model, we feel that the practitioner should specify the Mixed model as the appropriate alternative. We can certainly expect some loss in power from using the statistics from the Mixed model when in fact the break occurs according to the Crash or Changing Growth model. However, we are uncertain about the behaviour of the statistics from the Crash (Changing Growth) model when the break occurs according to the Changing Growth (Crash) or Mixed model. In order to assess the power properties of

$t_{DF}^{min}(i)$ and $\hat{t}_{DF}(i)$ (i=A,B,C) when the form of break is misspecified, we conduct a simple simulation experiment. We generate data according to the alternative hypothesis:

$$y_t = \mu_1\, DU_t^c + \mu_3\, DT_t^c + \alpha\, y_{t-1} + e_t\ , \qquad \mathrm{t} = 1, 2, \ldots, \mathrm{T} \tag{10}$$

where y_0=0, e_t are i.i.d. N(0,1), $DU_t^c = 1_{(t>T_b^c)}$, $DT_t^c = (t - T_b^c)\, 1_{(t>T_b^c)}$, T_b^c=50, T=100, α=0.8, μ_1= $\{0, 1, 2, 4, 6, 8\}$, and μ_3= $\{0, 0.1, 0.2, 0.3, 0.4, 0.5\}$. We consider all parameter combination of μ_1 and μ_3 (except μ_1=0 and μ_3=0). For each parameter combination, we estimate regressions (5)-(7) and calculate $t_{DF}^{min}(i)$ and $\hat{t}_{DF}(i)$ with λ_0=0.15 (i=A,B,C). We set the lag-truncation parameter equal to its true value, that is, k=0. Based on 2,000 replications, we calculate the power of $t_{DF}^{min}(i)$ and $\hat{t}_{DF}(i)$ (i=A,B,C) using the asymptotic critical values at the 5% significance level.[2] The break occurs according to the Crash model when $\mu_1 \neq$0 and μ_3=0, according to the Changing Growth model when μ_1=0 and $\mu_3 \neq$0, and according to the Mixed model when $\mu_1 \neq$0 and $\mu_3 \neq$0. The power of $t_{DF}^{min}(i)$ and $\hat{t}_{DF}(i)$ (i=A,B,C) when the true data generating process is given by the Crash model, the Changing Growth model, and the Mixed model are presented in Tables 1, 2, and 3 respectively. We briefly discuss the results from these simulations.[3]

---

[2] The critical values for $t_{DF}^{min}(i)$ (i=A,B,C) were obtained from Zivot and Andrews (1992), see their Tables 2-4. The asymptotic critical values for $\hat{t}_{DF}(i)$ (i=A,B,C) were obtained using simulations.

[3] Additional simulation evidence for $t_{DF}^{min}(i)$ and $\hat{t}_{DF}(i)$ (i=A,B,C) is presented in Section 4, see Tables 6-8, where the appropriate lag-truncation parameter is determined using the

We find that the power of $t_{DF}^{min}(i)$ is greater than its counterpart $\hat{t}_{DF}(i)$ (i=A,B,C) for all cases considered here. The difference between $t_{DF}^{min}(i)$ and $\hat{t}_{DF}(i)$ is smallest for the Crash model statistics and largest for the Mixed model statistics. Consider the results from the Crash model simulations in Table 1. The statistics $t_{DF}^{min}(A)$ and $\hat{t}_{DF}(A)$ exhibit the best power, and their power increase with the size of the intercept-break magnitude (μ_1). In comparison, the power of $t_{DF}^{min}(C)$ and $\hat{t}_{DF}(C)$ is less, but this difference diminishes as μ_1 increases. The statistics $t_{DF}^{min}(B)$ and $\hat{t}_{DF}(B)$ have power close to zero in most cases. Next, consider the results from the Changing Growth model simulations in Table 2. $t_{DF}^{min}(B)$ and $\hat{t}_{DF}(B)$ exhibit the best power, and their power increase with the slope-break magnitude (μ_3). There is some loss in power from using $t_{DF}^{min}(C)$, but this difference in power diminishes as μ_3 increases. The loss in power from using $t_{DF}^{min}(C)$ is more severe compared to the Crash model simulations. In all cases, $t_{DF}^{min}(A)$ and $\hat{t}_{DF}(A)$ have zero power.

Finally, consider the results from the Mixed model simulations in Table 3. For a fixed μ_3, the power of $t_{DF}^{min}(A)$ and $\hat{t}_{DF}(A)$ increase with the size of μ_1, but this increase is dampened for large values of μ_3. For a fixed μ_1, the power of $t_{DF}^{min}(A)$ and $\hat{t}_{DF}(A)$ falls with the size of μ_3. For example, when μ_3=0.1, the power of $t_{DF}^{min}(A)$ increases from 0.2985 for μ_1=4 to

---

data-dependent k(t-sig) procedure of Perron and Vogelsang (1992). The use of the data-dependent procedure to determine the appropriate lag-truncation parameter does not alter the main results concerning the performance of $t_{DF}^{min}(i)$ and $\hat{t}_{DF}(i)$ (i=A,B,C). A few minor differences in the power properties are discussed in Section 4.

0.9980 for μ_1=6. With μ_3=0.3, the power of $t_{DF}^{min}(A)$ falls to zero for μ_1=4 and 0.1160 for μ_1=6. The statistics for the Changing Growth model, $t_{DF}^{min}(B)$ and $\hat{t}_{DF}(B)$, exhibit a similar behaviour. For a fixed μ_1, the power of $t_{DF}^{min}(B)$ and $\hat{t}_{DF}(B)$ increases with the size of μ_3, but this increase is dampened for large values of μ_1. Further, for a fixed μ_3, the power of $t_{DF}^{min}(B)$ and $\hat{t}_{DF}(B)$ falls with the size of μ_1. For example, when μ_1=2, the power of $t_{DF}^{min}(B)$ increases from 0.0685 for μ_3=0.1 to 0.9515 for μ_3=0.3. With μ_1=6, the power of $t_{DF}^{min}(B)$ falls to zero with μ_3=0.1 and 0.0005 for μ_3=0.3. Although the power of $t_{DF}^{min}(C)$ and $\hat{t}_{DF}(C)$ does not always increase with the size of μ_1 and μ_3, there is general tendency for their power to increase. For example, with μ_3=0.2, the power of $t_{DF}^{min}(C)$ falls from 0.3505 with μ_1=1 to 0.2630 with μ_1=2, and then increases with μ_1. With μ_1=4, the power of $t_{DF}^{min}(C)$ falls from 0.7820 with μ_3=0.1 to 0.5435 with μ_3=0.3, but increase with further increase in μ_3.

We can summarize the results from our simulations as follows. First, the Crash and Changing Growth model simulations illustrate that substantial loss in power can occur if the Changing Growth (Crash) model is used when in fact the true data generating process follows the Crash (Changing Growth) model. In either case, the power of the Mixed model statistics increase with the size of break, but there is some loss in power. This loss in power diminishes with the size of the break. It is interesting to observe that the loss in power from using the Mixed model statistics is larger if the break occurs according to the Changing Growth model compared to the Crash model. Second, when the true data generating process occurs according to the Mixed model, the statistics from the Crash (Changing Growth) model have the undesirable property that their power falls as the size of the slope-break (intercept-break) increases. On the other hand, the power of Mixed model statistics in general increase with the size of the slope-break and intercept-break. Therefore, using the Crash (Changing Growth) model statistics will lead to incorrect acceptance of the null when the size of the slope-break (intercept-break) is large. Our results indicate that misspecification of the form of break can be crucial. We should point out that the use of the Crash (Changing Growth) model may lead to higher power when the intercept-break (slope-break) is very large and the slope-break (intercept-break) is small. In practice, however, one does not know the size or form of the break and so we recommend using the test statistics from the Mixed model so as to guard against misspecification.

3. TEST FOR THE JOINT NULL OF A RANDOM WALK AND NO BREAK

In this section we propose a test statistic for the joint null hypothesis of a unit root and no break in the intercept and slope of the trend function. The results in this section are direct extensions of the results in Banerjee, Lumsdaine, and Stock (1992). The model under the alternative hypothesis allows for a simultaneous break in the intercept and the slope of the trend function, that is:

$$y_t = \mu_0 + \mu_1 \, DU_t(T_b) + \mu_2 \, t + \mu_3 \, DT_t(T_b) + \alpha \, y_{t-1} + \sum_{j=1}^{k} c_j \, \Delta y_{t-j} + e_t \tag{11}$$

where $\mathrm{DU_t(T_b)}$ and $\mathrm{DT_t(T_b)}$ are defined above, and e_t satisfies the following assumption

ASSUMPTION 1: $\{e_t\}$ is a martingale difference sequence and satisfies $E(e_t|e_{t-1},\ldots) = \sigma^2$, $E(|e_t|^i|e_{t-1},\ldots) = \kappa_i$ (i=3,4), and $sup_t E(|e_t|^{4+\gamma}|e_{t-1},\ldots) = \kappa < \infty$ for some $\gamma > 0$.

The specification of the Mixed model in (11) assumes that the lag-truncation parameter is known. The joint null hypothesis of interest is $\mathrm{H}_0^{\mathrm{J}} : \alpha = 1,\ \mu_1 = 0, \text{and}\ \mu_3 = 0$. Assumption 1 has been taken from Banerjee, Lumsdaine and Stock (1992). Although the location of break is unknown, we assume that the break-fraction lies in $\Lambda = (\lambda_0, 1-\lambda_0)$. To test the joint null hypothesis H_0^J, we proceed as follows. For each break-date corresponding to $\lambda \in \Lambda$, that is, $\mathrm{T_b} \in \{[\lambda_0 \mathrm{T}], [\lambda_0 \mathrm{T}]+1, \ldots, \mathrm{T} - [\lambda_0 \mathrm{T}]\}$, we estimate the regression (11) and calculate the Wald statistic corresponding to H_0^J as:

$$\hat{F}_T(T_b) = \frac{(R\,\hat{\mu}(T_b) - r)\left[R\left(\sum_{t=1}^{T} x_t(T_b)\, x_t(T_b)'\right)^{-1} R'\right]^{-1}(R\,\hat{\mu}(T_b) - r)}{q\,\hat{\sigma}^2(T_b)} \tag{12}$$

where $\hat{\mu}(T_b)$ is the Ordinary Least Squares (OLS) estimator of $\mu = (\mu_0, \mu_1, \mu_2, \mu_3, \alpha, c_1, \ldots, c_k)'$, $x_t = (1, DU_t(T_b), t, DT_t(T_b), y_{t-1}, \Delta y_{t-1}, \ldots, \Delta y_{t-k})'$, r $= (0,0,1)'$, $\hat{\sigma}^2(T_b) = \frac{1}{(T-5-k)} \sum_{t=1}^{T}(y_t - x_t\,(T_b)'\hat{\mu})^2$, q is the number of restrictions, and R is defined so that $R\,\mu = r$ corresponds to the restrictions imposed on the parameter vector μ by the joint null H_0^J. Specifically,

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0_k \\ 0 & 0 & 0 & 1 & 0 & 0_k \\ 0 & 0 & 0 & 0 & 1 & 0_k \end{bmatrix}$$

Using the sequence of Wald statistics $\{F_T(T_b)\}_{T_b=[\lambda_0 T]}^{T-[\lambda_0 T]}$, we calculate the maximum-F-statistic for H_0^J as follows:

$$F_T^{max} = Max_{T_b \in \{[\lambda_0 T],[\lambda_0 T]+1,\ldots,T-[\lambda_0 T]\}} F_T(T_b) \tag{13}$$

In order to determine the asymptotic distribution of the test statistic we transform (11) according to Banerjee, Lumsdaine, and Stock (1992). Define the transformed regressors as $Z_{t-1}(T_b) = [Z_{t-1}^1, 1, y_{t-1} - \bar{\mu}_0\,(t-1), 1_{(t>T_b)}, (t-T_b)\,1_{(t>T_b)}, t]'$ with $Z_{t-1}^1 = [\Delta y_{t-1} - \bar{\mu}_0, \Delta y_{t-2} - \bar{\mu}_0, \ldots, \Delta y_{t-k} - \bar{\mu}_0]$, $\bar{\mu}_0 = E[\Delta y_t] = \mu_0/(1 - \sum_{j=1}^k c_j)$. The corresponding transformed parameter vector is defined as $\theta = (\theta_1', \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$ with $\theta_1 = (c_1, c_2, \ldots, c_k)'$, $\theta_2 = \mu_0 + [\sum_{j=1}^k c_j - \alpha]\,\bar{\mu}_0$, $\theta_3 = \alpha$, $\theta_4 = \mu_1$, $\theta_5 = \mu_3$, and $\theta_6 = \mu_2 + \alpha\,\bar{\mu}_0$. Using this transformation, we can re-write (11) as: $y_t = Z_{t-1}(T_b)'\,\theta + e_t$ for t $= 1, 2, \ldots,$ T. The OLS estimator of the transformed parameter vector is

$$\hat{\theta}(T_b) = \left[\sum_{t=1}^T Z_{t-1}(T_b)\, Z_{t-1}(T_b)'\right]^{-1} \left[\sum_{t=1}^T Z_{t-1}(T_b)\, y_t\right] \tag{14}$$

Using the scaling matrix $\Upsilon_T = \mathrm{Diag}(\mathrm{T}^{1/2}\,\mathrm{I_k}, \mathrm{T}^{1/2}, \mathrm{T}, \mathrm{T}^{1/2}, \mathrm{T}^{3/2}, \mathrm{T}^{3/2})$, we can get the following expression:

$$\Upsilon_T \left[\hat{\theta}(T_b) - \theta_0\right] = [\Gamma_T(T_b)]^{-1}\,[\Psi_T(T_b)] \tag{15}$$

where $\Gamma_T(T_b) = \Upsilon_T^{-1}\left(\sum_{t=1}^T Z_{t-1}(T_b)\, Z_{t-1}(T_b)'\right)\Upsilon_T^{-1}$ and $\Psi_T(T_b) = \Upsilon_T^{-1}\left(\sum_{t=1}^T Z_{t-1}(T_b)\, e_t\right)$. The following result establishes the asymptotic distribution of the scaled parameter vector.

THEOREM 1 : Suppose y_t is generated according to (11) with $\mu_1 = \mu_2 = \mu_3 = 0$ and $\alpha = 1$ and $\{e_t\}$ satisfies Assumption 1. Then, $\Upsilon_T \left[\hat{\theta}([\lambda T]) - \theta_0\right] \Rightarrow [\Gamma(\lambda)]^{-1}\ \Psi(\lambda)$, where $\Psi(\lambda) = \sigma\{B(1), W(1), \int_0^1 J(\lambda)\, dW(\lambda), W(1) - W(\lambda), (1-\lambda)\, W(1) - \int_\lambda^1 W(\delta)\, d\delta, W(1) - \int_0^1 W(\delta)\, d\delta\}$ and $\Gamma_{11}(\lambda) = \Omega_k$, $\Gamma_{22}(\lambda) = 1$, $\Gamma_{33}(\lambda) = \int_0^1 J(\delta)^2\, d\delta$, $\Gamma_{44}(\lambda) = 1 - \lambda$, $\Gamma_{55}(\lambda) = \frac{1}{3}(1-\lambda)^3$, $\Gamma_{66}(\lambda) = \frac{1}{3}$, $\Gamma_{1j}(\lambda) = 0_k$ for i $= 2, 3, \ldots, 6$, $\Gamma_{23}(\lambda) = \int_0^1 J(\delta)\, d\delta$, $\Gamma_{24}(\lambda) = 1 - \lambda$, $\Gamma_{25}(\lambda) = \frac{1}{2}(1-\lambda)^2$, $\Gamma_{26}(\lambda) = \frac{1}{2}$, $\Gamma_{34}(\lambda) = \int_\lambda^1 J(\delta)\, d\delta$, $\Gamma_{35}(\lambda) = \int_\lambda^1 (\lambda - \delta) J(\delta)\, d\delta$, $\Gamma_{36}(\lambda) = \int_0^1 \delta J(\delta)\, d\delta$, $\Gamma_{45}(\lambda) = \frac{1}{2}(1-\lambda)^2$, $\Gamma_{46}(\lambda) = \frac{1}{2}(1-\lambda^2)$, $\Gamma_{56}(\lambda) = \frac{1}{3} - \frac{1}{2}\lambda + \frac{1}{6}\lambda^3$, $W(\lambda)$ is a standard Brownian motion on [0,1], $J(\lambda) \equiv \left(1 - \sum_{i=1}^k c_i\right)^{-1} \sigma W(\lambda)$, $B(\lambda)$ is a p-dimensional Brownian motion with covariance matrix Ω_k, W and B are independent.

The proof of this result follows as a straightforward extension of the results in Banerjee, Lumsdaine, and Stock (1992). For a fixed break-date $T_b = [\lambda T]$, the Wald statistic for testing the joint null hypothesis $H_0^{J*} : \theta_4 = 0, \theta_5 = 0, \theta_3 = 1$ is:

$$\hat{F}_T([\lambda T]) = \frac{\left(R^*\, \hat{\theta}([\lambda T]) - r\right) \left[R^* \left(\sum_{t=1}^{T} Z_{t-1}([\lambda T])\, Z_{t-1}([\lambda T])'\right)^{-1} R^{*'}\right]^{-1} \left(R^*\, \hat{\theta}([\lambda T]) - r\right)}{q\, \hat{\sigma}^{2*}([\lambda T])}$$

where $\hat{\sigma}^{2*}([\lambda T]) = (T - 5 - k)^{-1} \sum_{t=1}^{T} (y_t - Z_{t-1}([\lambda T])'\, \hat{\theta}([\lambda T]))^2$, q is the number of restrictions under H_0^{J*}, r $= (0, 0, 1)'$, and R^* is chosen so that $R^*\theta = r$ under H_0^{J*}. Specifically, we set

$$R^* = \begin{bmatrix} 0_k & 0 & 1 & 0 & 0 & 0 \\ 0_k & 0 & 0 & 1 & 0 & 0 \\ 0_k & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The limiting distribution of $\hat{\theta}([\lambda T])$ given in Theorem 1 implies that:

$$\hat{F}_T([\lambda T]) \Rightarrow \frac{(R^*[\Gamma(\lambda)]^{-1}\Psi(\lambda))' \left(R^*[\Gamma(\lambda)]^{-1}R^{*'}\right)^{-1} (R^*[\Gamma(\lambda)]^{-1}\Psi(\lambda))}{q\,\sigma^2} \equiv F(\lambda) \qquad (16)$$

and using the continuous mapping theorem, we get:

$$F_T^{max} \Rightarrow sup_{\lambda\in\Lambda}\, F(\lambda) \qquad (17)$$

The limiting distribution of F_T^{max} in (17) depends on the choice of $\Lambda = (\lambda_0, 1-\lambda_0)$. The critical values for $sup_{\lambda\in\Lambda}F(\lambda)$ are tabulated for λ_0=0.15, 0.10, 0.05 in Table 4. The asymptotic critical values are obtained by simulation methods. First, we approximate one realization of the Brownian Motion, denoted by W(.), on a sufficiently find grid [1/n,2/n,...,(n-1)/n,1]. We generate n=1000 independent random draws from a N(0,1/$\sqrt{1000}$) distribution and approximate W(.) by the cumulative sum of this sequence. We use the realization of W(.) to calculate the various elements in $\Gamma(\lambda)$ and $\Psi(\lambda)$, the integrals are approximated by sums over the grid on the unit interval. This in turn yields one realization from the distribution of F_T^{max} with λ_0. We generate 5,000 realizations from the limiting distribution and calculate the critical values from the sorted vector of replicated statistics. We also calculate finite sample critical values for F_T^{max} with λ_0=0.15,0.10,0.05 for sample size T=100, 200. The finite sample critical values are simulated with data generated according to the null model as:

$$y_t = y_{t-1} + e_t \ , \qquad \mathrm{t} = 1, 2, \ldots, \mathrm{T} \tag{18}$$

with y_0=0, e_t are i.i.d. N(0,1), T is the sample size. The finite sample critical values are calculated when the true value of the lag-truncation is chosen (k=0) and when the lag-truncation parameter is determined using the data-dependent method k(t-sig) for choosing the appropriate lag-truncation parameter. With the data dependent method k(t-sig), we use kmax=5 and 2,000 replications. When the true value of the lag-truncation parameter is used (k=0), we use 10,000 replications.

4. FINITE SAMPLE SIZE AND POWER

In this section, we present finite sample size and power results for the F_T^{max} statistic. We draw attention to the differences in the power of F_T^{max} and the minimum-t-statistics from the Mixed model, namely, $t_{DF}^{min}(C)$ and $\hat{t}_{DF}(C)$. We follow the experimental design in Vogelsang and Perron (1998), and generate data according to:

$$\left[1 - (\alpha + \rho)\, L + \rho\, L^2\right]\, y_t = (1 + \psi\, L)\, [\mu_1\, DU_t^c + \mu_3\, DT_t^c + e_t] \ , \ \ \mathrm{t} = 1, 2, \ldots, \mathrm{T} \tag{19}$$

For the size simulations, we set α=1, and for the power simulations we set α=0.8. The sample size for all simulations is T=100, the correct break-date T_b^c=50 , $DU_t^c = DU_t(T_b^c)$ and $DT_t^c = DT_t(T_b^c)$, and $e_t \sim$ i.i.d. N(0,1). We use the following combinations of (ρ, ψ):

{(0,0);(0.6,0);(-0.6,0);(0,0.5);(0,-0.5)}. For the size simulations, we set $\mu_1 = \mu_3$=0. For the power simulations, we consider the data generating process under: (a) the Crash model with a break in the intercept for μ_1={1,2,4,6,8}; (b) the Changing Growth model with a break in the slope for μ_3={0.1,0.2,0.3,0.4,0.5}; (c) the Mixed model with a simultaneous break in the intercept and slope for μ_1={1,2,4,6} and μ_3={0.1,0.2,0.3,0.4}. For each parameter combination we generated 2,000 replications. We calculate the size and power of following statistics at the 5% significance level using the appropriate finite sample critical values: $t_{DF}^{min}(i)$ and $\hat{t}_{DF}(i)$ for i=A,B,C, and F_T^{max}. The lag-truncation parameter (k) in the regressions (5)-(7) is chosen according to the k(t-sig) procedure mentioned in section 2 with kmax=5. The power of all statistics under the Crash model, the Changing Growth model, and the Mixed model are given in Tables 6, 7, and 8 respectively.

The finite sample size for all statistics is presented in Table 5. The exact size of F_T^{max} is close to the nominal size in most cases, except when there is a negative MA component, that is, (ρ, ψ)=(0,-0.5). The exact size of F_T^{max} is lower compared to $t_{DF}^{min}(C)$, but greater than the size of all other statistics. Let us now turn to the results pertaining to the power simulations. The power properties of the minimum-t-statistics is very similar to their behaviour in the simulations reported in section 2, see Tables 1-3.[4] First, consider the results from the Crash

[4] One minor difference in the results is that there are numerous instances when the power of $\hat{t}_{DF}(i)$ is greater than the power of the corresponding $t_{DF}^{min}(i)$ for i=A,B, but these differences are quite small.

Model simulations given in Table 6. Substantial loss in power can occur if the Changing Growth model is specified since the power of $t_{DF}^{min}(B)$ and $\hat{t}_{DF}(B)$ is close to zero except when the magnitude of the intercept-break (μ_1) is very small. The power of F_T^{max}, $t_{DF}^{min}(C)$ and $\hat{t}_{DF}(C)$ increases with μ_1. The power of F_T^{max} is greater than the power of $t_{DF}^{min}(C)$ and $\hat{t}_{DF}(C)$, except for small μ_1. When F_T^{max} has lower power, the largest difference in the power of F_T^{max} and $t_{DF}^{min}(C)$ occur when (ρ, ψ)=(0,0.6). In all other cases this difference is less than 0.0435. We find that the power of F_T^{max} is greater than that of $t_{DF}^{min}(A)$ and $\hat{t}_{DF}(A)$, except for small values of μ_1. In these instances, the median difference in power is 0.0505.

Second, a similar pattern emerges in the Changing Growth simulations in Table 7. There is considerable loss in power if the Crash model statistics are used when the true data generating process follows the Changing Growth model, that is, the power of $t_{DF}^{min}(A)$ and $\hat{t}_{DF}(A)$ is close to zero in most cases. In all cases, the power of $\hat{t}_{DF}(B)$ is greater than the power of $t_{DF}^{min}(B)$. The power of F_T^{max}, $t_{DF}^{min}(C)$ and $\hat{t}_{DF}(C)$ increases with the size of the slope-break (μ_3). The power of F_T^{max} is greater than the power of $t_{DF}^{min}(C)$ and $\hat{t}_{DF}(C)$, except for small values of μ_3. When F_T^{max} has lower power, the largest difference in the power of F_T^{max} and $t_{DF}^{min}(C)$ occur when (ρ, ψ)=(0,0.6). In all other cases this difference is less than 0.0430. The loss in power from using F_T^{max} diminishes with the size of μ_3.

Finally, consider the results of the Mixed model power simulations in Table 8. For a fixed μ_3, the power of the Crash model statistics increase with μ_1, and the power of the Changing Growth model statistics decrease with μ_1. For a fixed μ_1, the power of the Crash model statistics decreases with μ_3 and the power of the Changing Growth statistics increases with μ_3. Therefore, $t_{DF}^{min}(i)$ and $\hat{t}_{DF}(i)$, i=A,B, can fail to reject the unit root null if the true data generating process is given by the Mixed model with a relatively large μ_3 (μ_1). For example, if μ_1=4 and μ_3=0.1 with (ρ, ψ)=(0,0), then the power of $t_{DF}^{min}(A)$ is 0.2545, the power of $t_{DF}^{min}(B)$ is 0, and the power of F_T^{max} is 0.9890. The power of F_T^{max} is in most cases greater than the power of $t_{DF}^{min}(C)$ and $\hat{t}_{DF}(C)$, except in cases when μ_1 and μ_3 are both small. In this case the difference is less than 0.0280. The power of F_T^{max} increases with the size of both μ_1 and μ_3.[5] The power of F_T^{max} is greater than the power of $t_{DF}^{min}(A)$ and $\hat{t}_{DF}(A)$ in all cases. However, the power of F_T^{max} is less than the power of $t_{DF}^{min}(B)$ and $\hat{t}_{DF}(B)$ when μ_1 is small.

Our simulations indicate that the power of F_T^{max} is greater than the power of $t_{DF}^{min}(C)$ and $\hat{t}_{DF}(C)$ except when the magnitude of intercept-break and/or slope-break is relatively small. In these cases, the use of F_T^{max} does not result in large loss in power. In addition, F_T^{max} has

[5] The behaviour of the Mixed model minimum-t-statistics is sometimes erratic, in that, their power does not always increase with the size of μ_1 and μ_3, except when (ρ, ψ)=(0,0.6). The power of $t_{DF}^{min}(C)$ and $\hat{t}_{DF}(C)$ tend to fall slightly with the size of μ_3 when μ_1 is large. For example, consider the power of $t_{DF}^{min}(C)$ when (ρ, ψ)=(0,0). When μ_1=1, the power of $t_{DF}^{min}(C)$ increases from 0.2115 with μ_3=0.1 to 0.4750 with μ_3=0.3. But with μ_1=4, the power of $t_{DF}^{min}(C)$ falls from 0.6180 with μ_3=0.1 to 0.4235 with μ_3=0.3. The behaviour of $t_{DF}^{min}(C)$ and $\hat{t}_{DF}(C)$ is even more erratic when μ_3 is fixed and μ_1 increases.

the desirable property that its power increases with the magnitude of the intercept-break and slope-break.

5. EMPIRICAL RESULTS FOR NELSON AND PLOSSER (1982) DATA

In this section, we revaluate the evidence for the presence of a unit root in all Nelson and Plosser (1982) series, except Unemployment, and post-war U.S. Quarterly Real GNP.[6] We analyze the natural logarithm of all series except the Interest Rate series which is analyzed in level form. Empirical evidence for the Nelson-Plosser data can be found in Zivot and Andrews (1992), Perron (1997), and Nunes, Newbold, and Kuan (1997). While these studies do not pre-specify the location of break, they retain Perron's (1989) characterization of the form of break under the alternative hypothesis. In particular, Perron (1989) specified the Changing Growth model for Quarterly Real GNP, the Mixed model for Real Wages and Common Stock Prices, and the Crash model the remaining series. However, Perron's (1989) specification of the form of break is based on his selection of the Great Crash of 1929 for all Nelson-Plosser series and the Oil Price Shock of 1973 for Quarterly Real GNP series as appropriate break-dates. We feel that once the location of break is treated as unknown, the form of break should also be treated as unknown. So one must proceed with the most general

[6] The Nelson and Plosser (1982) data was kindly provided by Herman Bierens. The data for Quarterly Real GNP (1947:I - 2000:II) was retrieved from the Federal Reserve Economic Data (FRED) database.

characterization of the break under the alternative hypothesis, namely, a simultaneous break in the intercept and/or slope.

We present empirical evidence for all Nelson and Plosser (1982) series and Quarterly Real GNP when the Mixed model is used as the appropriate alternative. We use the k(t-sig) method of Perron and Vogelsang (1992) to determine the appropriate order of the lag-truncation parameter in regression (7). Following Zivot and Andrews (1992), we use kmax=8 for the Nelson-Plosser (1982) series and kmax=12 for Quarterly Real GNP. In Table 9, we report the calculated statistics $t_{DF}^{min}(C)$, $\hat{t}_{DF}(C)$ and F_T^{max} for each series, and the corresponding estimated break-dates denoted by $\hat{T}_b(t_{DF}^{min})$, $\hat{T}_b(\hat{t}_{DF})$ and $\hat{T}_b(F_T^{max})$ respectively. The results for regression (7) corresponding to the different estimated break-dates is presented in Table 10.[7]

First, consider the results for $t_{DF}^{min}(C)$ and $\hat{T}_b(t_{DF}^{min})$. If we use the asymptotic critical

[7] We found that the results for Common Stock Prices and Real Wages given in Table 6 of Zivot and Andrews (1992) are incorrect. The estimated regression results for Real Wages with T_b= 1940 and k=8 yields the results reported in Zivot and Andrews (1992), but the t-statistic on the Δy_{t-8} was found to be 1.2277 which is less than 1.6. While the estimated regression results for Common Stock Prices with T_b=1936 and k=1 yield the results reported in Zivot and Andrews (1992), we found that with k=3 the t-statistic on Δy_{t-3} is 1.96 which is greater than 1.6. Therefore, in both cases the k(t-sig) procedure for choosing the lag-truncation parameter was not applied correctly.

values for $t_{DF}^{min}(C)$ given in Zivot and Andrews (1992), we reject the unit root null at the 1% significance level for Real GNP, Nominal GNP, Industrial Production, at the 2.5% significance level for Real Per Capita GNP, Common Stock Prices, Real Wages, at the 5% significance level for Employment, Nominal Wages, Quarterly Real GNP, and the 10% significance level for Money Stock. We fail to reject the unit root null for the GNP Deflator, Consumer Prices, Velocity, and Interest Rate series. Of the series for which the unit root null is rejected, the estimated break-date $\hat{T}_b(T_{DF}^{min})$ does not coincide with the Great Crash (1929) for Real Per Capita GNP, Money Stock, Common Stock Prices, and Real Wages. In addition, the estimated break-date for Quarterly Real GNP does not coincide with the Oil Price shock of 1973. The evidence against the unit root null is weakened if we use the finite sample critical values given in Perron (1997) for $t_{DF}^{min}(C)$ with lag-truncation parameter selected using the k(t-sig) method. With the finite sample critical values, we cannot reject the unit root null for Employment, Nominal Wages, Money Stock, and Quarterly Real GNP.

Since we do not allow for a break under the unit root null hypothesis, our results should be compared to the results in Zivot and Andrews (1992). We reject the unit root null hypothesis for Money Stock, Real Wages, and Quarterly Real GNP in addition to all series for which Zivot and Andrews (1992) rejected the unit root null. Further, our evidence is stronger for Real Per Capita GNP. Unlike Zivot and Andrews (1992), our estimated break-date for Real Per Capita GNP is 1938, for Money Stock is 1930, for Common Stock Prices is 1939, and for Quarterly Real GNP is 1964.IV. Our estimated break-date coincides with the Great Crash of

1929 for Real GNP, Nominal GNP, Industrial Production, Employment, and Nominal Wages. In cases for which the unit root null is rejected, we can determine the significance of constant (μ_0), the intercept-break coefficient (μ_1), the trend (μ_2), and the slope-break coefficient (μ_3) using the standard normal distribution. The estimated coefficients and their respective t-statistics are presented in Table 10. The t-statistics for μ_0, μ_1, and μ_2 are significant for all series, with the sole exception of μ_2 for Real Per Capita GNP. The slope-break coefficient μ_3 is significant for Nominal GNP, Real Per Capita GNP, Money Stock, Common Stock Prices, Real Wages, and Quarterly Real GNP.

Second, the results for $\hat{t}_{DF}(C)$ are qualitatively similar to those of $\hat{t}_{DF}^{min}(C)$. The critical values for $\hat{t}_{DF}(C)$, both asymptotic and finite sample with lag-truncation parameter chosen according to the k(t-sig) method, can be found in Vogelsang and Perron (1998). Based on the asymptotic critical values, we reject the unit root null at the 1% significance level for Nominal GNP, Industrial Production, at the 2.5% significance level for Real GNP, Common Stock Prices, Real Wages, at the 5% significance level for Real Per Capita GNP, Employment, and at the 10% significance level for Money Stock, and Quarterly Real GNP. The unit root null cannot be rejected for GNP Deflator, Consumer Prices, Nominal Wages, Velocity, and Interest Rate. The results for all series are in agreement with the results corresponding to $t_{DF}^{min}(C)$, with the exception of Nominal Wages. The estimated break-date $\hat{T}_b(\hat{t}_{DF})$ is the same as $\hat{T}_b(t_{DF}^{min})$ for all series for which the unit root is rejected, except Common Stock Prices for which the estimated break-date is 1936. The use of finite sample critical values weakens the evidence against the unit root null somewhat.

Finally, consider the results with F_T^{max}. The critical values for F_T^{max}, both asymptotic and finite sample, are obtained from Table 4 above. Based on the asymptotic critical values, we can reject the joint null hypothesis at the 1% significance level for Nominal GNP, Industrial Production, Real Wages, at the 2.5% level for Real GNP, Real Per capita GNP, Common Stock Prices, at the 5% significance level for Employment, Nominal Wages, Interest Rate. The use of finite sample critical values leads to a rejection of the joint null hypothesis for these series at higher significance levels. Although F_T^{max} fails to reject H_0^J for the Money Stock and Quarterly Real GNP series, it is borderline significant at the 10% significance level. Also, F_T^{max} is significant for the Interest Rate series. The estimated break-date $\hat{T}_b(F_T^{max})$ is the same as $\hat{T}_b(t_{DF}^{min})$ for all series for which the joint null hypothesis is rejected, except for Nominal Wages and Interest Rate. The break-date, $\hat{T}_b(F_T^{max})$, for Nominal Wages is 1920 and 1964 for Interest Rate.

The failure of $t_{DF}^{min}(C)$ and $\hat{t}_{DF}(C)$ to reject $H_0 : \alpha$=1 for the Interest Rate series, and the significance of F_T^{max} can be explained as follows. It is reasonable to rule out a non-zero trend ($\mu_2 \neq 0$ and $\mu_3 \neq 0$) in the presence of a unit root, see Perron (1988), pp 304. Therefore, F_T^{max} will be significant if either $|\alpha| < 1$ or if α=1 and $\mu_1 \neq 0$. Since Vogelsang and Perron (1998) have shown that the limiting distribution of $t_{DF}^{min}(C)$ is invariant to a shift in the intercept under the null, the rejection of H_0^J can be attributed to the presence of a break under the unit root null hypothesis. To sum up, our results indicate that the unit root null hypothesis cannot be rejected for GNP Deflator, Consumer Prices, Velocity, and Interest Rate. For all other series, we reject the unit root null hypothesis. Since we have not imposed the form of break under the alternative hypothesis, our results are robust to possible misspecification and reveal important information regarding the location and form of break. Consider, for example, the results with Real Per Capita GNP. In this case, the evidence against a unit root is strengthened by using the Mixed model as the appropriate alternative, and we find that unlike previous empirical evidence the estimated break-date does not coincide with the Great Crash but rather the break occurs considerably later in 1938. Also, we uncover evidence against the unit root null for Money Stock and Quarterly Real GNP when the Mixed model is used. While the estimated break-date for Money Stock is 1930, that for Quarterly Real GNP is 1964.IV. The estimated break date of 1964 for the Quarterly Real GNP series coincides with the tax-cut mentioned in Christiano (1992). In contrast to earlier studies, our results indicate that a slope-break coefficient should be included for Nominal GNP, Real Per Capita GNP, Money Stock, and Quarterly Real GNP.

6. CONCLUSION

In this paper, we have argued that the practitioner should treat the form of break as unknown when testing for the presence of a unit root. Earlier studies have considered three different characterizations of the form of break under the alternative of trend-break stationarity, namely, the Crash model, the Changing Growth model, and the Mixed model.

The Crash model allows for a break in the intercept alone, and the Changing Growth model allows for a break in the slope with the two segments joined at the break-date. On the other hand, a simultaneous break in the intercept and slope is permitted under the Mixed model. Minimum-t-statistics to test for a unit root have been developed for all three characterizations of the alternative when the location of break is unknown. Once the form of break is treated as unknown, the Mixed model is the appropriate alternative.

Misspecification of the form of break can have serious implications for the power of the minimum-t-statistics. Our simulations indicate that the Crash (Changing Growth) model minimum-t-statistics fail to reject the unit root null if the break occurs according to the Changing Growth (Crash) model. However, there is some loss in power from using the Mixed model minimum-t-statistics when the break occurs according to the Crash or Changing Growth models. While the loss in power when the break occurs according to the Crash is negligible, the loss in power can be fairly large when the break occurs according to the Changing Growth model. We propose a maximal-F-statistic for the joint null hypothesis that there is a unit root and no break in the intercept and slope. We derive the limiting distribution of the maximal-F-statistic, and also tabulate its finite sample and asymptotic critical values. We find that the maximal-F-statistic can have greater power than the Mixed model minimum-t-statistics, and its power properties are less erratic. The use of Mixed model with both the minimal-t-statistics and the maximal-F-statistic yield a testing procedure that is robust to misspecification of the form of the break. In addition, these statistics may reveal important characteristics pertaining to the location and form of break. We test for the presence of a unit root in all Nelson and Plosser (1982) series except unemployment, and post-war U.S. Quarterly Real GNP. Unlike previous studies, we specify the alternative according to the Mixed model. Our results are most directly comparable to those in Zivot and Andrews (1992) since we do not allow for a break under the unit root null. We find evidence against the presence of a unit root in all series except GNP Deflator, Consumer Prices, Velocity, and Interest Rate.

REFERENCES

Banerjee, A., R. L. Lumsdaine, and J. H. Stock, 1992, Recursive and Sequential Tests of the Unit-Root and Trend-Break Hypothesis: Theory and International Evidence, Journal of Business and Economic Statistics, 10, 271-287.

Christiano, L. J., 1992, Searching for a break in GNP, Journal of Business and Economic Statistics, 10, 237-250.

Dickey, D. A. and W. A. Fuller, 1979, Distribution of the Estimator for Autoregressive Time Series With a Unit Root, Journal of the American Statistical Association 74, 427-431.

Dickey, D. A. and W. A. Fuller, 1981, Likelihood Ratio Statistics For Autoregressive Time Series With a Unit Root, Econometrica 49, 1057-1072.

Hall, A. R., 1992a, Joint Hypothesis Tests For a Random Walk Based on Instrumental Variable Estimators, Journal of Time Series Analysis 13, 29-45.

Hall, A. R., 1992b, Unit Root Tests After Data Based Model Selection, Proceedings of the American Statistical Association, Economics and Business Section.

Hall, A. R., 1994, Testing for a Unit Root in Time Series With Pretest Data-Based Model Selection, Journal of Business and Economic Statistics, 12, 4, 461-470.

Nelson, C. R. and C. I. Plosser, 1982, Trends and Random Walks in Macroeconomic Time Series, Some Evidence and Implications, Journal of Monetary Economics 10, 139-162.

Ng, S., and P. Perron, 1995, Unit Root Tests in ARMA Models With Data-Dependent Methods for the Selection of Eth Truncation Lag, Journal of the American Statistical Association, 90, 268-281.

Nunes, L. C., P. Newbold, and C.-M. Kuan, 1997, Testing for Unit Roots With Breaks: Evidence on the Great Crash and the Unit Root Hypothesis Reconsidered, Oxford Bulletin of Economic and Statistics, 59, 435-448.

Perron, P., 1988, Trends and Random Walks in Macroeconomic Time Series. Further Evidence from a New Approach, Journal of Economic Dynamics and Control, 12, 297-332.

Perron, P., 1989, The Great Crash, the Oil Price Shock and the Unit Root Hypothesis, Econometrica, 57, 1361-1401.

Perron, P., 1990, Tests of Joint Hypothesis For Time Series Regression With a Unit Root, in: Thomas Fomby and George Rhodes, eds., Advances in Econometrics, Vol. 8, Cointegration, Spurious Regression, and Unit Roots, (Greenwich, London: JAI Press) 135-159.

Perron, P., 1997, Further Evidence on Breaking Trend Functions in Macroeconomic Variables, Journal of Econometrics, 80, 355-385.

Perron, P. and T. J. Vogelsang, 1992, Nonsationarity and Level Shifts With an Application to Purchasing Power Parity, Journal of Business and Economic Statistics, 10, 301-320.

Phillips, P. C. B., 1987, Time Series Regression With Unit Roots, Econometrica 55, 277-302.

Phillips, P. C. B. and P. Perron, 1988, Testing For a Unit Root in Time Series Regression, Biometrika 75, 335-346.

Vogelsang, T. J. and P. Perron, 1998, Additional Tests for a Unit Root Allowing for a Break in the Trend Function at an Unknown Time, International Economic Review, 39, 1073-1100.

Zivot, E. and D. W. K. Andrews, 1992, Further Evidence on the Great Crash, the Oil-Price Shock, and the Unit-Root Hypothesis, Journal of Business and Economic Statistics, 10, 251-270.

Table 1: Finite Sample Power of $t_{DF}^{min}(i)$ and $\hat{t}_{DF}(i)$ (i=A,B,C), 5% nominal size
Crash Model DGP: $y_t = \mu_1 \, DU_t^c + \alpha \, y_{t-1} + e_t$, $e_t \sim$ i.i.d. N(0,1)

		$t_{DF}^{min}(A)$	$\hat{t}_{DF}(A)$	$t_{DF}^{min}(B)$	$\hat{t}_{DF}(B)$	$t_{DF}^{min}(C)$	$\hat{t}_{DF}(C)$
μ_1=1.0	μ_3=0	0.3000	0.2285	0.1075	0.0730	0.2430	0.2210
μ_1=2.0		0.5135	0.4035	0.0010	0.0005	0.3140	0.2870
μ_1=4.0		0.9960	0.9865	0.0000	0.0000	0.8790	0.8650
μ_1=6.0		1.0000	1.0000	0.0000	0.0000	1.0000	0.9990
μ_1=8.0		1.0000	1.0000	0.0000	0.0000	1.0000	1.0000

Table 2: Finite Sample Power of $t_{DF}^{min}(i)$ and $\hat{t}_{DF}(i)$ (i=A,B,C), 5% nominal size
Changing Growth Model DGP: $y_t = \mu_3\, DT_t^c + \alpha\, y_{t-1} + e_t$, $e_t \sim$ i.i.d. N(0,1)

		$t_{DF}^{min}(A)$	$\hat{t}_{DF}(A)$	$t_{DF}^{min}(B)$	$\hat{t}_{DF}(B)$	$t_{DF}^{min}(C)$	$\hat{t}_{DF}(C)$
μ_1=0.0	μ_3=0.1	0.0015	0.0015	0.4345	0.3700	0.2325	0.2060
	μ_3=0.2	0.0000	0.0000	0.6330	0.5650	0.2740	0.2175
	μ_3=0.3	0.0000	0.0000	0.8795	0.8420	0.4010	0.2735
	μ_3=0.4	0.0000	0.0000	0.9785	0.9720	0.6430	0.4115
	μ_3=0.5	0.0000	0.0000	0.9975	0.9965	0.8460	0.5200

Table 3: Finite Sample Power of $t_{DF}^{min}(i)$ and $\hat{t}_{DF}(i)$ (i=A,B,C), 5% nominal size
Mixed Model DGP: $y_t = \mu_1\, DU_t^c + \mu_3\, DT_t^c + \alpha\, y_{t-1} + e_t$, $e_t \sim$ i.i.d. N(0,1)

		$t_{DF}^{min}(A)$	$\hat{t}_{DF}(A)$	$t_{DF}^{min}(B)$	$\hat{t}_{DF}(B)$	$t_{DF}^{min}(C)$	$\hat{t}_{DF}(C)$
μ_1=1.0	μ_3=0.1	0.0020	0.0015	0.3570	0.3040	0.2390	0.2040
μ_1=2.0		0.0000	0.0000	0.0685	0.0535	0.2440	0.2155
μ_1=4.0		0.2985	0.2080	0.0000	0.0000	0.7820	0.7565
μ_1=6.0		0.9980	0.9965	0.0000	0.0000	0.9970	0.9960
μ_1=8.0		1.0000	1.0000	0.0000	0.0000	1.0000	1.0000
μ_1=1.0	μ_3=0.2	0.0000	0.0000	0.7420	0.6750	0.3505	0.2340
μ_1=2.0		0.0000	0.0000	0.5285	0.4580	0.2630	0.1725
μ_1=4.0		0.0000	0.0000	0.0020	0.0010	0.6580	0.6175
μ_1=6.0		0.7485	0.6440	0.0000	0.0000	0.9895	0.9875
μ_1=8.0		1.0000	0.9995	0.0000	0.0000	1.0000	1.0000
μ_1=1.0	μ_3=0.3	0.0000	0.0000	0.9480	0.9315	0.5670	0.3175
μ_1=2.0		0.0000	0.0000	0.9515	0.9355	0.5230	0.1750
μ_1=4.0		0.0000	0.0000	0.2920	0.2170	0.5435	0.4850
μ_1=6.0		0.1160	0.0650	0.0005	0.0000	0.9710	0.9635
μ_1=8.0		0.9865	0.9725	0.0000	0.0000	1.0000	1.0000
μ_1=1.0	μ_3=0.4	0.0000	0.0000	0.9910	0.9855	0.7745	0.4150
μ_1=2.0		0.0000	0.0000	0.9980	0.9975	0.8505	0.2175
μ_1=4.0		0.0000	0.0000	0.9710	0.9505	0.5800	0.3480
μ_1=6.0		0.0055	0.0020	0.2415	0.1440	0.9405	0.9290
μ_1=8.0		0.7765	0.6725	0.0000	0.0000	1.0000	1.0000
μ_1=1.0	μ_3=0.5	0.0000	0.0000	0.9995	0.9995	0.9310	0.5190
μ_1=2.0		0.0000	0.0000	1.0000	1.0000	0.9740	0.2695
μ_1=4.0		0.0000	0.0000	1.0000	1.0000	0.8915	0.2280
μ_1=6.0		0.0005	0.0005	0.9715	0.9205	0.8825	0.8270
μ_1=8.0		0.3040	0.2020	0.2210	0.0770	0.9990	0.9990

Table 4: Critical Values for F_T^{max} with $\Lambda_0 = [\lambda_0, 1-\lambda_0]$

		1%	2.5%	5%	10%
			λ_0=0.15		
T=100	k=0	12.0157	10.8744	10.0248	9.0628
	k(t-sig)	13.0842	11.8875	10.7809	9.7820
T=200	k=0	11.6450	10.4785	9.6730	8.8708
	k(t-sig)	12.6811	11.0691	10.3038	9.3692
T=∞		10.9288	10.1691	9.4376	8.6958
			λ_0=0.10		
T=100	k=0	12.1731	10.9391	10.0970	9.1282
	k(t-sig)	13.0842	11.8875	10.8409	9.8127
T=200	k=0	11.7058	10.5348	9.7310	8.9164
	k(t-sig)	12.7662	11.1565	10.3460	9.3941
T=∞		10.9841	10.2152	9.4931	8.7353
			λ_0=0.05		
T=100	k=0	12.1958	10.9940	10.1425	9.1871
	k(t-sig)	13.0842	11.9215	10.8752	9.8674
T=200	k=0	11.7236	10.5892	9.7763	8.9535
	k(t-sig)	12.7662	11.1565	10.4089	9.4368
T=∞		11.0364	10.2414	9.5427	8.7946

Table 5: Finite Sample Size of F_T^{max}, $t_{DF}^{min}(i)$ and $\hat{t}_{DF}(i)$ (i=A,B,C), 5% nominal size
DGP: $y_t = (\alpha+\rho)y_{t-1} - \alpha\rho y_{t-2} + (1+\psi L)\, e_t$, α=1, $e_t \sim$ i.i.d N(0,1)

		$t_{DF}^{min}(A)$	$\hat{t}_{DF}(A)$	$t_{DF}^{min}(B)$	$\hat{t}_{DF}(B)$	$t_{DF}^{min}(C)$	$\hat{t}_{DF}(C)$	F_T^{max}
ρ=0.0	ψ=0.0	0.0420	0.0405	0.0495	0.0530	0.0585	0.0535	0.0525
ρ=0.6	ψ=0.0	0.0550	0.0505	0.0605	0.0615	0.0580	0.0545	0.0695
ρ=-0.6	ψ=0.0	0.0405	0.0390	0.0530	0.0545	0.0540	0.0520	0.0510
ρ=0.0	ψ=0.5	0.0640	0.0585	0.0785	0.0790	0.0845	0.0800	0.0865
ρ=0.0	ψ=-0.5	0.3160	0.2815	0.3180	0.3080	0.4320	0.4030	0.4125

Table 6: Finite Sample Power of F_T^{max}, $t_{DF}^{min}(i)$ and $\hat{t}_{DF}(i)$ (i=A,B,C), 5% nominal size
Crash Model DGP: $y_t = (\alpha+\rho)y_{t-1} - \alpha\rho y_{t-2} + (1+\psi L)\,[\mu_1\, DU_t^c + e_t]$, $e_t \sim$ i.i.d N(0,1)

	$t_{DF}^{min}(A)$	$\hat{t}_{DF}(A)$	$t_{DF}^{min}(B)$	$\hat{t}_{DF}(B)$	$t_{DF}^{min}(C)$	$\hat{t}_{DF}(C)$	F_T^{max}
α=0.8, ρ=0.0, ψ=0.0							
μ_1=1	0.2380	0.2440	0.0740	0.0725	0.1995	0.1885	0.1560
μ_1=2	0.3660	0.3800	0.0010	0.0010	0.2305	0.2270	0.3640
μ_1=4	0.9825	0.9815	0.0000	0.0000	0.7875	0.7860	0.9850
μ_1=6	1.0000	1.0000	0.0000	0.0000	0.9970	0.9945	1.0000
μ_1=8	1.0000	1.0000	0.0000	0.0000	1.0000	1.0000	1.0000
α=0.8, ρ=0.6, ψ=0.0							
μ_1=1	0.8030	0.7655	0.5695	0.5510	0.7620	0.7150	0.6875
μ_1=2	0.8800	0.8690	0.1965	0.1955	0.8200	0.7920	0.7870
μ_1=4	0.9985	0.9985	0.0060	0.0075	0.9825	0.9790	0.9915
μ_1=6	1.0000	1.0000	0.0000	0.0000	1.0000	1.0000	1.0000
μ_1=8	1.0000	1.0000	0.0000	0.0000	1.0000	1.0000	1.0000
α=0.8, ρ=-0.6, ψ=0.0							
μ_1=1	0.0780	0.0825	0.0205	0.0215	0.0670	0.0640	0.0770
μ_1=2	0.2100	0.2100	0.0005	0.0005	0.0585	0.0575	0.3440
μ_1=4	0.9685	0.9545	0.0000	0.0000	0.4515	0.4355	0.9940
μ_1=6	1.0000	1.0000	0.0000	0.0000	0.9530	0.9305	1.0000
μ_1=8	1.0000	1.0000	0.0000	0.0000	1.0000	0.9975	1.0000
α=0.8, ρ=0.0, ψ=0.5							
μ_1=1	0.2245	0.2185	0.0750	0.0730	0.2080	0.1965	0.1740
μ_1=2	0.3710	0.3655	0.0015	0.0010	0.2280	0.2160	0.3755
μ_1=4	0.9820	0.9810	0.0000	0.0000	0.7795	0.7790	0.9850
μ_1=6	1.0000	1.0000	0.0000	0.0000	0.9975	0.9970	1.0000
μ_1=8	1.0000	1.0000	0.0000	0.0000	1.0000	1.0000	1.0000
α=0.8, ρ=0.0, ψ=-0.5							
μ_1=1	0.7265	0.6935	0.4345	0.4145	0.7580	0.7065	0.7325
μ_1=2	0.7215	0.6775	0.0770	0.0755	0.6840	0.6275	0.7060
μ_1=4	0.9820	0.9810	0.0000	0.0000	0.8685	0.8525	0.9830
μ_1=6	1.0000	1.0000	0.0000	0.0000	0.9965	0.9960	0.9995
μ_1=8	1.0000	1.0000	0.0000	0.0000	1.0000	1.0000	1.0000

Table 7: Finite Sample Power of F_T^{max}, $t_{DF}^{min}(i)$ and $\hat{t}_{DF}(i)$ (i=A,B,C), 5% nominal size
Changing Growth Model DGP: $y_t = (\alpha+\rho)y_{t-1} - \alpha\rho y_{t-2} + (1+\psi L)\left[\mu_3\, DT_t^c + e_t\right]$, $e_t \sim$ i.i.d N(0,1)

	$t_{DF}^{min}(A)$	$\hat{t}_{DF}(A)$	$t_{DF}^{min}(B)$	$\hat{t}_{DF}(B)$	$t_{DF}^{min}(C)$	$\hat{t}_{DF}(C)$	F_T^{max}
α=0.8, ρ=0.0, ψ=0.0							
μ_3=0.1	0.0000	0.0000	0.3110	0.3340	0.1870	0.1720	0.1440
μ_3=0.2	0.0000	0.0000	0.5150	0.5400	0.2120	0.1720	0.2350
μ_3=0.3	0.0000	0.0000	0.7470	0.7680	0.3153	0.2390	0.4400
μ_3=0.4	0.0000	0.0000	0.9365	0.9445	0.5190	0.3475	0.7655
μ_3=0.5	0.0000	0.0000	0.9915	0.9930	0.7385	0.4495	0.9450
α=0.8, ρ=0.6, ψ=0.0							
μ_3=0.1	0.1010	0.1085	0.8225	0.8315	0.7215	0.7040	0.6265
μ_3=0.2	0.0000	0.0000	0.8625	0.8765	0.7030	0.6740	0.6360
μ_3=0.3	0.0000	0.0000	0.9225	0.9305	0.7430	0.6815	0.7165
μ_3=0.4	0.0000	0.0000	0.9675	0.9725	0.7845	0.7020	0.7925
μ_3=0.5	0.0000	0.0000	0.9880	0.9930	0.8190	0.7000	0.8790
α=0.8, ρ=-0.6, ψ=0.0							
μ_3=0.1	0.0000	0.0000	0.1985	0.2105	0.0680	0.0590	0.0775
μ_3=0.2	0.0000	0.0000	0.4055	0.4215	0.1085	0.0825	0.2350
μ_3=0.3	0.0000	0.0000	0.7940	0.8000	0.3220	0.2105	0.7140
μ_3=0.4	0.0000	0.0000	0.9800	0.9780	0.6755	0.4380	0.9600
μ_3=0.5	0.0000	0.0000	0.9990	0.9955	0.9165	0.6150	0.9870
α=0.8, ρ=0.0, ψ=0.5							
μ_3=0.1	0.0010	0.0010	0.3155	0.3415	0.1860	0.1725	0.1555
μ_3=0.2	0.0000	0.0000	0.5375	0.5625	0.2220	0.1840	0.2570
μ_3=0.3	0.0000	0.0000	0.7450	0.7660	0.3160	0.2340	0.4495
μ_3=0.4	0.0000	0.0000	0.9360	0.9430	0.5040	0.3310	0.7200
μ_3=0.5	0.0000	0.0000	0.9910	0.9910	0.7230	0.4610	0.9045
α=0.8, ρ=0.0, ψ=-0.5							
μ_3=0.1	0.0100	0.0095	0.7600	0.7700	0.7210	0.7215	0.7060
μ_3=0.2	0.0000	0.0000	0.8125	0.8205	0.7110	0.7050	0.7000
μ_3=0.3	0.0000	0.0000	0.8940	0.9015	0.7000	0.6840	0.7280
μ_3=0.4	0.0000	0.0000	0.9565	0.9630	0.7460	0.7025	0.8150
μ_3=0.5	0.0000	0.0000	0.9920	0.9925	0.8095	0.7035	0.9245

Table 8: Finite Sample Power of F_T^{max}, $t_{DF}^{min}(i)$ and $\hat{t}_{DF}(i)$ (i=A,B,C), 5% nominal size
Mixed Model DGP: $y_t = (\alpha+\rho)y_{t-1} - \alpha\rho y_{t-2} + (1+\psi L)\,[\mu_1\, DU_t^c + \mu_3\, DT_t^c + e_t]$, $e_t \sim$ i.i.d N(0,1)

		$t_{DF}^{min}(A)$	$\hat{t}_{DF}(A)$	$t_{DF}^{min}(B)$	$\hat{t}_{DF}(B)$	$t_{DF}^{min}(C)$	$\hat{t}_{DF}(C)$	F_T^{max}
α=1, ρ=0.0, ψ=0.0								
μ_1=1	μ_3=0.1	0.0005	0.0005	0.3070	0.3345	0.2115	0.1900	0.2100
μ_1=2		0.0015	0.0015	0.0760	0.0845	0.1770	0.1640	0.4350
μ_1=4		0.2545	0.2615	0.0000	0.0000	0.6180	0.6175	0.9890
μ_1=6		0.9920	0.9805	0.0000	0.0000	0.9930	0.9915	1.0000
μ_1=1	μ_3=0.2	0.0000	0.0000	0.6450	0.6720	0.3100	0.2335	0.3650
μ_1=2		0.0000	0.0000	0.5835	0.6130	0.2735	0.1525	0.6625
μ_1=4		0.0110	0.0115	0.0380	0.0455	0.5065	0.4995	0.9960
μ_1=6		0.5835	0.5720	0.0000	0.0000	0.9645	0.9605	1.0000
μ_1=1	μ_3=0.3	0.0000	0.0000	0.8705	0.8820	0.4750	0.3325	0.6075
μ_1=2		0.0000	0.0000	0.9480	0.9545	0.5885	0.2540	0.8525
μ_1=4		0.0005	0.0005	0.4920	0.5290	0.4235	0.3610	0.9995
μ_1=6		0.1360	0.1350	0.0250	0.0320	0.9245	0.9190	1.0000
μ_1=1	μ_3=0.4	0.0000	0.0000	0.9740	0.9770	0.6530	0.3930	0.8595
μ_1=2		0.0000	0.0000	0.9950	0.9970	0.8010	0.2950	0.9660
μ_1=4		0.0005	0.0000	0.9600	0.9695	0.5945	0.2545	0.9985
μ_1=6		0.0330	0.0350	0.3915	0.4430	0.8350	0.8285	1.0000
α=1, ρ=0.6, ψ=0.0								
μ_1=1	μ_3=0.1	0.1025	0.1110	0.7845	0.7945	0.7480	0.7295	0.7015
μ_1=2		0.0635	0.0690	0.4970	0.5185	0.8035	0.7875	0.8150
μ_1=4		0.3505	0.3610	0.0825	0.0890	0.9720	0.9630	0.9905
μ_1=6		0.9905	0.9905	0.0040	0.0045	1.0000	1.0000	1.0000
μ_1=1	μ_3=0.2	0.0000	0.0000	0.8875	0.9015	0.7695	0.7240	0.7375
μ_1=2		0.0005	0.0010	0.7855	0.8040	0.8000	0.7645	0.8625
μ_1=4		0.0025	0.0030	0.2525	0.2775	0.9650	0.9585	0.9960
μ_1=6		0.3135	0.3505	0.0500	0.0590	0.9985	0.9985	1.0000
μ_1=1	μ_3=0.3	0.0000	0.0000	0.9565	0.9610	0.8005	0.7140	0.7980
μ_1=2		0.0000	0.0000	0.9470	0.9550	0.8390	0.7605	0.9205
μ_1=4		0.0005	0.0005	0.5760	0.6085	0.9655	0.9510	0.9985
μ_1=6		0.0110	0.0150	0.2205	0.2375	0.9995	0.9995	1.0000
μ_1=1	μ_3=0.4	0.0000	0.0000	0.9885	0.9895	0.8500	0.7155	0.8860
μ_1=2		0.0000	0.0000	0.9900	0.9915	0.8935	0.7540	0.9615
μ_1=4		0.0000	0.0000	0.8835	0.9065	0.9685	0.9415	0.9995
μ_1=6		0.0005	0.0005	0.5025	0.5360	0.9990	0.9990	1.0000
α=1, ρ=-0.6, ψ=0.0								
μ_1=1	μ_3=0.1	0.0010	0.0010	0.2710	0.2935	0.1265	0.1105	0.1465
μ_1=2		0.0005	0.0000	0.1365	0.1545	0.0740	0.0545	0.4850
μ_1=4		0.2660	0.2365	0.0000	0.0000	0.2205	0.2155	0.9950
μ_1=6		0.9910	0.9675	0.0000	0.0000	0.8315	0.8190	1.0000
μ_1=1	μ_3=0.2	0.0000	0.0000	0.6715	0.6930	0.2700	0.2025	0.4265
μ_1=2		0.0000	0.0000	0.8605	0.8745	0.4295	0.1530	0.7825
μ_1=4		0.0130	0.0075	0.2205	0.2525	0.1395	0.1050	0.9980
μ_1=6		0.6400	0.5860	0.0035	0.0040	0.6130	0.5965	1.0000

Table 8 (Continued): Finite Sample Power of F_T^{max}, $t_{DF}^{min}(i)$ and $\hat{t}_{DF}(i)$ (i=A,B,C), 5% nominal size
Mixed Model DGP: $y_t = (\alpha+\rho)y_{t-1} - \alpha\rho y_{t-2} + (1+\psi L)\left[\mu_1\, DU_t^c + \mu_3\, DT_t^c + e_t\right]$, $e_t \sim$ i.i.d N(0,1)

		$t_{DF}^{min}(A)$	$\hat{t}_{DF}(A)$	$t_{DF}^{min}(B)$	$\hat{t}_{DF}(B)$	$t_{DF}^{min}(C)$	$\hat{t}_{DF}(C)$	F_T^{max}
α=1, ρ=-0.6, ψ=0.0								
μ_1=1	μ_3=0.3	0.0000	0.0000	0.9145	0.9245	0.5390	0.3500	0.8140
μ_1=2		0.0000	0.0000	0.9765	0.9795	0.7645	0.2760	0.9455
μ_1=4		0.0000	0.0000	0.9515	0.9580	0.5210	0.0460	0.9995
μ_1=6		0.1585	0.1205	0.2765	0.3170	0.3380	0.3245	1.0000
μ_1=1	μ_3=0.4	0.0000	0.0000	0.9865	0.9870	0.8080	0.4850	0.9625
μ_1=2		0.0000	0.0000	0.9980	0.9980	0.9215	0.3870	0.9930
μ_1=4		0.0000	0.0000	0.9995	0.9945	0.9295	0.0625	1.0000
μ_1=6		0.0075	0.0050	0.9555	0.9535	0.4935	0.1620	1.0000
α=1, ρ=0.0, ψ=0.5								
μ_1=1	μ_3=0.1	0.0000	0.0010	0.3285	0.3530	0.2220	0.2010	0.2405
μ_1=2		0.0010	0.0010	0.1120	0.1240	0.2065	0.1875	0.4995
μ_1=4		0.2775	0.2920	0.0005	0.0005	0.6005	0.5900	0.9860
μ_1=6		0.9920	0.9890	0.0000	0.0000	0.9805	0.9805	1.0000
μ_1=1	μ_3=0.2	0.0000	0.0000	0.6875	0.7145	0.3720	0.2855	0.4170
μ_1=2		0.0000	0.0000	0.6685	0.6965	0.3495	0.1865	0.6750
μ_1=4		0.0180	0.0165	0.0500	0.0620	0.4805	0.4640	0.9940
μ_1=6		0.6055	0.6355	0.0005	0.0015	0.9530	0.9505	1.0000
μ_1=1	μ_3=0.3	0.0000	0.0000	0.8910	0.8995	0.4930	0.3300	0.6260
μ_1=2		0.0000	0.0000	0.9635	0.9700	0.6435	0.2905	0.8640
μ_1=4		0.0025	0.0015	0.6265	0.6620	0.4140	0.3190	0.9965
μ_1=6		0.1820	0.1965	0.0530	0.0690	0.8820	0.8810	1.0000
μ_1=1	μ_3=0.4	0.0000	0.0000	0.9735	0.9785	0.6630	0.4255	0.7200
μ_1=2		0.0000	0.0000	0.9915	0.9920	0.8160	0.3515	0.8395
μ_1=4		0.0000	0.0000	0.9795	0.9845	0.6635	0.2335	0.9405
μ_1=6		0.0430	0.0470	0.5100	0.5610	0.7755	0.7605	1.0000
α=1, ρ=0.0, ψ=-0.5								
μ_1=1	μ_3=0.1	0.0075	0.0085	0.7275	0.7385	0.7410	0.7325	0.7265
μ_1=2		0.0015	0.0015	0.4005	0.4110	0.7120	0.6880	0.7785
μ_1=4		0.1755	0.1950	0.0030	0.0030	0.7960	0.7680	0.9845
μ_1=6		0.9780	0.9810	0.0000	0.0000	0.9890	0.9890	1.0000
μ_1=1	μ_3=0.2	0.0000	0.0000	0.8565	0.8660	0.7565	0.7405	0.7575
μ_1=2		0.0000	0.0000	0.7580	0.7755	0.7450	0.7125	0.8375
μ_1=4		0.0030	0.0025	0.0770	0.0840	0.7500	0.6945	0.9935
μ_1=6		0.3810	0.4205	0.0000	0.0000	0.9860	0.9855	1.0000
μ_1=1	μ_3=0.3	0.0000	0.0000	0.9365	0.9445	0.7760	0.7330	0.7950
μ_1=2		0.0000	0.0000	0.9635	0.9705	0.8380	0.7425	0.9250
μ_1=4		0.0000	0.0000	0.4025	0.4405	0.7205	0.6240	0.9975
μ_1=6		0.0650	0.0745	0.0035	0.0045	0.9640	0.9640	1.0000
μ_1=1	μ_3=0.4	0.0000	0.0000	0.9840	0.9855	0.8250	0.7555	0.8925
μ_1=2		0.0000	0.0000	0.9925	0.9940	0.9000	0.7680	0.9680
μ_1=4		0.0000	0.0000	0.9030	0.9175	0.7665	0.5975	0.9995
μ_1=6		0.0075	0.0080	0.1200	0.1450	0.9210	0.9185	1.0000

Table 9: Empirical Results for the Nelson-Plosser (1982) Data and Quarterly Real GNP
Mixed Model Regression: $y_t = \hat{\mu}_0 + \hat{\mu}_1\, DU_t(T_b) + \hat{m}u_2\, t + \hat{\mu}_3\, DT_t(T_b) + \hat{\alpha}\, y_{t-1} + \hat{e}_t$

Series	$t_{DF}^{min}(C)$	$\hat{T}_b(t_{DF}^{min})$	$\hat{t}_{DF}(C)$	$\hat{T}_b(\hat{t}_{DF})$	F_T^{max}	$T_b(F_T^{max})$
Real GNP	$-5.6577^{(a,c)}$	1929	$-5.6577^{(b,b)}$	1929	$10.9804^{(b,c)}$	1929
Nominal GNP	$-6.1741^{(a,b)}$	1929	$-6.1741^{(a,a)}$	1929	$13.0122^{(a,b)}$	1929
Real Per Capita GNP	$-5.2983^{(b,d)}$	1938	$-5.2983^{(c,c)}$	1938	$10.2524^{(b,d)}$	1938
Industrial Production	$-5.8192^{(a,c)}$	1929	$-5.8192^{(a,b)}$	1929	$12.0478^{(a,b)}$	1929
Employment	$-5.1995^{(c,.)}$	1929	$-5.1995^{(c,d)}$	1929	$10.2154^{(c,d)}$	1929
GNP Deflator	-4.1723	1929	-4.1723	1920	8.5103	1920
Consumer Prices	-3.5120	1893	-2.1086	1869	8.3915	1863
Nominal Wages	$-5.2147^{(c,.)}$	1929	-3.6338	1920	$9.9920^{(c,d)}$	1920
Money Stock	$-4.9709^{(d,.)}$	1930	$-4.9709^{(d,d)}$	1930	8.7065	1930
Velocity	-3.9737	1929	-3.9737	1929	5.5704	1929
Interest Rate	-1.8785	1963	-1.3131	1964	$9.8964^{(c,d)}$	1964
Common Stock Prices	$-5.5152^{(b,d)}$	1939	$-5.5015^{(b,c)}$	1936	$10.2942^{(b,d)}$	1939
Real Wages	$-5.4509^{(b,d)}$	1940	$-5.4509^{(b,c)}$	1940	$11.3206^{(a,c)}$	1940
Quarterly Real GNP	$-5.1135^{(c,.)}$	1964.IV	$-5.1135^{(d,d)}$	1964.IV	8.7649	1964.IV

NOTE: The small letters in parenthesis that appear as superscript indicate the significance of these statistics. The first letter in the parenthesis indicates significance with respect to the asymptotic critical values, and the second letter indicates significance with respect to the appropriate finite sample critical values. 'a', 'b', 'c', and 'd' indicate significance at the 1%, 2.5%, 5%, and 10% significance level, and '.' appears if the statistic is not significant at the 10% significance level. The asymptotic and finite sample critical values of $t_{DF}^{min}(C)$ are taken from Table 4 in Zivot and Andrews (1992) and Table 1 in Perron (1997) respectively. The asymptotic and finite sample critical values for $\hat{t}_{DF}(C)$ were obtained from Table 3 in Vogelsang and Perron (1998). The asymptotic and finite sample critical values for F_T^{max} are taken from Table 4 above.

Table 10: Estimated Regressions for the Nelson-Plosser (1982) Data and Quarterly Real GNP
Mixed Model Regression: $y_t = \hat{\mu}_0 + \hat{\mu}_1\, DU_t(T_b) + \hat{m}u_2\, t + \hat{\mu}_3\, DT_t(T_b) + \hat{\alpha}\, y_{t-1} + \hat{e}_t$

Series	T_b	k*	$\hat{\mu}_0$	$\hat{\mu}_1$	$\hat{\mu}_2$	$\hat{\mu}_3$	$\hat{\alpha}$	$S(\hat{e})$
Real GNP	1929	8	3.6921	-0.1793	0.0241	0.0045	0.2352	0.0503
			(5.6850)	(-4.2164)	(4.1459)	(1.0059)	(-5.6577)	
Nominal GNP	1929	8	5.4047	-0.2799	0.0224	0.0115	0.5057	0.0680
			(6.2240)	(-4.5282)	(3.0520)	(1.7811)	(-6.1741)	
Real Per Capita GNP	1938	2	3.2484	0.1326	0.0010	0.0072	0.5490	0.0522
			(5.3017)	(4.0687)	(0.7412)	(3.1672)	(-5.2983)	
Industrial Production	1929	8	0.1339	-0.3239	0.0331	0.0012	0.3005	0.0876
			(4.3925)	(-5.1854)	(5.6933)	(0.9001)	(-5.8192)	
Employment	1929	8	5.1709	-0.0747	0.0098	-0.0015	0.4914	0.0289
			(5.2386)	(-3.8435)	(4.5430)	(-1.6187)	(-5.1995)	
GNP Deflator	1929	5	0.6601	-0.0899	0.0061	0.0010	0.7831	0.0435
			(4.2638)	(3.1706)	(3.6491)	(1.0358)	(-4.1723)	
	1920	5	0.2720	-0.0997	0.0060	-0.0026	0.9042	0.0418
			(2.4134)	(4.2269)	(4.0137)	(-2.0290)	(-2.6100)	
Consumer Prices	1893	5	0.3813	-0.0205	-0.0009	0.0031	0.8921	0.0361
			(3.3150)	(-1.1960)	(-0.8578)	(2.2153)	(-3.5120)	
	1869	0	0.3109	-0.0077	-0.0193	0.0204	0.9528	0.0520
			(3.7658)	(-0.2035)	(-2.8023)	(2.9934)	(-2.1086)	
Nominal Wages	1929	7	2.1348	-0.1640	0.0177	-0.0004	0.6581	0.0542
			(5.2841)	(-3.8359)	(4.4476)	(-0.1847)	(-5.2147)	
	1920	7	0.9733	-0.1516	0.0211	-0.0133	0.8357	0.0535
			(3.4607)	(-4.3287)	(4.4541)	(-3.0775)	(-3.6338)	
Money Stock	1930	8	0.4831	-0.1053	0.0212	-0.0023	0.6893	0.0419
			(5.4325)	(-3.4871)	(4.6864)	(-1.9341)	(-4.9709)	
Velocity	1929	1	0.3540	-0.0473	-0.0041	0.0060	0.7635	0.0636
			(3.6305)	(-1.7643)	(-3.3186)	(3.6149)	(-3.9737)	
Interest Rate	1963	3	0.4049	-0.0231	-0.0016	0.1594	0.9047	0.2481
			(1.8000)	(-0.0987)	(-0.7641)	(2.8927)	(-1.8785)	
	1964	0	0.2403	-0.1287	-0.0004	0.2157	0.9421	0.2475
			(1.2593)	(-0.5393)	(-0.1989)	(3.4392)	(-1.3131)	
Common Stock Prices	1939	1	0.5320	-0.1942	0.0078	0.0261	0.6038	0.1397
			(5.2391)	(-2.5958)	(4.7365)	(4.6067)	(-5.5152)	
	1936	3	0.5743	-0.2783	0.0090	0.0252	0.5702	0.1392
			(5.2399)	(-3.6409)	(4.8941)	(4.9949)	(-5.5015)	
Real Wages	1940	3	1.8163	0.0842	0.0086	0.0047	0.3892	0.0307
			(5.4785)	(4.3846)	(5.3509)	(3.3869)	(-5.4509)	
Quarterly Real GNP	1964.IV	12	1.3289	0.0171	0.0015	-0.0001	0.8218	0.0086
			(5.1524)	(4.1128)	(4.8642)	(-1.7395)	(-5.1135)	

NOTE: T_b is the chosen break-date, k* is the value of the lag-truncation parameter chosen according to the k(t-sig) procedure of Perron and Vogelsang (1992) with kmax=8 for all series except Quarterly Real GNP for which kmax=12. The t-statistics appear in parenthesis. The t-statistic for $\hat{\alpha}$ tests the hypothesis that α=1. $S(\hat{e})$ is the estimated standard deviation of the regression error.

APPENDIX

Outline of Proof for Theorem 1: For any $\lambda \in \Lambda$, we would like to show that: $[\Gamma_T([\lambda\, T])]^{-1}\,[\Psi_T([\lambda\, T])] \Rightarrow [\Gamma(\lambda)]^{-1}\,[\Psi(\lambda)]$

First consider: $\Psi_T([\lambda\, T]) = [\Psi_{1,T}([\lambda\, T]), \Psi_{2,T}([\lambda\, T]), \Psi_{3,T}([\lambda\, T]), \Psi_{4,T}([\lambda\, T]), \Psi_{5,T}([\lambda\, T]), \Psi_{6,T}([\lambda\, T])]'$. Using the results in Banerjee, Lumsdaine, Stock (1992), it follows that:

$$\Psi_{1,T}([\lambda\, T]) = T^{-1/2}\, \sum_{t=1}^{T} Z_{t-1}^{1}\, e_t \Rightarrow \sigma B(1)$$

$$\Psi_{2,T}([\lambda\, T]) = T^{-1/2}\, \sum_{t=1}^{t} e_t \Rightarrow \sigma W(1)$$

$$\Psi_{3,T}([\lambda\, T]) = T^{-1}\, \sum_{t=1}^{T}[y_{t-1} - \bar{\mu}_0(t-1)]e_t \Rightarrow \sigma \int_0^1\, J(\delta)\, dW(\delta)$$

$$\Psi_{4,T}([\lambda\, T]) = T^{-1/2}\, \sum_{t=[\lambda\, T]+1}^{T} e_t \Rightarrow \sigma[W(1) - W(\lambda)]$$

$$\Psi_{5,T}([\lambda\, T]) = T^{-3/2} \sum_{t=[\lambda\, T]+1}^{T}(t - [\lambda\, T])e_t$$

$$= T^{-3/2} \sum_{t=1}^{T} t\, e_t - T^{-3/2} \sum_{t=1}^{[\lambda\, T} t\, e_t - ([\lambda\, T]/T) \left[T^{-1/2} \sum_{t=1}^{T} e_t - T^{-1/2} \sum_{t=1}^{[\lambda\, T] e_t}\right]$$

$$\Rightarrow \sigma\, [W(1) - \int_0^1 W(\delta)\, d\delta - \lambda\, W(\lambda) + \int_0^\lambda W(\delta)\, d\delta - \lambda W(1) + \lambda W(\lambda)]$$

$$= \sigma\, [(1-\lambda)\, W(\lambda) - \int_\lambda^1 W(\delta)\, d\delta]$$

$$\Psi_{6,T}([\lambda\, T]) = T^{-3/2} \sum_{t=1}^{T} te_t \Rightarrow \sigma[W(1) - \int_0^1 W(\delta)\, d\delta\ ]$$

where $J(\lambda) \equiv [1 - \sum_{i=1}^{k} c_i]^{-1} \sigma\, W(\lambda)$. Therefore, $\Psi_T([\lambda\, T]) \Rightarrow \Psi(\lambda)$

The convergence of $\Psi_{2,T}([\lambda\, T])$, $\Psi_{4,T}([\lambda\, T])$, $\Psi_{5,T}([\lambda\, T])$, and $\Psi_{6,T}([\lambda\, T])$ follows from the Assumption 1 and the Functional Central Limit Theorem. The convergence of $\Psi_{3,T}([\lambda\, T])$ follows from $T^{-1} \sum_{t=1}^{T} [y_{t-1} - \bar{\mu}_0(t-1)]\, e_t \Rightarrow \sigma \int_0^1 J(\delta)\, dW(\delta)$, and the convergence of $\Psi_{1,T}([\lambda\, T])$ follows from $T^{-1/2} \sum_{t=1}^{[\lambda\, T]} Z_{t-1}^{1}\, e_t \Rightarrow \sigma\, B(\lambda)$ where $B(\lambda)$ is a k-dimensional Brownian motion with covariance matrix $E[Z_t^1\, Z_t^{1}{}'] = \Omega_k$.

Next, consider $[\Gamma_T([\lambda\, T])] = \Upsilon_T^{-1} \left[\sum_{t=1}^{T} Z_{t-1}([\lambda\, T])\, Z_{t-1}([\lambda\, T])'\right] \Upsilon_T^{-1}$. Let $\Gamma_{i,j,T}(.)$ denoted the (i,j)th element of $\Gamma_T(.)$. Straightforward calculations yield the results for $\Gamma_{i,j,T}([\lambda\, T])$ for i,j=2,4,5,6. The convergence of $\Gamma_{2,3,T}([\lambda\, T])$ and $\Gamma_{3,i,T}([\lambda\, T])$ for j=3,4,5,6 follows from $T^{-1/2} \sum_{t=1}^{[\lambda\, T]}(\Delta\, y_t - \bar{\mu}_0) \Rightarrow (1 - \sum_{j=1}^{k} c_j)^{-1} \sigma W(\lambda) \equiv J(\lambda)$. The convergenc of $\Gamma_{1,1,T}([\lambda\, T])$ follows from $T^{-1} \sum_{t=1}^{T} Z_{t-1}^{1}\, Z_{t-1}^{1}{}' \xrightarrow{p} \Omega_k$, the convergence of $\Gamma_{1,j,T}([\lambda\, t])$ for j=2,4,5,6 follows from the Functional Central Limit Theorem, and the convergence of $\Gamma_{1,3,T}([\lambda\, T])$ follows from $T^{-3/2}\sum_{t=1}^{T} Z_{t-1}^{1}\, y_t \Rightarrow 0_k$. Combining these results, we obtain $\Gamma_T([\lambda\, T]) \Rightarrow \Gamma(\lambda)$, and the result in Theorem 1 follows, that is:

$$\Gamma_{1,1,T}([\lambda\, T]) = T^{-1}\sum_{t=1}^{T} Z_{t-1}^{1}\, Z_{t-1}^{1\,\prime} \Rightarrow \Omega_k,\ \Gamma_{1,2,T}([\lambda\, T]) = T^{-1}\sum_{t=1}^{T} Z_{t-1}^{1} \to 0_k$$

$$\Gamma_{1,3,T}([\lambda\, T]) = T^{-3/2}\sum_{t=1}^{T} Z_{t-1}^{1}\,[y_{t-1} - \bar{\mu}_0(t-1)] \to 0_k, \Gamma_{1,4,T}([\lambda\, T]) = T^{-1}\sum_{t=[\lambda\, T]+1}^{T} Z_{t-1}^{1} \to 0_k$$

$$\Gamma_{1,5,T}([\lambda\, T]) = T^{-2}\sum_{t=[\lambda\, T]+1}^{T} (t - [\lambda\, T])\, Z_{t-1}^{1} \to 0_k\ \Gamma_{1,6,T}([\lambda\, T]) = T^{-2}\sum_{t=1}^{T} t\, Z_{t-1}^{1} \to 0_k$$

$$\Gamma_{2,2,T}([\lambda\, T]) = 1,\ \Gamma_{2,3,T}([\lambda\, T]) = T^{-3/2}\sum_{t=1}^{T}[y_{t-1} - \bar{\mu}_0\,(t-1)] \Rightarrow \int_0^1 J(\delta)\, d\delta$$

$$\Gamma_{2,4,T}([\lambda\, T]) = (T - [\lambda\, T])/T \to (1-\lambda)\ \Gamma_{2,5,T}([\lambda\, T]) = T^{-2}\sum_{t=[\lambda\, T]+1}^{T} (t - [\lambda\, T]) \to (1-\lambda)^2/2$$

$$\Gamma_{2,6,T}([\lambda\, T]) = T^{-2}\sum_{t=1}^{T} t \to 1/2\ \Gamma_{3,3,T}([\lambda\, T]) = T^{-2}\sum_{t=1}^{T}[y_{t-1} - \bar{\mu}_0\,(t-1)]^2 \Rightarrow \int_0^1 J(\delta)^2\, d\delta$$

$$\Gamma_{3,4,T}([\lambda\, T]) = T^{-3/2}\sum_{t=[\lambda\, T]+1}^{T}[y_{t-1} - \bar{\mu}_0\,(t-1)] \Rightarrow \int_\lambda^1 J(\delta)\, d\delta$$

$$\Gamma_{3,5,T}([\lambda\, T]) = T^{-5/2}\sum_{t=[\lambda\, T]+1}^{T}[y_{t-1} - \bar{\mu}_0\,(t-1)]\,(t - [\lambda\, T]) \Rightarrow \int_\lambda^1 (\delta - \lambda)\, J(\delta)\, d\delta$$

$$\Gamma_{3,6,T}([\lambda\, T]) = T^{-5/2}\sum_{t=1}^{T}[y_{t-1} - \bar{\mu}_0\,(t-1)]\, t \Rightarrow \int_0^1 \delta\, J(\delta)\, d\delta$$

$$\Gamma_{4,4,T}([\lambda\, T]) = (T - [\lambda\, T])/T \to 1-\lambda,\ \Gamma_{4,5,T}([\lambda\, T]) = T^{-2}\sum_{t=[\lambda\, T]+1}^{T} (t - [\lambda\, T]) \to (1-\lambda)^2/2$$

$$\Gamma_{4,6,T}([\lambda\, T]) = T^{-2}\sum_{t=[\lambda\, T]+1}^{T} t \to (1-\lambda^2)/2,\ \Gamma_{5,5,T}([\lambda\, T]) = T^{-3}\sum_{t=[\lambda\, T]+1}^{T} (t - [\lambda\, T])^2 \to (1-\lambda)^3/3$$

$$\Gamma_{5,6,T}([\lambda\, T]) = T^{-3}\sum_{t=[\lambda\, T]+1}^{T} (t - [\lambda\, T])\, t \to 1/3 - \lambda/2 + \lambda^3/6,\ \Gamma_{6,6,T}([\lambda\, T]) = T^{-2}\sum_{t=1}^{T} t^2 \to 1/3$$